Inverse Reynolds-Dominance approach to transient fluid dynamics

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Motivation and goal

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Hydrodynamics: Conservation equations

$$\partial_\mu T^{\mu\nu} = 0 \,, \qquad \partial_\mu N^\mu = 0 \tag{1}$$

- Hydrodynamics: based on $(4+1=5)$ conservation equations
  - Ideal case: Sufficient (if equation of state is supplied)
    - $\rightarrow$ Variables: ϵ, n, u^μ
  - Dissipative case: Underdetermined
    - $\rightarrow$ Variables: ϵ, n, u^μ, Π, n^μ, $\pi^{\mu\nu}$
- **Fundamental question of dissipative hydrodynamics**: How to obtain information about the dissipative components of N^μ and $T^{\mu\nu}$?

Decomposition of conserved currents (Landau frame)

$$N^\mu = nu^\mu + n^\mu \tag{2}$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \tag{3}$$

Projectors: $\Delta^{\mu\nu} := g^{\mu\nu} - u^\mu u^\nu$, $\Delta^{\mu\nu}_{\alpha\beta} := (\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha)/2 - \Delta^{\mu\nu}\Delta_{\alpha\beta}/3$

First- and second-order hydrodynamics

- First-order hydro: Relate dissipative quantities to fluid-dynamical gradients

$$\Pi = -\zeta\theta \,, \quad n^\mu = \kappa I^\mu \,, \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} \tag{4}$$

- (In standard hydrodynamic frame): **Acausal** and **unstable**!
- Second-order hydro: Treat dissipative quantitites as dynamical, provide relaxation equations

Relaxation equations

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \text{h.o.t.} \tag{5a}$$

$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa I^\mu + \text{h.o.t.} \tag{5b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \text{h.o.t.} \tag{5c}$$

- Needs input from **microscopic theory**
- This talk: Take **kinetic theory** as the foundation
  - Only consider the shear sector for simplicity

$\theta := \partial^\mu u_\mu$, $\sigma^{\mu\nu} := \nabla^{\langle\mu} u^{\nu\rangle}$, $\nabla^\mu := \Delta^{\mu\nu}\partial_\nu$, $I^\mu := \nabla^\mu(\mu/T)$, $A^{\langle\mu} B^{\nu\rangle} := \Delta^{\mu\nu}_{\alpha\beta} A^\alpha B^\beta$

Hydrodynamics from kinetic theory

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- Kinetic theory: describe evolution of one-particle distribution function $f(x,k)$ in phase space
  - Split into local-equilibrium part $f_0(x,k)$ and deviation $\delta f(x,k)$
- Hydrodynamic quantities described through irreducible moments

$$\rho_r^{\mu_1\cdots\mu_\ell}(x) := \int \mathrm{d}K E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f(x,k) \tag{6}$$

- Related through e.g. $\pi^{\mu\nu} \equiv \rho_0^{\mu\nu}$
- Dynamics described through Boltzmann equation $k^\mu\partial_\mu f(x,k) = C[f]$

Linearized moment equations

$$\sum_{n=0}^{N_2} \tau_{rn}^{(2)}\dot{\rho}_n^{\langle\mu\nu\rangle} + \rho_r^{\mu\nu} = 2\eta_r\sigma^{\mu\nu} + \mathrm{h.o.t.} \tag{7}$$

- How to close this system?
  - $\rightarrow$ Power-counting scheme in Knudsen number $\mathrm{Kn} := \lambda_{\mathsf{mfp}}/\lambda_{\mathsf{hydro}}$ and inverse Reynolds numbers $\mathrm{Re}^{-1} := \delta f/f_0$ to second order

$\mathrm{d}K := \mathrm{d}^3k/[(2\pi)^3k^0]$, $E_{\mathbf{k}} := u^\mu k_\mu$, matching conditions: $\rho_1 = \rho_2 = \rho_1^\mu = 0$

The DNMR approach

G. S. Denicol, H. Niemi, E. Molnar, D. H. Rischke, Phys. Rev. D **85**, 114047 (2012)

- General idea: **Diagonalize** the collision matrices $\tau^{(\ell)} \equiv (\Omega^{(\ell)})^{-1}\mathrm{diag}(\tau_1^{(\ell)}, \tau_2^{(\ell)}, \cdots)\Omega^{(\ell)}$, sort eigenvalues in decreasing order
  - *Separation of scales: The slowest microscopic modes are most important macroscopically*
- Relate irreducible moments to dissipative quantities, e.g. $\rho_r^{\mu\nu} \simeq a_r \pi^{\mu\nu} + b_r \sigma^{\mu\nu}$
  $\rightarrow \; \dot{\rho}_r^{\langle\mu\nu\rangle} \sim \dot{\pi}^{\langle\mu\nu\rangle}, \; \dot{\sigma}^{\langle\mu\nu\rangle}$
- Discard terms of order $\mathcal{O}(\mathrm{Kn}^2\mathrm{Re}^{-1})$ or higher

Shear-stress relaxation equation (DNMR)

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} \quad (8)$$

- First-order contributions $\sim \mathcal{O}(\mathrm{Re}^{-1})$ and $\sim \mathcal{O}(\mathrm{Kn})$
- Second-order contributions $\sim \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1})$ and $\sim \mathcal{O}(\mathrm{Kn}^2)$

The issue: Parabolic terms

- Consider the second-order terms of tensor-rank two:

$$\begin{aligned}\mathcal{J}^{\mu\nu} &= 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n} n^{\langle\mu}F^{\nu\rangle} \\ &\quad + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}I^{\nu\rangle}\ , \qquad (9)\\ \mathcal{K}^{\mu\nu} &= \tilde{\eta}_1 \omega^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda + \tilde{\eta}_2\theta\sigma^{\mu\nu} + \tilde{\eta}_3\sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \tilde{\eta}_4\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \tilde{\eta}_5 I^{\langle\mu}I^{\nu\rangle} \\ &\quad + \tilde{\eta}_6 F^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_7 I^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_8\nabla^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_9\nabla^{\langle\mu}F^{\nu\rangle} \qquad (10)\end{aligned}$$

- Second derivatives of fluid-dynamical quantities appear
  - → Equations become **parabolic**!
  - → Theory becomes acausal and thus unstable
- DNMR approach: **Ignore** terms of order $\mathcal{O}(\mathrm{Kn}^2)$
  - → Equations are hyperbolic again
- Is there a way to ensure $\mathcal{K}^{\mu\nu} = 0$ from the beginning?

$F^\mu := \nabla^\mu P_0$, $\omega^{\mu\nu} := (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$

The IReD approach

DW, A. Palermo, V. E. Ambruș, arXiv:2203.12608

- General idea: Relate moments through their Navier-Stokes solutions

$$\rho_r^{\mu\nu} = 2\eta_r \sigma^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) , \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) ,$$
$$\Rightarrow \rho_r^{\mu\nu} = \frac{\eta_r}{\eta}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) . \tag{11}$$

$\rightarrow$ $\dot{\rho}_r^{\langle\mu\nu\rangle} \sim \dot{\pi}^{\langle\mu\nu\rangle}$

- Discard terms of order $\mathcal{O}(\mathrm{Kn}^2\mathrm{Re}^{-1})$ or higher

Shear-stress relaxation equation (IReD)

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} \tag{12}$$

- Only terms $\sim \mathcal{O}(\mathrm{Re}^{-1})$, $\sim \mathcal{O}(\mathrm{Kn})$, $\sim \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1})$ appear
  - $\rightarrow$ Equations stay **hyperbolic**, no need to discard terms
  - $\rightarrow$ Transport coefficients change
- How are the methods related?

Approach also known as "order-of-magnitude approach" J. A. Fotakis, E. Molnár, H. Niemi, C. Greiner, D. H. Rischke arXiv: 2203.11549

Equivalence of DNMR and IReD

- Consider again the structure of the second-order terms of tensor-rank two:

$$\mathcal{J}^{\mu\nu}_{\text{DNMR}} = \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_{\lambda} + 2\tau_{\pi}\pi^{\langle\mu}_{\lambda}\omega^{\nu\rangle\lambda} + \lambda_{\pi n}n^{\langle\mu}I^{\nu\rangle} - \tau_{\pi n}n^{\langle\mu}F^{\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle}\,, \tag{13}$$

$$\mathcal{K}^{\mu\nu}_{\text{DNMR}} = \tilde{\eta}_1\omega^{\lambda\langle\mu}\omega^{\nu\rangle}{}_{\lambda} + \tilde{\eta}_2\theta\sigma^{\mu\nu} + \tilde{\eta}_3\sigma^{\lambda\langle\mu}\sigma^{\nu\rangle}_{\lambda} + \tilde{\eta}_4\sigma^{\langle\mu}_{\lambda}\omega^{\nu\rangle\lambda} + \tilde{\eta}_5 I^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_6 F^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_7 I^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_8\nabla^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_9\nabla^{\langle\mu}F^{\nu\rangle}\,. \tag{14}$$

- Use Navier-Stokes solutions of moment equations to absorb terms in $\mathcal{K}^{\mu\nu}_{\text{DNMR}}$ into $\mathcal{J}^{\mu\nu}_{\text{IReD}}$ via $\theta \simeq -\Pi/\zeta\,,\ I^{\mu} \simeq n^{\mu}/\kappa\,,\ \sigma^{\mu\nu} \simeq \pi^{\mu\nu}/2\eta$
  - "Trade one power of Kn for one power of Re^{-1}"
- The terms in red can be related to
$\dot{\sigma}^{\langle\mu\nu\rangle} \sim -\omega^{\lambda\langle\mu}\omega^{\nu\rangle}{}_{\lambda} - \frac{\tilde{\eta}_6}{\tilde{\eta}_1}F^{\langle\mu}F^{\nu\rangle} - \frac{\tilde{\eta}_9}{\tilde{\eta}_1}\nabla^{\langle\mu}F^{\nu\rangle}$
- Since $\dot{\sigma}^{\langle\mu\nu\rangle} = \frac{1}{2\eta}\dot{\pi}^{\langle\mu\nu\rangle} - \frac{1}{2\eta^2}\pi^{\mu\nu}\dot{\eta}$, $\tilde{\eta}_1$ leads to a modification of τ_π:

$$\tau_\pi^{\text{IReD}} = \tau_\pi^{\text{DNMR}} + \frac{\tilde{\eta}_1}{2\eta}. \tag{15}$$

- Result: IReD and DNMR equivalent up to order $\mathcal{O}(\text{Kn}^2, \text{Kn}\text{Re}^{-1}, \text{Re}^{-2})$

Test case: Ultrarelativistic hard spheres

- Simple model with constant cross-section: Linearized collision matrices can be calculated analytically

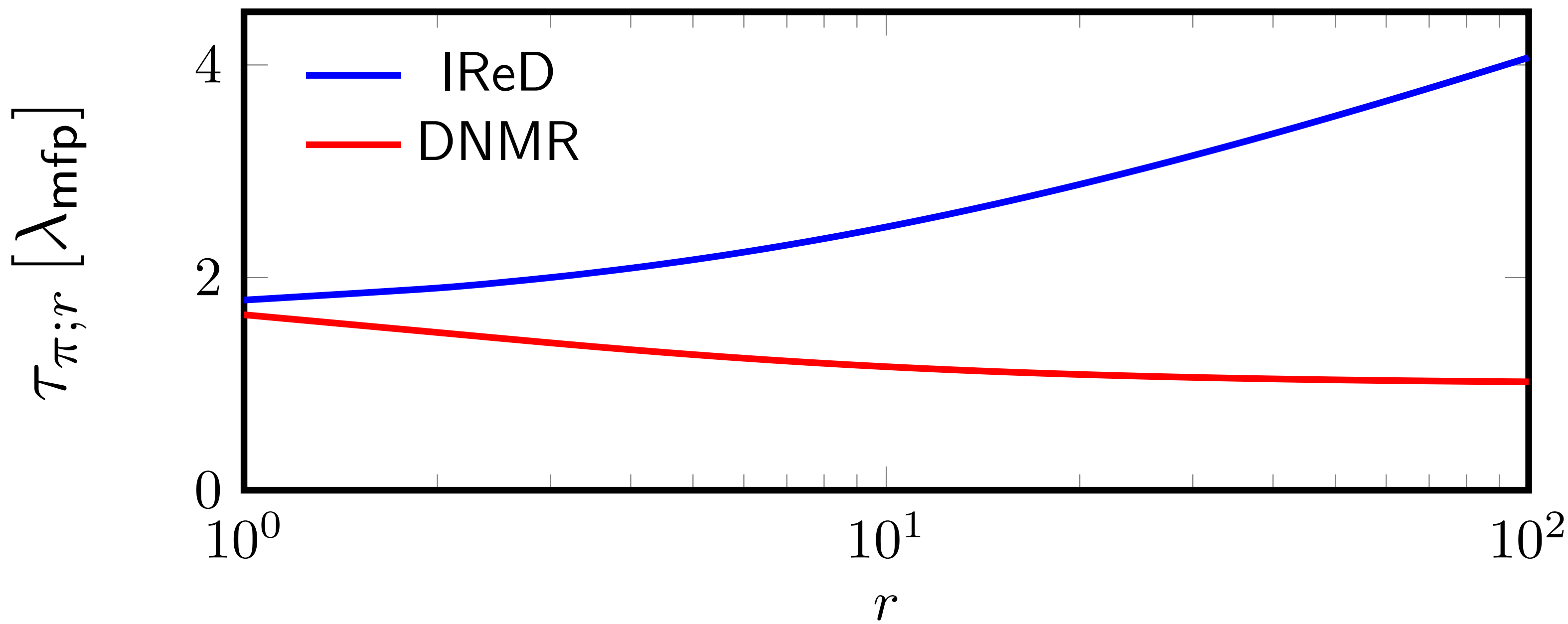
DW, V. E. Ambruș, E. Molnár, in preparation

IReD	Relation	DNMR
$\tau_\pi = 1.66\lambda_{\mathrm{mfp}}$	$\tau_\pi = \tilde{\tau}_\pi + \dfrac{\tilde{\eta}_1}{2\eta}$	$\tilde{\tau}_\pi = 2\lambda_{\mathrm{mfp}}$
$\tau_{\pi\pi} = 1.69\tau_\pi$	$\tau_{\pi\pi} = \tilde{\tau}_{\pi\pi} + \dfrac{\tilde{\eta}_1 - \tilde{\eta}_3}{2\eta}$	$\tilde{\tau}_{\pi\pi} = 1.69\tilde{\tau}_\pi$
$\ell_{\pi n} = -0.57\tau_\pi/\beta$	$\ell_{\pi n} = \tilde{\ell}_{\pi n} + \dfrac{\tilde{\eta}_8}{\kappa}$	$\tilde{\ell}_{\pi n} = -0.69\tilde{\tau}_\pi/\beta$

- Properly accounting for $\mathcal{K}^{\mu\nu}$ within IReD gives a 17% difference in τ_π, together with substantial differences in e.g. $\ell_{\pi n}/\tau_\pi$

Relaxation times: ~~Separation of scales~~

- Fundamental idea of DNMR: Separation of microscopic timescales
  - What happens to this idea in IReD?
- Consider relaxation equations for higher (non-hydrodynamic) moments
  - DNMR: Relaxation times given by eigenvalues of inverse collision matrix
  - IReD: Modification through absorption of $\mathcal{K}^{\mu\nu}$ into $\mathcal{J}^{\mu\nu}$



- Different behaviour in the two theories for $r \to \infty$:
  - DNMR: $\tau_{\pi;r} \to \lambda_{\text{mfp}}$
  - IReD: $\tau_{\pi;r} \sim \log(r)$

    $\rightarrow$ The *Separation of Scales* paradigm does not hold in IReD anymore!

Conclusion and Outlook

- The IReD approach to relativistic dissipative hydrodynamics relates irreducible moments ($\rho_r^{\mu\nu}$) directly to dissipative quantities ($\pi^{\mu\nu}$)
  - $\rightarrow$ No terms $\sim \mathcal{O}(\mathrm{Kn}^2)$ appear in equations of motion
  - $\rightarrow$ Equations stay **hyperbolic**, no modifications needed
- Relaxation times behave fundamentally different, *separation of scales* no longer valid
- IReD and DNMR are (perturbatively) equivalent
- **However**, in the regime where the $\mathcal{O}(\mathrm{Kn}^2)$ contributions are non-negligible, the IReD approach should do better
  - $\rightarrow$ Future plan: Compare performance in different setups

Appendix

Hard spheres collision matrix

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- The collision matrix is linked with the expansion of $\delta f_{\mathbf{k}}$ with respect to a complete basis,

$$\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \mathcal{H}_{\mathbf{k}n}^{(\ell)},$$

where $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$ is defined such that $\rho_n^{\mu_1 \cdots \mu_\ell} \equiv \int dK E_{\mathbf{k}}^n k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \delta f_{\mathbf{k}}$.

- The linearized collision integrals are given by

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu(2\ell+1)} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^{r-1} k^{\langle \nu_1} \cdots k^{\nu_\ell \rangle}$$
$$\times \left(\mathcal{H}_{\mathbf{k}n}^{(\ell)} k_{\langle \nu_1} \cdots k_{\nu_\ell \rangle} + \mathcal{H}_{\mathbf{k}'n}^{(\ell)} k'_{\langle \nu_1} \cdots k'_{\nu_\ell \rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)} p_{\langle \nu_1} \cdots p_{\nu_\ell \rangle} - \mathcal{H}_{\mathbf{p}'n}^{(\ell)} p'_{\langle \nu_1} \cdots p'_{\nu_\ell \rangle} \right) ,$$

- In the case of the UR ideal HS gas, $W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} = s(2\pi)^6 \delta^{(4)}(k + k' - p - p') \frac{\sigma_T \nu}{4\pi}$ and

$$\mathcal{A}_{r=0,n}^{(1)} = \frac{16(-\beta)^n g^2}{\lambda_{\rm mfp}(n+3)!} \left[S_n^{(1)}(N_1) - \frac{\delta_{n0}}{2} \right], \qquad \mathcal{A}_{r=0,n}^{(2)} = \frac{432 g^2 (-\beta)^n}{\lambda_{\rm mfp}(n+5)!} S_n^{(2)}(N_2),$$

$$\mathcal{A}_{r>0,n\le r}^{(1)} = \frac{g^2 \beta^{n-r} (r+2)! [n(r+4) - r]}{\lambda_{\rm mfp}(n+3)! r} \times \left(\delta_{nr} + \delta_{n0} - \frac{2}{r+1} \right),$$

$$\mathcal{A}_{r>0,n\le r}^{(2)} = \frac{g^2 \beta^{n-r} (r+4)! (n+1)}{\lambda_{\rm mfp}(n+5)! r(r+1)} \times (9n + nr - 4r) \left(\delta_{nr} - \frac{2}{r+2} \right),$$

while $\mathcal{A}_{r>0,n>r}^{(1)} = \mathcal{A}_{r>0,n>r}^{(2)} = 0$ and $S_n^{(\ell)}(N_\ell) = \sum_{m=n}^{N_\ell} \binom{m}{n} \frac{1}{(m+\ell)(m+\ell+1)}$.