



# New exact solutions of non relativistic, viscous hydrodynamics

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NTN 2022, GYÖNGYÖS

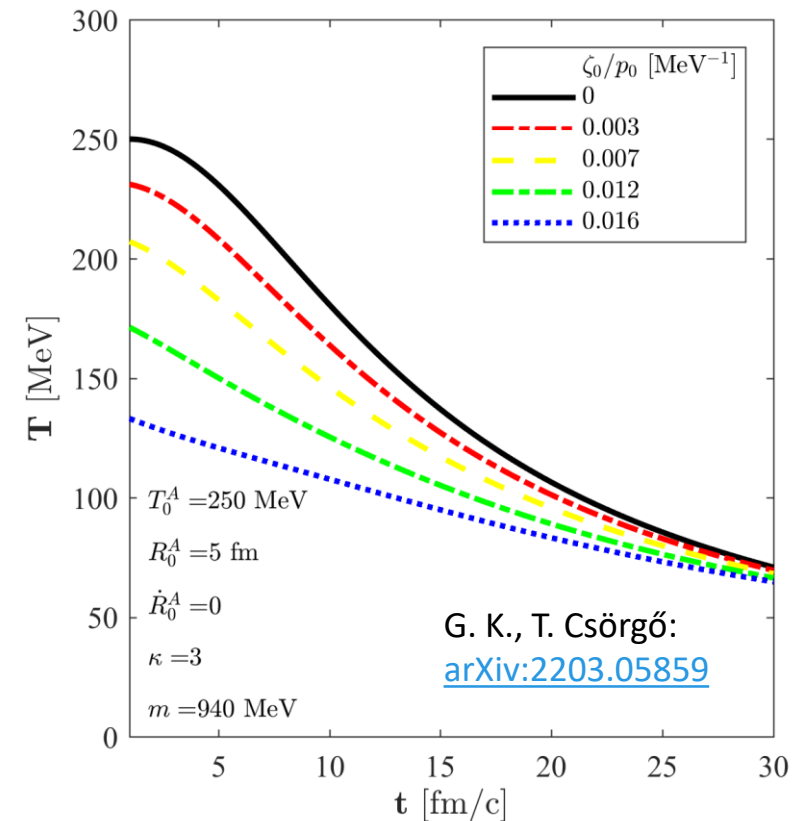
05/05/2022

PARTIALLY SUPPORTED BY:  
NKFIH K-133046, FK-123842, FK-123959

# Introduction

## Asymptotic perfection

- New exact solutions of viscous hydrodynamics have been found recently
- ***These solutions tends to perfect fluid solutions at asymptotically late times***
- **Perfect fluid attractor:** how to extract the effect of bulk and shear viscosity in final state measurements?



# Introduction

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## List of variety of viscous, asymptotically perfect fluid solutions:

Solutions were found...

- ...in relativistic regime: T. Csörgő, G. K.: [arXiv:2003.08859](https://arxiv.org/abs/2003.08859) M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: [arXiv:1909.02498](https://arxiv.org/abs/1909.02498)
- ...in non relativistic regime: G. K., T. Csörgő: [arXiv:2203.05859](https://arxiv.org/abs/2203.05859)  **Today's topic**

In the relativistic regime, we have found solutions of...

- ...Israel-Stewart theory: M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: [arXiv:1909.02498](https://arxiv.org/abs/1909.02498) T. Csörgő, G. K.: [arXiv:2003.08859](https://arxiv.org/abs/2003.08859)
- ...Navier-Stokes theory: M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: [arXiv:1909.02498](https://arxiv.org/abs/1909.02498) T. Csörgő, G. K.: [arXiv:2003.08859](https://arxiv.org/abs/2003.08859)  
G. K., T. Csörgő: [arXiv:2203.05859](https://arxiv.org/abs/2203.05859)

Solutions were found with different assumptions, with different symmetries:

- $\zeta \sim p, \eta \sim p \rightarrow$  spherical (only bulk), spheroidal, ellipsoidal T. Csörgő, G. K.: [arXiv:2003.08859](https://arxiv.org/abs/2003.08859) G. K., T. Csörgő: [arXiv:2203.05859](https://arxiv.org/abs/2203.05859)
- $\zeta \sim n \rightarrow$  spherical M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: [arXiv:1909.02498](https://arxiv.org/abs/1909.02498)
- $\zeta = \zeta(p(t)) \rightarrow$  spherical, spheroidal, ellipsoidal (draft) G. K., T. Csörgő: [arXiv:2203.05859](https://arxiv.org/abs/2203.05859) T. Csörgő, G. K.: [arXiv:2003.08859](https://arxiv.org/abs/2003.08859)

# Introduction

Common properties of these solutions: **Hubble-flow**

Scales of the fireball:  $X(t), Y(t), Z(t)$

Transverse scale:  $R(t)$

Spherical symmetry:  $\vec{v} = \frac{\dot{R}}{R}(r_x, r_y, r_z)$

Spheroidal symmetry:  $v_H(\vec{r}, t) = \left( \frac{\dot{R}}{R} r_x, \frac{\dot{Y}}{Y} r_y, \frac{\dot{R}}{R} r_z \right), \quad v_{rot}(\vec{r}, t) = \omega(r_z, 0, -r_x), \quad \dot{\vartheta} = \omega(t)$

Ellipsoidal symmetry: 
$$v_H(\vec{r}, t) = \begin{pmatrix} \left( \frac{\dot{X}}{X} \cos^2 \vartheta + \frac{\dot{Z}}{Z} \sin^2 \vartheta \right) r_x \\ \frac{\dot{Y}}{Y} r_y \\ \left( \frac{\dot{X}}{X} \sin^2 \vartheta + \frac{\dot{Z}}{Z} \cos^2 \vartheta \right) r_z \end{pmatrix} + \left( \frac{\dot{Z}}{Z} - \frac{\dot{X}}{X} \right) \frac{\sin(2\vartheta)}{2} \begin{pmatrix} r_z \\ 0 \\ r_x \end{pmatrix}$$
$$v_{rot}(\vec{r}, t) = \dot{\vartheta} \begin{pmatrix} r_z \\ 0 \\ -r_x \end{pmatrix} + \dot{\vartheta} \begin{pmatrix} \left( \frac{X}{Z} \cos^2 \vartheta + \frac{Z}{X} \sin^2 \vartheta \right) r_z \\ 0 \\ -\left( \frac{X}{Z} \sin^2 \vartheta + \frac{Z}{X} \cos^2 \vartheta \right) r_x \end{pmatrix} + \dot{\vartheta} \left( \frac{X}{Z} - \frac{Z}{X} \right) \frac{\sin(2\vartheta)}{2} \begin{pmatrix} r_x \\ 0 \\ -r_z \end{pmatrix}, \quad \dot{\vartheta} = \frac{\omega(t)}{2} = \frac{\omega_0}{2} \frac{R_0^2}{R(t)^2}$$

M. I. Nagy, T. Csörgő: [arXiv:1309.4390](https://arxiv.org/abs/1309.4390)

M. I. Nagy, T. Csörgő: [arXiv:1606.09160](https://arxiv.org/abs/1606.09160)

T. Csörgő, M. I. Nagy, I. F. Barna: [arXiv:1511.02593](https://arxiv.org/abs/1511.02593)

# Introduction

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Common properties of these solutions: **Self-similarity**  $(\partial_t + \vec{v}\nabla)s = 0$

Spherical symmetry:  $s = \frac{r^2}{R^2}$

Spheroidal symmetry:  $s = \frac{r_x^2 + r_z^2}{R^2} + \frac{r_y^2}{Y^2}$

Ellipsoidal symmetry:  $s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2} + \left(\frac{1}{Z^2} - \frac{1}{X^2}\right) [(r_x^2 - r_z^2) \sin^2 \vartheta + r_x r_z \sin(2\vartheta)]$

M. I. Nagy, T. Csörgő: [arXiv:1309.4390](https://arxiv.org/abs/1309.4390)

M. I. Nagy, T. Csörgő: [arXiv:1606.09160](https://arxiv.org/abs/1606.09160)

T. Csörgő, M. I. Nagy, I. F. Barna: [arXiv:1511.02593](https://arxiv.org/abs/1511.02593)

# Introduction

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## Non relativistic hydrodynamics

- The same effects can be understood in a much simpler formalism
- ***The basic equations of non relativistic, viscous hydro are fully clarified***

$$\partial_t n + \nabla(n\vec{v}) = 0$$

$$\partial_t \varepsilon + \nabla(\varepsilon\vec{v}) + p\nabla\vec{v} = \zeta(\nabla\vec{v})^2 + 2\eta \left[ \text{Tr}(D^2) - \frac{1}{3}(\nabla\vec{v})^2 \right]$$

$$(\varepsilon + p)(\partial_t + \vec{v}\nabla)\vec{v} + \nabla p = \nabla(\zeta\nabla\vec{v}) + \eta \left[ \Delta\vec{v} + \frac{1}{3}\nabla(\nabla\vec{v}) \right]$$

$$\partial_t \sigma + \nabla(\sigma\vec{v}) = \frac{\zeta}{T}(\nabla\vec{v})^2 + \frac{2\eta}{T} \left[ \text{Tr}(D^2) - \frac{1}{3}(\nabla\vec{v})^2 \right] \geq 0$$

Balance eq. of entropy

Local conservation laws

# Introduction

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To close the equation system:

$$\text{EoS: } \varepsilon = \kappa(T) \rho$$
$$p = nT$$

G. K., T. Csörgő: [arXiv:2203.05859](https://arxiv.org/abs/2203.05859)

## Relationship to the speed of sound

$\kappa$  is constant:

$$c_s^2(T) = (1 + \kappa^{-1}) \frac{T}{m}$$

$\kappa$  is temperature dependent:

$$c_s^2(T) = \underbrace{\left[ 1 + \left( \kappa + T \frac{d\kappa}{dT} \right)^{-1} \right]}_{\gamma(T)} \frac{T}{m}$$

$\gamma(T)$ : temperature dependent adiabatic index

# New solutions

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Self-similar solutions: product of a homogeneous term and arbitrary functions of  $s$

$$n(t, s) = n_H(t)\mathcal{V}(s) \longrightarrow n(t, s) = n_0 \frac{V_0}{V} \mathcal{V}(s)$$

$$T(t, s) = T_H(t)\mathcal{T}(s) \longrightarrow$$

$$p(t, s) = n(t, s)T(t, s) = p_H(t)\mathcal{V}(s)\mathcal{T}(s)$$

For constant  $\kappa$  we assume:

$$T_H(t) = T_P(t)T_D(t)$$

$v(s)$  and  $\tau(s)$  are not independent from each other:

$$-\frac{C_E}{2} = \mathcal{T}'(s) + \frac{\mathcal{T}(s)}{\mathcal{V}(s)} \mathcal{V}'(s)$$



# Parametric solutions with inhomogeneous pressure (Spherical symmetry)

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If the pressure is inhomogeneous, then  $\zeta$  has to be linear in pressure:  $\zeta(t, s) = \zeta_0 \frac{p(t, s)}{p_0}$

Assumption: 
$$T_H(t) = T_D(t) \left( \frac{R_0}{R} \right)^{\frac{d}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{T}_D}{T_D} = \underbrace{\frac{\zeta_0 d^2}{\kappa p_0} \left( \frac{\dot{R}}{R} \right)^2}_{\text{effect of bulk viscosity}}$$

The Euler equation is:

$$R\ddot{R} = C_E \frac{T_H}{m} \left( 1 - \underbrace{\frac{\zeta_0}{p_0} \frac{d\dot{R}}{R}}_{\text{effect of bulk viscosity}} \right)$$

# Parametric solutions with inhomogeneous pressure (Spheroidal symmetry)

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If the pressure is inhomogeneous, then  $\zeta$  and  $\eta$  has to be linear in pressure:  $\frac{\zeta(t, s)}{\zeta_0} = \frac{\eta(t, s)}{\eta_0} = \frac{p(t, s)}{p_0}$

Assumption:  $T_H(t) = T_D(t) \left( \frac{R_0^2 Y_0}{R^2 Y} \right)^{\frac{1}{\kappa}}$

With that, the energy conservation becomes:

$$\frac{\dot{T}_D}{T_D} = \underbrace{\frac{\zeta_0}{\kappa p_0} \left( \frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} \right)^2}_{\text{effect of bulk viscosity}} + \underbrace{\frac{4\eta_0}{3\kappa p_0} \left( \frac{\dot{R}}{R} - \frac{\dot{Y}}{Y} \right)^2}_{\text{effect of shear viscosity}}$$

The Euler equation is:

$$R\ddot{R} = Y\ddot{Y} = C_E \frac{T_P T_D}{m} \left[ 1 - \underbrace{\frac{\zeta_0}{p_0} \left( \frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} \right)}_{\text{effect of bulk viscosity}} \right]$$

# Parametric solutions with inhomogeneous pressure (Spheroidal symmetry, rotation)

If the pressure is inhomogeneous, then  $\zeta$  and  $\eta$  has to be linear in pressure:  $\frac{\zeta(t, s)}{\zeta_0} = \frac{\eta(t, s)}{\eta_0} = \frac{p(t, s)}{p_0}$

Assumption:  $T_H(t) = T_D(t) \left( \frac{R_0^2 Y_0}{R^2 Y} \right)^{\frac{1}{\kappa}}$

With that, the energy conservation becomes:

$$\frac{\dot{T}_D}{T_D} = \underbrace{\frac{\zeta_0}{\kappa p_0} \left( \frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} \right)^2}_{\text{effect of bulk viscosity}} + \underbrace{\frac{4\eta_0}{3\kappa p_0} \left( \frac{\dot{R}}{R} - \frac{\dot{Y}}{Y} \right)^2}_{\text{effect of shear viscosity}}$$

The Euler equation is: effect of bulk viscosity

effect of shear viscosity

$$R\ddot{R} - R^2\omega^2 = Y\ddot{Y} = C_E \frac{T_P T_D}{m} \left[ 1 - \underbrace{\frac{\zeta_0}{p_0} \left( \frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} \right)}_{\text{effect of bulk viscosity}} \right]$$

effect of rotation

# Parametric solutions with inhomogeneous pressure (Ellipsoidal symmetry, rotation)

Pressure is inhomogeneous  $\rightarrow \zeta$  and  $\eta$  are linear in pressure:  $\frac{\zeta(t, s)}{\zeta_0} = \frac{\eta(t, s)}{\eta_0} = \frac{p(t, s)}{p_0}$

Assumption:  $T_H(t) = T_D(t) \left( \frac{X_0 Y_0 Z_0}{XYZ} \right)^{\frac{1}{\kappa}}$

With that, the energy conservation becomes:

$$\frac{\dot{T}_D}{T_D} = \underbrace{\frac{\zeta_0}{\kappa p_0} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right)^2}_{\text{effect of bulk viscosity}} + \underbrace{\frac{2\eta_0}{\kappa p_0} \left[ \frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Z}^2}{Z^2} - \frac{1}{3} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right)^2 \right]}_{\substack{\text{the only effect of shear viscosity} \\ \text{for spheroidally symmetric fireball}}} + \underbrace{\frac{\eta_0 \omega_0^2 (X_0 + Z_0)^4}{4\kappa p_0 (X + Z)^4} \left( \frac{X}{Z} - \frac{Z}{X} \right)^2}_{\text{effect of rotation, if } \eta \neq 0}$$

effect of shear viscosity for ellipsoidally symmetric fireball

The Euler equation is (where  $R=(X+Z)/2$ ):

$$X \underbrace{\left( \ddot{X} - R\omega^2 \right)}_{\text{effect of rotation}} = Y \ddot{Y} = Z \underbrace{\left( \ddot{Z} - R\omega^2 \right)}_{\text{effect of rotation}} = C_E \frac{T_P T_D}{m} \left[ 1 - \underbrace{\frac{\zeta_0}{p_0} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right)}_{\text{effect of bulk viscosity}} \right]$$

# Parametric solutions with inhomogeneous pressure (Ellipsoidal symmetry, rotation and $T$ dependent $\kappa$ )

Pressure is inhomogeneous  $\rightarrow \zeta$  and  $\eta$  are linear in pressure:  $\frac{\zeta(t, s)}{\zeta_0} = \frac{\eta(t, s)}{\eta_0} = \frac{p(t, s)}{p_0}$

No assumption for  $f_T(t)$  and we define the scale volume:

$$T(t, s) = T_H(t)\mathcal{T}(s) \quad T_H(t) \not\propto T_P(t)T_D(t) \quad V \equiv V(t) = (2\pi)^{3/2}XYZ$$

The energy conservation becomes:

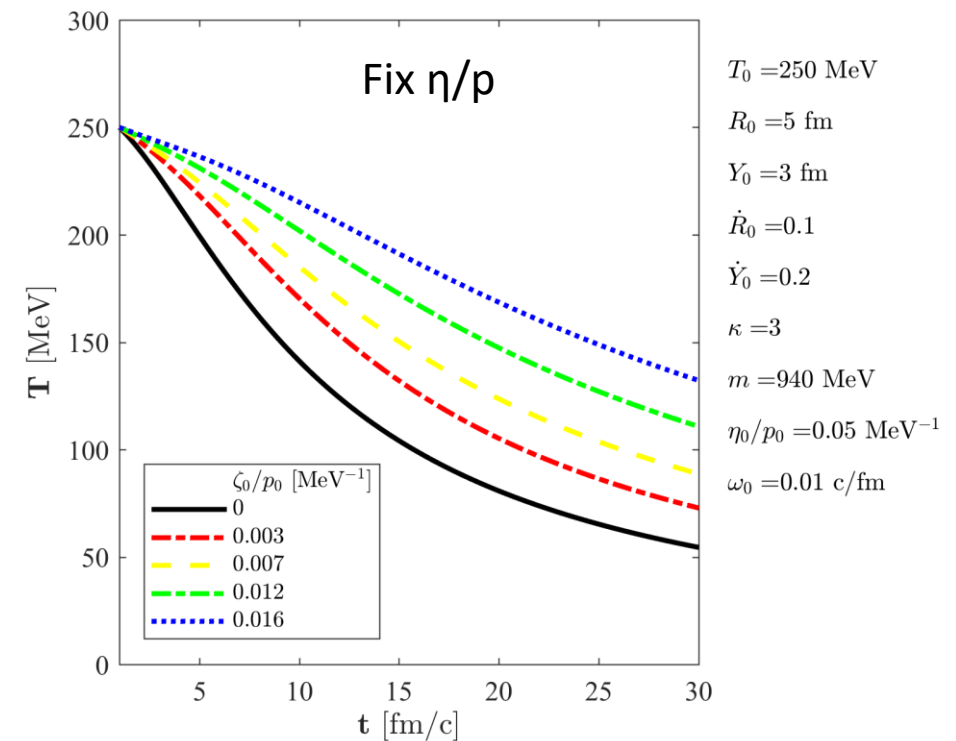
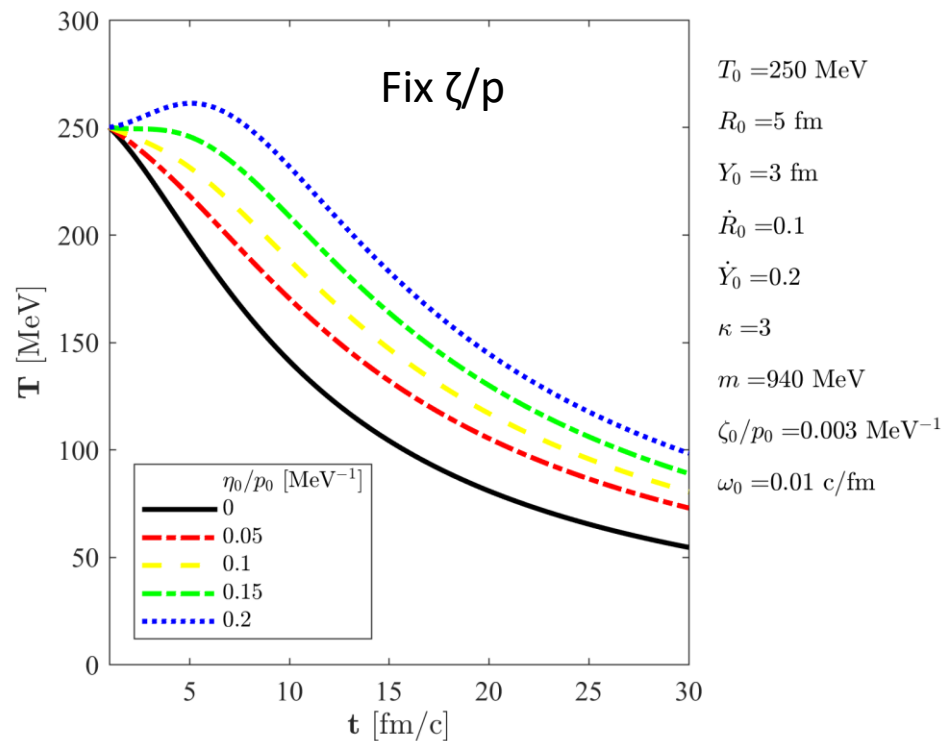
$$\left(\frac{1}{\gamma(T)-1}\right)\frac{\dot{T}_D}{T_D} + \frac{\dot{V}}{V} = \underbrace{\frac{\zeta_0}{p_0}\left(\frac{\dot{V}}{V}\right)^2}_{\text{effect of bulk viscosity}} + \underbrace{\frac{2\eta_0}{p_0}\left[\frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Z}^2}{Z^2} - \frac{1}{3}\left(\frac{\dot{V}}{V}\right)^2\right]}_{\substack{\text{the only effect of shear viscosity} \\ \text{for spheroidally symmetric fireball}}} + \underbrace{\frac{\eta_0\omega_0^2}{4p_0}\frac{(X_0+Z_0)^4}{(X+Z)^4}\left(\frac{X}{Z} - \frac{Z}{X}\right)^2}_{\text{effect of rotation, if } \eta \neq 0}$$

effect of shear viscosity for ellipsoidally symmetric fireball

The Euler equation:

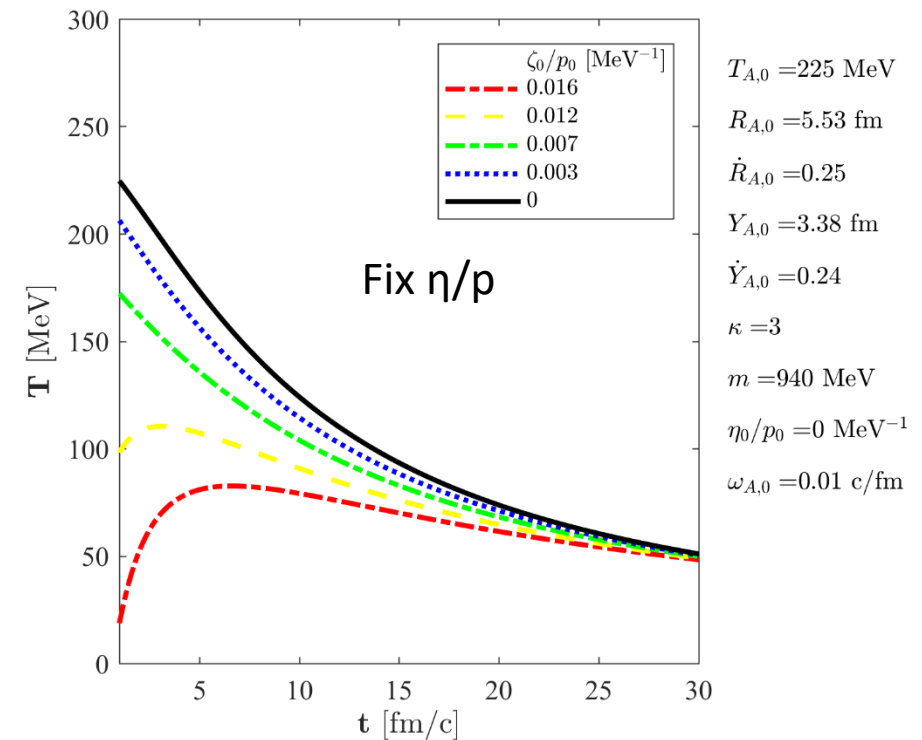
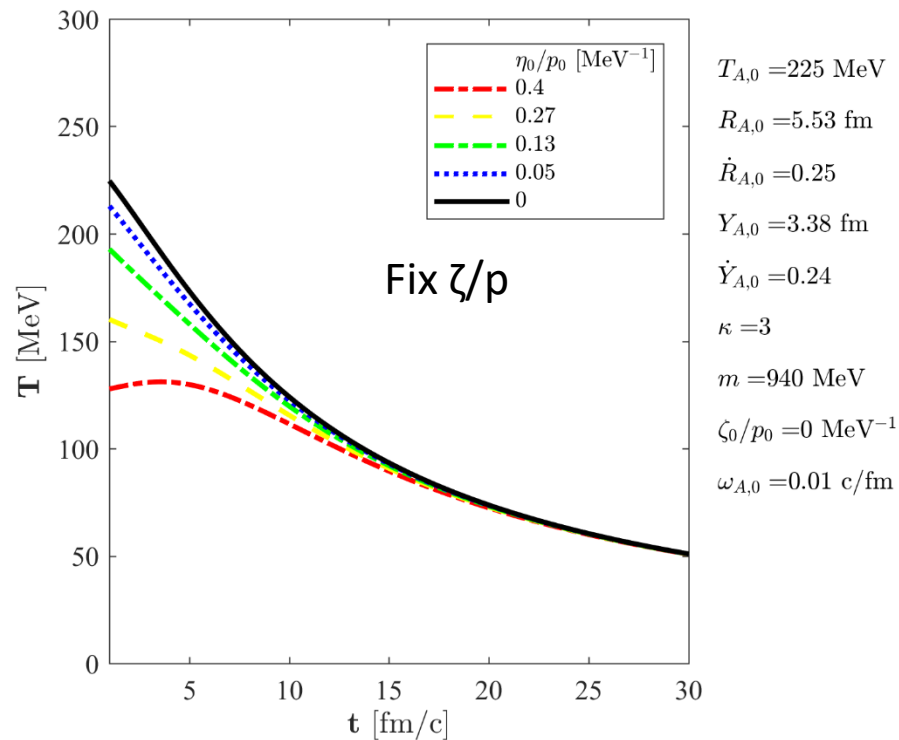
$$X \underbrace{\left(\ddot{X} - R\omega^2\right)}_{\text{effect of rotation}} = Y \ddot{Y} = Z \underbrace{\left(\ddot{Z} - R\omega^2\right)}_{\text{effect of rotation}} = C_E \frac{T_H}{m} \left[ 1 - \underbrace{\frac{\zeta_0}{p_0}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)}_{\text{effect of bulk viscosity}} \right]$$

# Spheroidally symmetric, parametric solution: Evolution of temperature



***Fixed initial conditions***

# Spheroidally symmetric, parametric solution: Evolution of temperature



***Fixed attractor***

# Spheroidally symmetric, analytic solution

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The pressure is homogeneous

$$R(t) = R_0 + \dot{R}_0(t - t_0),$$

There is no rotation in the system

$$Y(t) = Y_0 + \dot{Y}_0(t - t_0),$$

The solution of particle density is same as before

We use:  $T_H(t) = T_P(t)T_D(t)$

The temperature equation is:

$$\kappa_0 \partial_t \ln(T_D) = \frac{\xi_R}{(t + \Delta t_R)^2} + \frac{\xi_Y}{(t + \Delta t_Y)^2} + \frac{\xi_{RY}}{(t + \Delta t_R)(t + \Delta t_Y)}$$

New time-like parameters:

$$\Delta t_R = H_R^{-1} - t_0 = \frac{R_0}{\dot{R}_0} - t_0$$

$$\Delta t_Y = H_Y^{-1} - t_0 = \frac{Y_0}{\dot{Y}_0} - t_0$$

Assumptions for viscosities:

$$\frac{\zeta(t, s)}{\zeta_0} = \frac{\eta(t, s)}{\eta_0} = \frac{p(t, s)}{p_0}$$

The viscosities are carried by:

$$\xi_R = \frac{4\zeta_0}{p_0} + \frac{4\eta_0}{3p_0}$$

$$\xi_Y = \frac{\zeta_0}{p_0} + \frac{4\eta_0}{3p_0}$$

$$\xi_{RY} = \frac{4\zeta_0}{p_0} - \frac{8\eta_0}{3p_0}$$



# Spheroidally symmetric, analytic solution

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The solution of the dissipative correction is:

$$T_D(t) = \left( \frac{t_0 + \Delta t_R}{t + \Delta t_R} \right)^{\frac{\xi_{RY}}{\kappa_0(H_R^{-1} - H_Y^{-1})}} \left( \frac{t_0 + \Delta t_Y}{t + \Delta t_Y} \right)^{\frac{\xi_{RY}}{\kappa_0(H_Y^{-1} - H_R^{-1})}} \exp \left[ \frac{\xi_R H_R}{\kappa_0} \left( 1 - \frac{t_0 + \Delta t_R}{t + \Delta t_R} \right) + \frac{\xi_Y H_Y}{\kappa_0} \left( 1 - \frac{t_0 + \Delta t_Y}{t + \Delta t_Y} \right) \right]$$

For asymptotically late times the dissipative correction is constant:

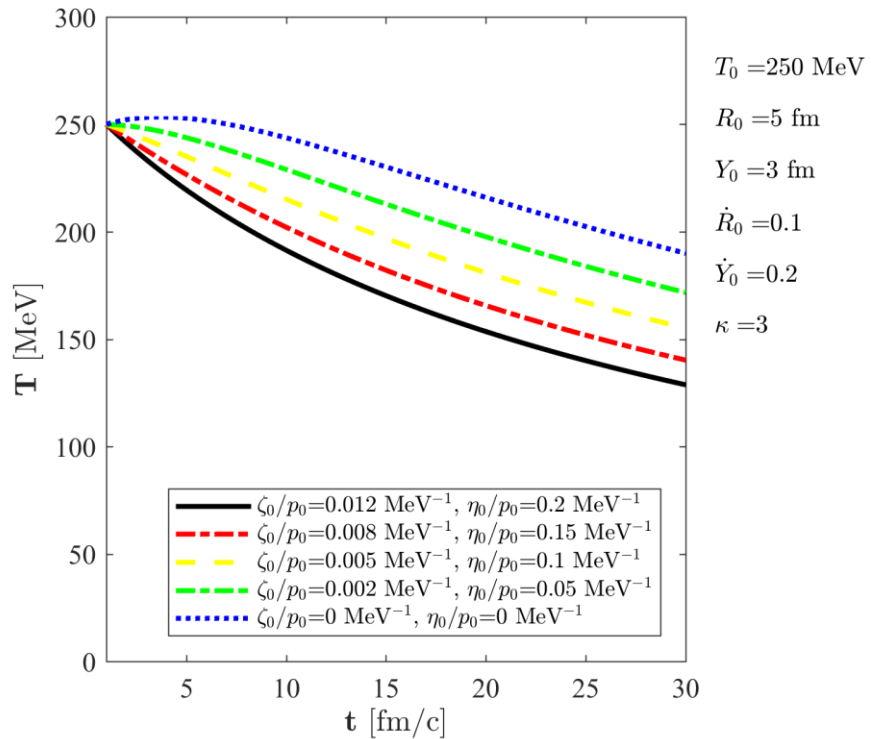
$$T_D(\kappa_0) = \left( \frac{t_0 + \Delta t_R}{t_0 + \Delta t_Y} \right)^{\frac{\xi_{RY}}{\kappa_0(H_R^{-1} - H_Y^{-1})}} \exp \left( \frac{\xi_R H_R}{\kappa_0} + \frac{\xi_Y H_Y}{\kappa_0} \right)$$

For late time,  $T_D$  does not affect the time evolution, but rescale the initial temperature

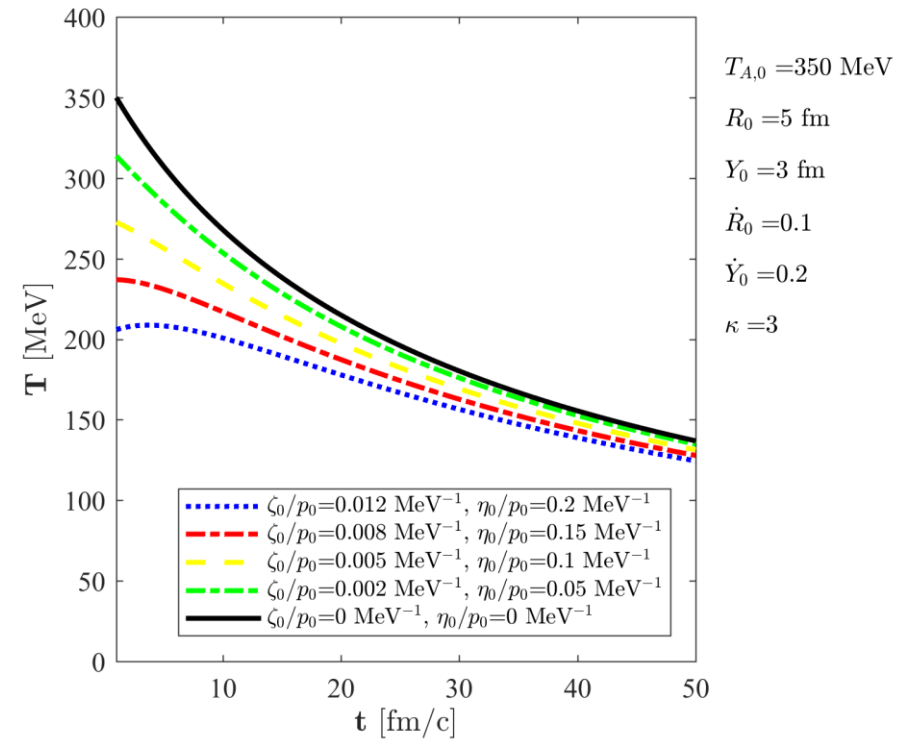
**This analytic solution tends to a perfect fluid solution:**

$$T_A(t, s) = T_{A,0}(\kappa_0) \left( \frac{R_0^2 Y_0}{R^2 Y} \right)^{\frac{1}{\kappa_0}} \mathcal{T}(s)$$

# Spheroidally symmetric, analytic solution



**Fixed initial conditions**



**Fixed attractor**

# Spheroidally symmetric, analytic solution

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## Analytic solution with T dependent $\kappa$

Temperature equation:

$$\left[ \frac{d}{dT}(\kappa T) \right] \partial_t \ln(T_H) + \frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} = \frac{\zeta_0}{p_0} \left( \frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} \right)^2 + \frac{4\eta_0}{3p_0} \left( \frac{\dot{R}}{R} - \frac{\dot{Y}}{Y} \right)^2$$

We have seen that it can be solved analytically, if  $\kappa$  is constant

There is another possibility:

$$\frac{d}{dT}[\kappa(T)T] = \Delta\kappa_T = \text{constant}$$

$$\Delta\kappa_T \partial_t \ln(T_D) = \frac{\xi_R}{(t + \Delta t_R)^2} + \frac{\xi_Y}{(t + \Delta t_Y)^2} + \frac{\xi_{RY}}{(t + \Delta t_R)(t + \Delta t_Y)}$$

The same temperature eq. as before with different constant coefficient:

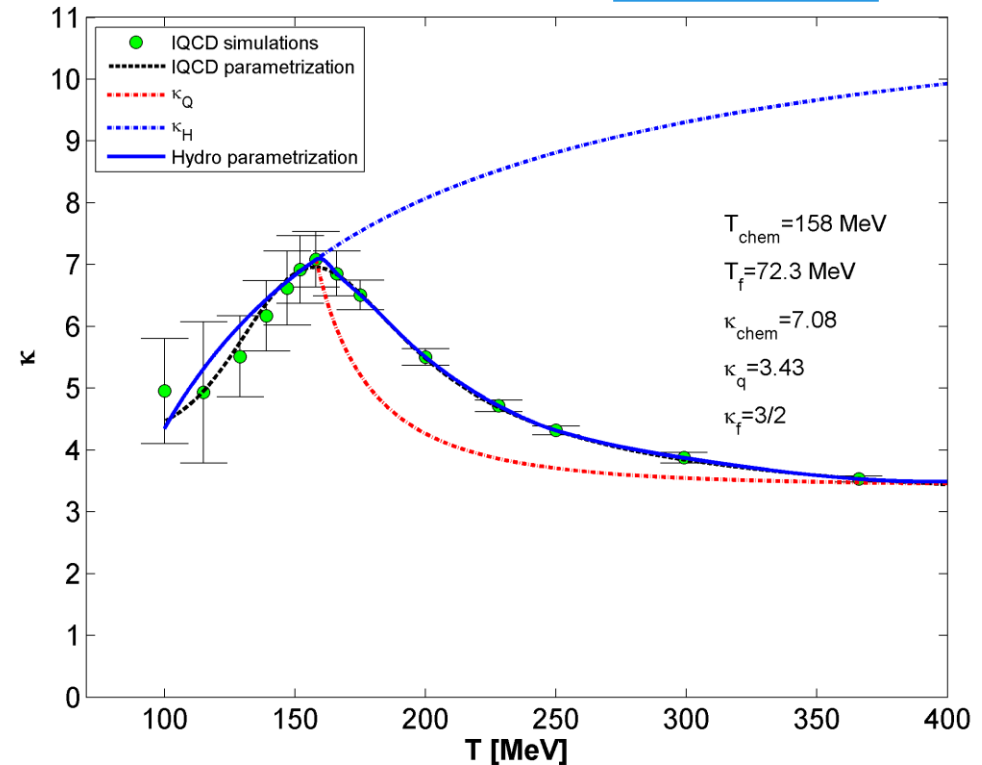
$$\kappa_0 \rightarrow \Delta\kappa_T$$

# Spheroidally symmetric, analytic solution

Temperature dependent function for  $\kappa$  is obtained earlier  $\rightarrow$  lattice QCD EoS was parametrized

[arXiv:1610.02197](https://arxiv.org/abs/1610.02197) and my MSc thesis

*We have provided an analytic, viscous solution with lattice QCD EoS!*



# Spheroidally symmetric, analytic solution

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## One more interesting property...

For fixed attractor and kinematic shear viscosity:

→ *the kinematic bulk viscosity is a non monotonic function of the initial temperature*

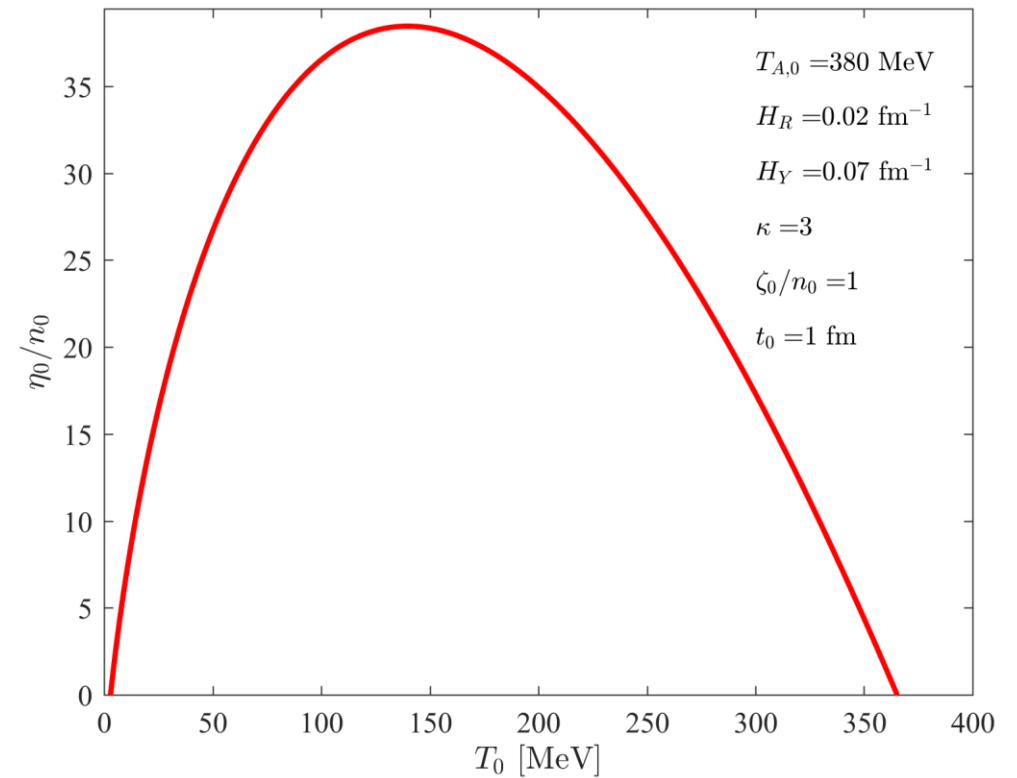
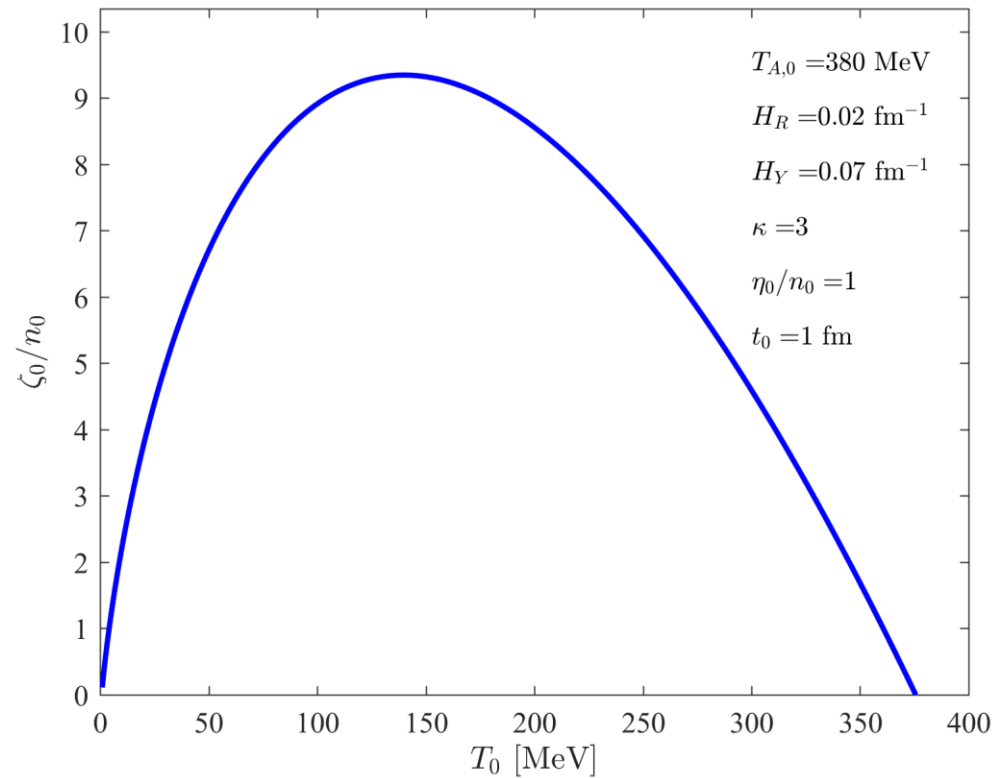
For fixed attractor and kinematic bulk viscosity:

→ *the kinematic shear viscosity is a non monotonic function of the initial temperature*

$$\kappa_0 T_0 \ln\left(\frac{T_{A,0}(\kappa_0)}{T_0}\right) = \frac{\zeta_0}{n_0} \left[ 4H_R + H_Y - \frac{4H_R H_Y}{H_R + H_Y} \ln\left(\frac{t_0 + \Delta t_Y}{t_0 + \Delta t_R}\right) \right] + \frac{4\eta_0}{3n_0} \left[ H_R + H_Y + \frac{2H_R H_Y}{H_R + H_Y} \ln\left(\frac{t_0 + \Delta t_Y}{t_0 + \Delta t_R}\right) \right]$$

# Spheroidally symmetric, analytic solution

Non monotonic shear and bulk viscosity as functions of the initial temperature



# Summary

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New, parametric and analytic solutions of non relativistic hydrodynamics have been found

Common property of these solutions: **Hubble-flow**

***Only academic results, not plan to describe measurements***

The parametric solutions with inhomogeneous pressure are fully developed

- *(ellipsoidal symmetry, rotation is included, temperature dependent  $\kappa$  is allowed, non-constant  $\zeta$  and  $\eta$ )*

Recent result: Spheroidally symmetric, analytic solutions with homogeneous pressure

- *(temperature dependent  $\kappa$  is allowed with a certain condition, non-constant  $\zeta$  and  $\eta$ )*

**All of the presented solutions are asymptotically perfect and tend to perfect fluid solutions**

The analytic solution has indicated **non monotonic behaviour of the kinematic viscosities**

*Thank you for your attention!*