Nonextensive statistics and multipatricle production processes

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Outline

- Part 1: Correspondence of multiplicity and energy distributions
- Part 2: Relaxation and correlation times of nonequilibrium multiparticle systems

Correspondence of multiplicity and energy distributions

For the count probability distribution, $P(N)$, the generating function $G(z)$ is defined as:

$$G(z) = \sum_{N=0}^{\infty} P(N) z^N. \tag{1}$$

We shall discuss multiplicity distributions $P(N)$ in quasi power-law ensembles and their generating functions $G(z)$. They are connected with the energy distributions $F(E)$ of elements in the ensemble.

Correspondence of multiplicity and energy distributions

Distributions $P(N)$ used in this talk: Poisson (PD), negative binomial (NBD) and binomial (BD) and their generating functions $G(z)$.

	$P(N)$	$G(z)$
PD	$\frac{\lambda^N}{N!}\exp(-\lambda)$	$\exp[\lambda(z-1)]$
NBD	$\frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)}p^N(1-p)^k$	$\left[1-\frac{p}{1-p}(z-1)\right]^{-k}$
BD	$\frac{K!}{N!(K-N)!}p^N(1-p)^{K-N}$	$[1+p(z-1)]^K$

Note, that generating functions of NBD and BD are in fact some quasi-power functions of z and as such can be written in the form of the corresponding Tsallis distributions.

Correspondence of multiplicity and energy distributions

The multiplicity generating function

$$\begin{aligned} G(z) &= \exp_q\left[\langle N\rangle(1-z)\right] \\ &= \left[1+(q-1)\langle N\rangle(1-z)\right]^{\frac{1}{1-q}}, \end{aligned} \tag{2}$$

where $q-1=1/k$ for NBD, $q-1=-1/K$ for BD, and $q-1\to 0$ for PD.
For

$$z = 1-\frac{E}{U} \tag{3}$$

with the total available energy

$$U=\sum_{i=1}^{N}E_i, \tag{4}$$

Eq. (2) gives the energy distribution

$$F(E) = G(z=1-E/U) = \left[1+(q-1)\frac{E}{T}\right]^{\frac{1}{1-q}} \tag{5}$$

which is the well known Tsallis distribution, and which for $q\to 1$ becomes Boltzmann-Gibbs distribution.

Correspondence of multiplicity and energy distributions

For fixed number of particles N, energy distribution emerges directly from the calculus of probability for a situation known as *induced partition*. In short: $N-1$ randomly chosen independent points $\{U_1, \ldots, U_{N-1}\}$ split a segment $(0, U)$ into N parts, whose length is distributed according to:

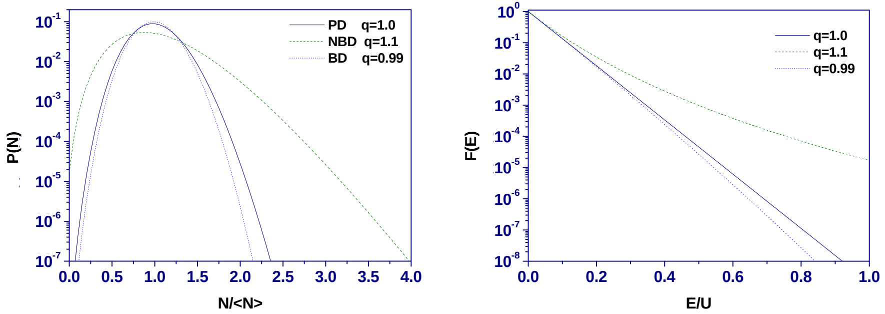
$$F(E|N) = \frac{N-1}{U}\left(1 - \frac{E}{U}\right)^{N-2}. \tag{6}$$

The length of the kth part corresponds to the value of energy $E_k = U_{k+1} - U_k$ (for ordered U_k). Whereas for fixed N one have equation (6), then for N fluctuating according to $P(N)$, the resulting energy distribution is

$$F(E) = \sum_{N=2}^{\infty} P(N) F(E|N). \tag{7}$$

For $P(N)$ given by BD, PD, and NBD, equation (7) leads to Tsallis distribution. Relationships between Poissonian multiplicity distribution and Boltzmann-Gibbs energy distribution are discussed in [Eur. Phys. J. A **57** 3].

Correspondence of multiplicity and energy distributions



Multiplicity distributions and corresponding energy distributions. Figure taken from [Eur. Phys. J. A **57** 3].

Correspondence of multiplicity and energy distributions

- The statistical properties of the energy division between a set of particles are completely characterized by the generating function $G(z)$.

- Despite correspondence between multiplicity and energy distributions, the multiplicity distribution gives in practice complementary information to the energy distribution, because $P(N)$ is defined by the N^{th} derivative of $G(z) = F(E)$ at $E = U$, i.e., in the region not available experimentally in measurements at collider experiments.

Relaxation and correlation times of nonequilibrium multiparticle systems

The evolution of the particle distribution can be studied through the Boltzmann transport equation (BTE),

$$\frac{df(r,p,t)}{dt} = \frac{\partial f}{\partial t} + \vec{u} \cdot \nabla_r f + \vec{F} \cdot \nabla_p f = C[f], \tag{8}$$

where $f(r,p,t)$ is the distribution of particles which depends on position r, momentum p and time t, $\vec{F}$ is the external force, $\vec{u}$ is the velocity and $C[f]$ is the collision term. Assuming in what follows homogeneity of the system ($\nabla_r f = 0$) and absence of external forces ($\vec{F} = 0$) Eq. (8) reduces to

$$\frac{df(r,p,t)}{dt} = \frac{\partial f}{\partial t} = C[f]. \tag{9}$$

Relaxation and correlation times of nonequilibrium multiparticle systems

In the relaxation time approximation (RTA) the collision term is assumed to be equal to

$$C[f] = \frac{f_{eq} - f}{\tau_{rel}}, \tag{10}$$

where f_{eq} is the local equilibrium distribution and τ_{rel} is the relaxation time, understood as the time taken by the non-equilibrium system to reach equilibrium. In this approximation BTE simplifies to

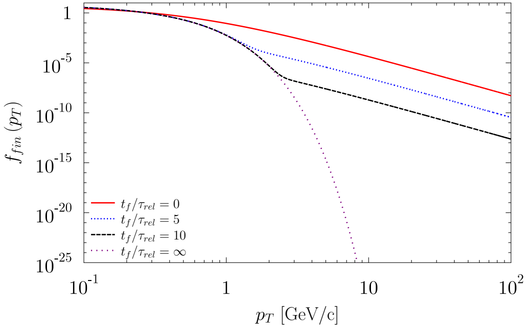
$$\frac{\partial f}{\partial t} = \frac{f_{eq} - f}{\tau_{rel}}. \tag{11}$$

Solving this equation for the initial conditions such that at $t = 0$ one has initial distribution, $f = f_{in}$, and at freeze-out time, $t = t_f$ one has final distribution, $f = f_{fin}$ (to be identified with the actually measured distribution) one gets that

$$f_{fin} = f_{in} \exp\left(-\frac{t_f}{\tau_{rel}}\right) + f_{eq}\left[1 - \exp\left(-\frac{t_f}{\tau_{rel}}\right)\right]. \tag{12}$$

Relaxation and correlation times of nonequilibrium multiparticle systems

Using f_{in} and f_{eq} in the form of Tsallis distribution (with, respectively, q_{in} and q_{eq}) we get f_{fin} for different values of t_f/τ_{rel}.



Schematic transverse momenta distributions f_{fin} resulting from the relaxation time approximation scenario for $q_{in} = 1.25$, $q_{eq} = 1.0$, $T = 0.14$ GeV, and for $t_f/\tau_{rel} = 0,\ 5,\ 10,\ \infty$ (curves from top to down). Figure taken from [Phys. Rev. D **103** 114026].

Relaxation and correlation times of nonequilibrium multiparticle systems

However, if we would require that all distributions $f(t)$ in Eq. (12) have the form of Tsallis distributions **depending on time entirely via the time dependence of the corresponding nonextensivity parameters**, $f(t) = f[q(t)]$, then the time evolution would be given by

$$\frac{\partial f(t)}{\partial t} = F[q(t)] \tag{13}$$

Assuming further that the dependence of q on time is given by

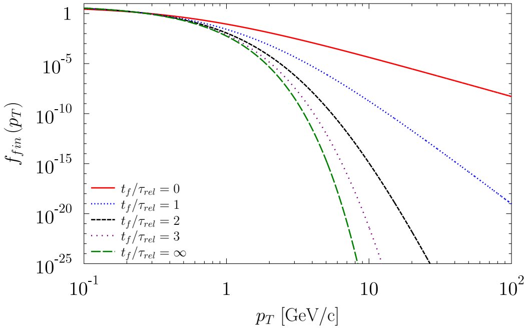
$$\frac{\partial q}{\partial t} = \frac{q_{eq} - q}{\tau_{rel}}, \tag{14}$$

and remembering that we always assume that $q_{eq} = 1$, we have that

$$q - 1 = (q_{in} - 1)\exp\left(-\frac{t_f}{\tau_{rel}}\right) \tag{15}$$

Relaxation and correlation times of nonequilibrium multiparticle systems

Figure below shows the resultant schematic distributions f_{fin} for different t_f/τ_{rel}; they all have form of Tsallis distribution with $q = q\left(t = t_f\right)$ as given by Eq. (15).



Schematic transverse momenta distributions for f_{fin} calculated from Eq. (13) for $t_f/\tau_{rel} = 0,\ 1,\ 2,\ 3,\ \infty$ (curves from top to down). The values for T, q_{in}, q_{eq} are the same as for previous figure. Figure taken from [Phys. Rev. D **103** 114026].

Relaxation and correlation times of nonequilibrium multiparticle systems

To deduce energy dependence of t_f/τ_{rel} from data on p_T distributions we use the formula fitting p_T at all energies available:

$$f_{fin}(p_T) = \frac{2-q}{T}\left[1+(q-1)\frac{p_T}{T}\right]^{\frac{1}{1-q}} \qquad (16)$$

characterized by the energy dependent Tsallis q parameter and the temperature parameter T.
Also f_{in} can be selected in this form, but with q characteristic for hard scattering. The high p_T differential cross section $d\sigma/dp_T \propto p_T^{-\gamma}$ with $\gamma = 4$, what translates to $q_{in} - 1 = 1/\gamma$.

Relaxation and correlation times of nonequilibrium multiparticle systems

We calculate the relation between temperatures deduced from different components of Eq. (12) using the fact that for Tsallis distribution

$$\langle p_T \rangle = \frac{T}{3-2q}. \tag{17}$$

Using this in Eq. (12) one obtains that

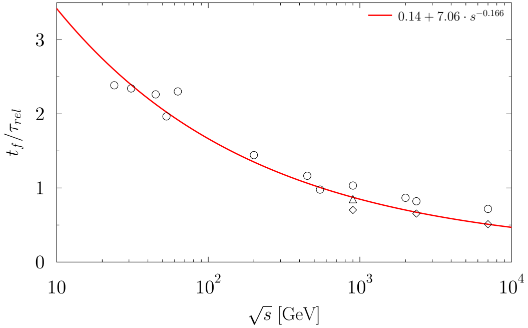
$$\langle p_T \rangle (3-2q) = \langle p_T \rangle + \left[\langle p_T \rangle (3-2q_{in}) - \langle p_T \rangle \right] \cdot \exp\left(-\frac{t_f}{\tau_{rel}} \right) \tag{18}$$

and assuming that $\langle p_T \rangle = const$ during the time evolution one gets that

$$\frac{t_f}{\tau_{rel}} = \ln\left(\frac{q_{in}-1}{q-1} \right). \tag{19}$$

Relaxation and correlation times of nonequilibrium multiparticle systems

Using for $q = q(s)$ values obtained from the experimental data on transverse momentum distributions for different energies [J. Phys. G **37** 115009], [Phys. Rev. D **91** 114027], [Eur. Phys. J. A **51** 80] we obtain the ratio t_f/τ_{rel}.



Energy dependence of t_f/τ_{rel} obtained from the experimental data. Based on data from: [Phys. Rev. D **91** 114027] (triangle), [J. Phys. G **37** 115009] (circles) and [Eur. Phys. J. A **51** 80] (diamonds). Figure taken from [Phys. Rev. D **103** 114026].

Relaxation and correlation times of nonequilibrium multiparticle systems

We will now move on to correlation time τ_{cor} which determines multiplicity distribution $P(N)$. Its scaled variance is given by the correlation function $\nu_2(t_1, t_2) = \nu_2(t = |t_1 - t_2|)$ by the relation

$$\frac{\mathrm{Var}(N)}{\langle N \rangle} = 1 + \langle N \rangle \langle \nu_2 \rangle, \tag{20}$$

where

$$\langle \nu_2 \rangle = \int\int \nu_2(t_1, t_2)\, dt_1 dt_2 = \frac{2}{t_f^2} \int_0^{t_f} \left(t_f - t\right) \nu_2(t) dt. \tag{21}$$

Relaxation and correlation times of nonequilibrium multiparticle systems

For the correlation function of the the form

$$\nu_2(t) = \exp\left(-\frac{2t}{\tau_{cor}}\right) \tag{22}$$

one gets

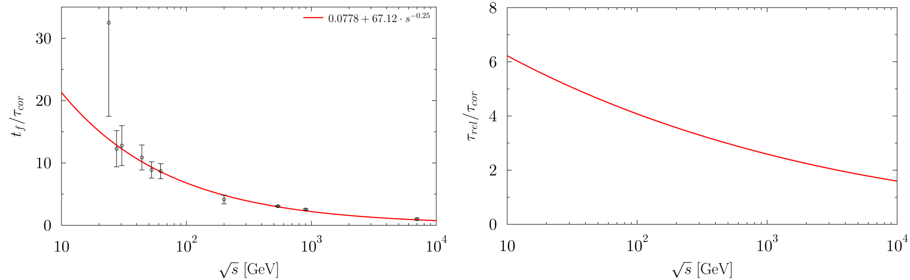
$$\langle \nu_2 \rangle = \left(\frac{\tau_{cor}}{t_f}\right)^2 \left[\exp\left(-\frac{t_f}{\tau_{cor}}\right) - 1 + 2\frac{t_f}{\tau_{cor}}\right] \tag{23}$$

and the scaled variance is equal to

$$\frac{\mathrm{Var}\,(N)}{\langle N \rangle} = 1 + \frac{\langle N \rangle}{2}\left(\frac{\tau_{cor}}{t_f}\right)^2 \left[\exp\left(-\frac{t_f}{\tau_{cor}}\right) - 1 + 2\frac{t_f}{\tau_{cor}}\right]. \tag{24}$$

Relaxation and correlation times of nonequilibrium multiparticle systems

Using $\mathrm{Var}\,(N)$ and $\langle N \rangle$ values evaluated from the charged-particle multiplicity distributions for non-single-diffractive proton-proton (antiproton) collisions we obtain the ratio t_f/τ_{cor}. Combining the results of both approaches, we present the ratio τ_{rel}/τ_{cor} in the energy range from 10 GeV to 7 TeV.



Energy dependence of t_f/τ_{cor} and τ_{rel}/τ_{cor} obtained from experimental data: [J. Phys. G **37** 083001], [Acta Phys. Polon. B **19** 763]. Figure taken from [Phys. Rev. D **103** 114026].

Conclusions

- We demonstrated that energy distributions are connected with multiplicity distributions by their generating functions.
- We have shown that if all the distributions occurring in the RTA have the same functional form of a quasi-power Tsallis distribution the time evolution of which depends on the time evolution of its control parameter, nonextensivity $q(t)$, then it is more convenient to consider only the time evolution of this control parameter.
- This talk is based on [Eur. Phys. J. A **57** 3] and [Phys. Rev. D **103** 114026].

Additional slides

Correspondence of multiplicity and energy distributions: Boltzmann-Gibbs energy distribution and Poissonian multiplicity distribution

Suppose that one has N independently produced particles with energies $\{E_{1,\dots,N}\}$, distributed according to Boltzmann distribution,

$$F(E) = \frac{1}{T}\exp\left(-\frac{E}{T}\right) \tag{25}$$

with "temperature" parameter $T = \langle E \rangle$. The sum of energies, $U = \sum_{i=1}^{N} E_i$ is then distributed according to gamma distribution

$$\begin{aligned} F_N(U) &= \frac{1}{T(N-1)!}\left(\frac{U}{T}\right)^{N-1}\exp\left(-\frac{U}{T}\right) \\ &= F_{N-1}(U)\frac{U}{N-1} \end{aligned} \tag{26}$$

with cumulative distribution equal to:

$$F_N(>U) = 1 - \sum_{i=1}^{N-1}\frac{1}{(i-1)!}\left(\frac{U}{T}\right)^{i-1}\exp\left(-\frac{U}{T}\right). \tag{27}$$

Correspondence of multiplicity and energy distributions: Boltzmann-Gibbs energy distribution and Poissonian multiplicity distribution

Looking for such N that $\sum_{i=0}^{N} E_i \leq U \leq \sum_{i=0}^{N+1} E_i$ we find its distribution which has known Poissonian form

$$\begin{aligned} P(N) &= F_{N+1}(> U) - F_N(> U) \\ &= \frac{(U/T)^N}{N!} \exp\left(-\frac{U}{T}\right) \\ &= \frac{\langle N \rangle^N}{N!} \exp\left(-\langle N \rangle\right) \end{aligned} \tag{28}$$

with $\langle N \rangle = U/T$.

Correspondence of multiplicity and energy distributions: Boltzmann-Gibbs energy distribution and Poissonian multiplicity distribution

For the constrained systems (if the available energy is limited, $U = \mathrm{const}$), whenever we have independent variables $\{E_{1,\dots,N}\}$ taken from the exponential distribution (25), the corresponding multiplicity N has Poissonian distribution. However, if the multiplicity is limited, $N = \mathrm{const}$, the resulting *conditional probability* becomes:

$$\begin{aligned} F(E|N) &= \frac{F_1(E)\,F_{N-1}(U-E)}{F_N(U)} \\ &= \frac{N-1}{U}\left(1-\frac{E}{U}\right)^{N-2} \end{aligned} \tag{29}$$

and only in the limit $N \to \infty$ the energy distribution goes to the Boltzmann distribution (25). For fluctuating multiplicity according to Poisson distribution, the energy distribution is given by the Boltzmann distribution.
In the same way Tsallis energy distribution is connected with the NBD of multiplicity [Physica A **376** 279].

Correspondence of multiplicity and energy distributions: Imprints of acceptance

In experiments, particle multiplicity is measured usually only within some window of phase-space. Let us assume that the detection process is a Bernoulli process described by the BD ($K = 1$ and $p = \alpha$ for a fixed experimental acceptance $\alpha < 1$). The number of registered particles is

$$M = \sum_{i=1}^{N} n_i, \tag{30}$$

where n_i follows the BD with the generating function $G_{BD}(z)$ and N comes from $P(N)$ with the generating function $G(z)$.

Correspondence of multiplicity and energy distributions: Imprints of acceptance

The measured multiplicity distribution

$$P(M) = \frac{1}{M!} \frac{d^M H(z)}{dz^M}\bigg|_{z=0} \tag{31}$$

is therefore given by generating function $H(z) = G(G_{BD}(z))$. Such rough procedure applied to NBD, BD or PD gives again the same distributions but with modified parameters: $p \to \alpha p / [1 - p(1-\alpha)]$ for NBD, $p \to \alpha p$ for BD, and $\lambda \to \alpha\lambda$ for PD. The measured multiplicity distribution is given by

$$P(M) = \sum_{N=M}^{\infty} P(N) P(M|N) \tag{32}$$

with the acceptance function

$$P(M|N) = \frac{N!}{M!(N-M)!} \alpha^M (1-\alpha)^{N-M} \tag{33}$$

Detection process extend $P(M)$ distribution to multiplicities $M = 0$ and $M = 1$, namely: $P(0) = \sum_{N=2}^{\infty} P(N)(1-\alpha)^N$ and $P(1) = \sum_{N=2}^{\infty} P(N) N\alpha (1-\alpha)^{N-1}$.

Relaxation and correlation times of nonequilibrium multiparticle systems: RTA and beyond

The form of the function F from Eq. (13) can be deduced by taking $f(t)$ given by Tsallis distribution with $q = q(t)$ and calculating df/dt. As a result, we get that

$$F\left[p_T, q(t)\right] = f\left[p_T, q(t)\right] \cdot \left\{ \ln\left[1 + (q(t) - 1)\frac{p_T}{T}\right] + \frac{T}{T + [q(t) - 1]p_T} - \frac{[q(t) - 1]^2 + 1}{2 - q(t)} \right\} \cdot \frac{1}{[q(t) - 1]^2} \frac{dq(t)}{dt}. \tag{34}$$

Note that approximately (because $\ln x + 1/x \approx 1$) we get that

$$\frac{\partial f}{\partial t} = \frac{-f}{(2 - q)} \frac{dq(t)}{dt} = \frac{f}{\tau_{rel}} \frac{q - 1}{2 - q}. \tag{35}$$