Field-Theoretical Formulation of the Thermodynamical Bethe Ansatz

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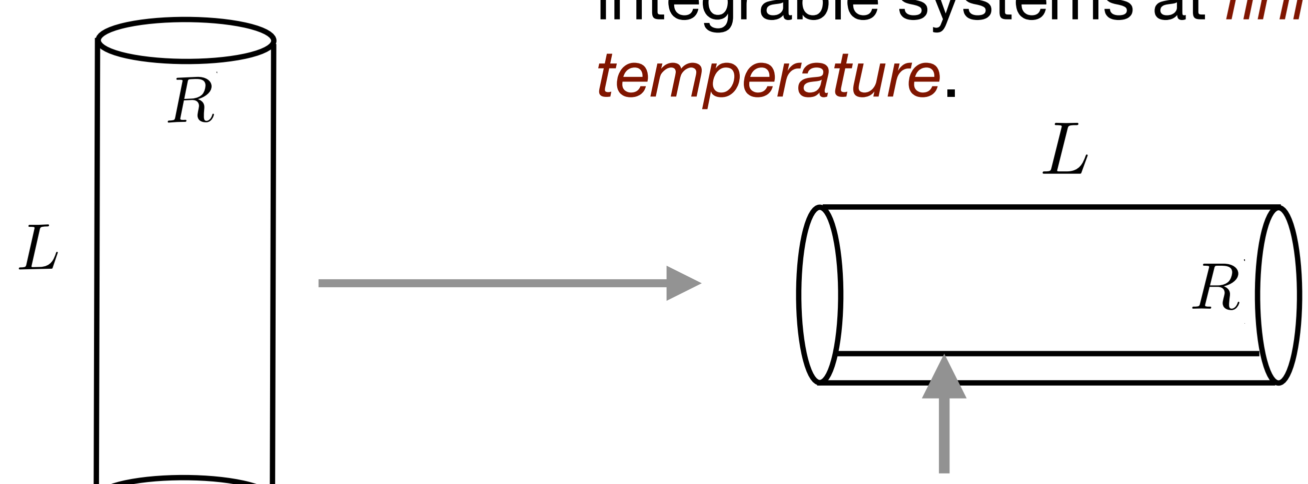
I.K., arXiv[hep-th]1909xxxx

I.K., Didina Serban, D. L. Vu, arXiv[hep-th]1805.02591, 1809.05705, 1906.01909

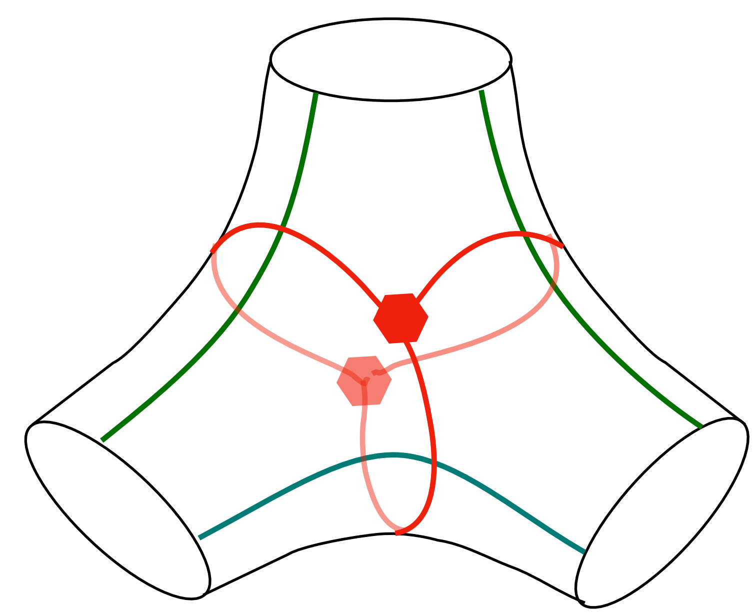
Thermodynamic Bethe Ansatz (TBA) [Yang&Yang, 1969]
— thermodynamics of 1-dim. integrable systems at *finite temperature*.

Since [Al. Zamolodchikov (1990) TBA became the main tool to compute *finite size* effects in 1+1 dim. relativistic field theories



Cut the cylinder and glue it back by inserting a complete set of virtual states (wrapping particles)

More recently TBA related methods are used in computation of correlation functions, e.g. *hexagonalization method* in N=4 SYM.



Need to learn how to evaluate efficiently the sum over the virtual particles in different problems.

Is it possible to replace the original TBA arguments by a more refined QFT/statistical formulation?

The question was posed decades ago and the answer is in principle yes although the effective QFT has not been yet formulated

Balog'94, Saleur 1999

Woynarovich, 2004: gaussian fluctuations around the saddle point of the Y-Y potential.

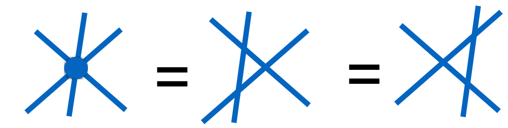
Pozsgay, 2010: showed that there is another O(1) contribution from the measure.

Kato&Wadati, 2004: exact cluster expansion.
I.K., Serban, Vu 2018 graph expansion for the free energy with periodic and open b.c.

In this talk we construct from scratch an effective QFT generating the exact cluster expansion for TBA

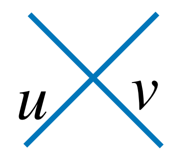
— For simplicity we take an integrable theory with one single neutral particle.
— In view of applications to N=4 SYM the scattering matrix is not supposed to be of difference type and no relativistic symmetry is assumed, only a mirror transformation.

Euclidean 1+1 dimensional integrable field theory with factorized scattering

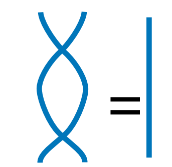


Rapidity variable: $p = p(u),\ E = E(u)$ $\qquad (\theta \equiv \pi u)$

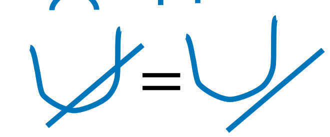
Two-particle S-matrix:



unitarity: $S(u,v)\ S(v,u) = 1$



crossing $S(u,v)S(u,v^{2\gamma}) = 1$

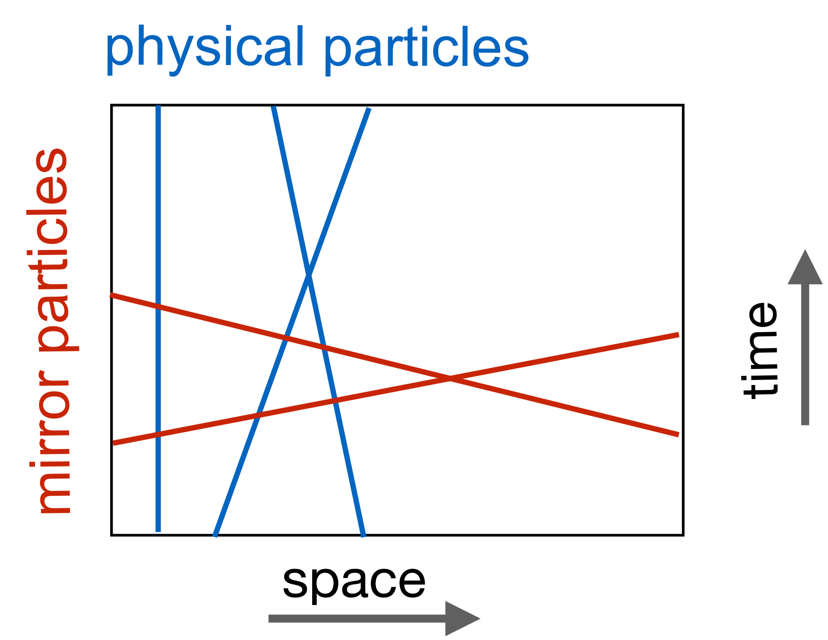


$S(u,u) = -1$

— no Lorentz invariance assumed, only a **mirror** transformation exchanging space and time

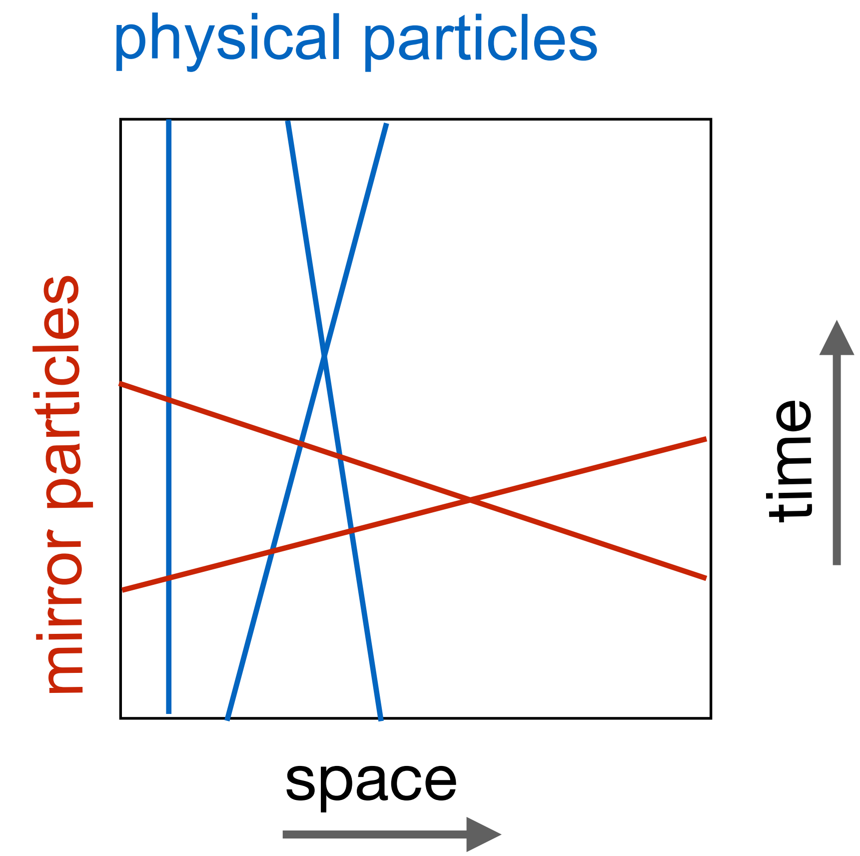
$$E \to i\tilde{p},\ p \to i\tilde{E}$$

Physical theory ➜ Mirror theory



Mirror transformation as analytical continuation

For an observer living in the physical space, a mirror particle looks as physical particle with complex rapidity

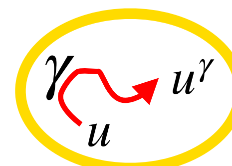


Physical theory $\rightarrow$ Mirror theory

$$S(u,v) \rightarrow \tilde{S}(u,v) \equiv S(u^\gamma, v^\gamma)$$

$$p(u) \rightarrow \tilde{p}(u) = -iE(u^\gamma)$$

$$E(u) \rightarrow \tilde{E}(u) = -ip(u^\gamma)$$



$$\gamma : u \rightarrow u^\gamma$$

Example 1: Lorentz-invariant massive integrable QFT

$$E^2 - p^2 = m^2$$

$$E = m\cosh(\pi u)$$

$$p = m\sinh(\pi u)$$

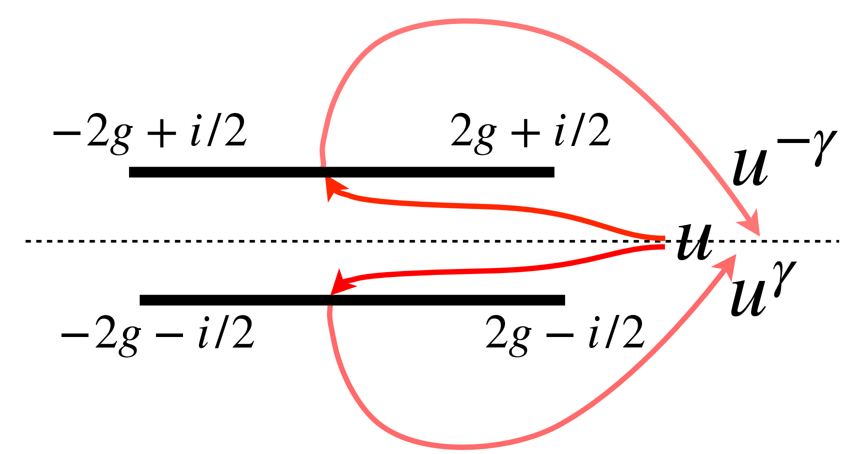
$$u^\gamma \equiv u + i/2$$

$$u$$

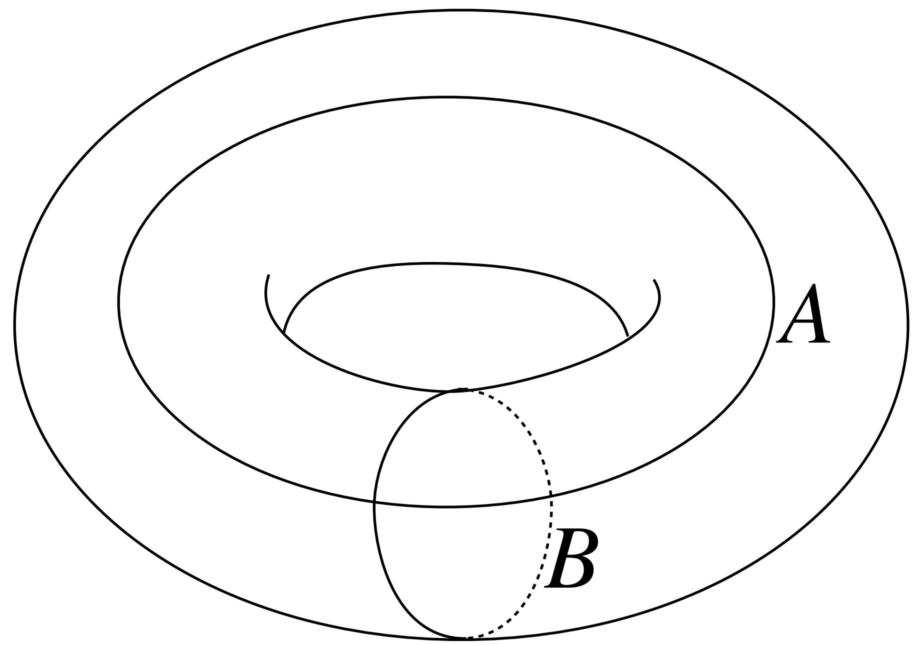
$$u^{-\gamma} \equiv u - i/2$$

Phys=Mirror

Example 2: N=4 SYM



EFFECTIVE QFT FOR PERIODIC BOUNDARY CONDITIONS

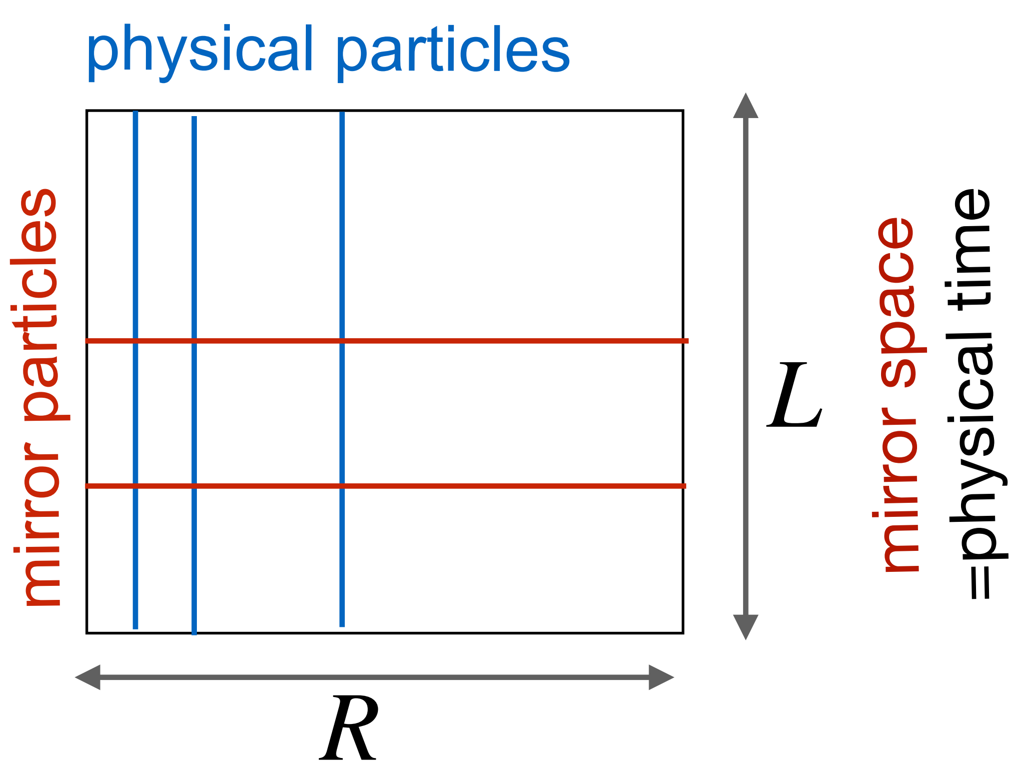


Physical theory defined on the B-cycle of length R,
Mirror theory defined on the A-cycle of length L

Degrees of freedom of the effective QFT:
particles winding around the A and B cycles

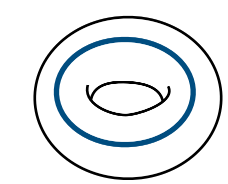
1. Wrapping operators

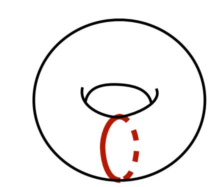
The Hilbert space of the effective QFT is spanned on the elementary excitations on a torus with asymptotically large space and time circles. Two types of them: time-wrapping particles in the physical theory and and space-wrapping particles in the mirror theory



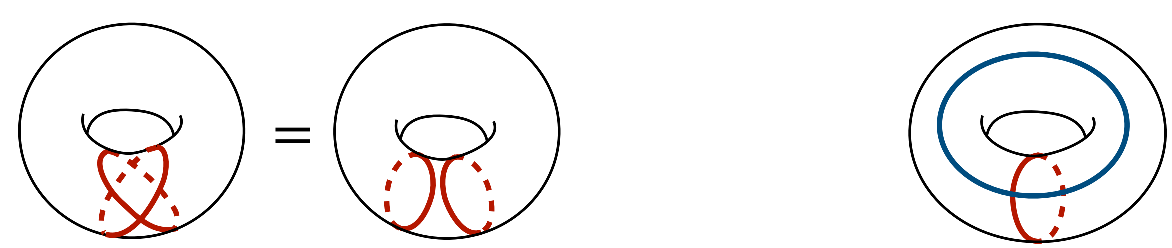
physical space
= mirror time

Operators creating wrapping particles:

$\mathbf{A}(u)$ = time-wrapping operator 

$\mathbf{B}(u)$ = space-wrapping operator 

Particles wrapping the same cycle do not scatter but particles wrapping different cycles do:



$\mathbf{B}(u)$ creates a mirror particle wrapping the B-cycle

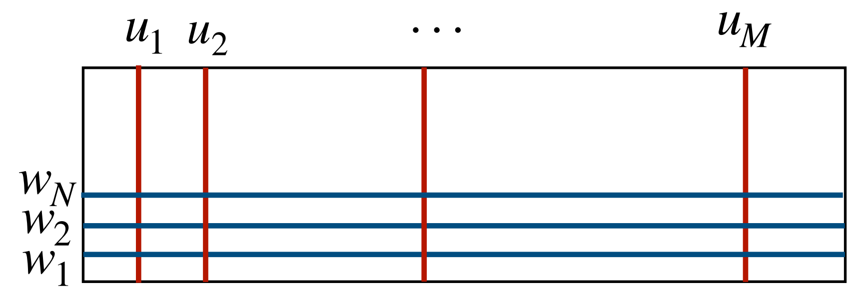
$\mathbf{A}(u^{\gamma})$ creates a mirror particle wrapping the A-cycle

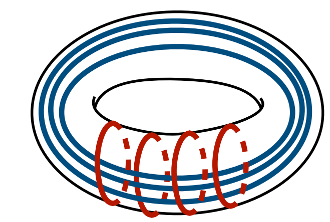
$\mathbf{A}(u)$ creates a physical particle wrapping the B-cycle

$\mathbf{B}(u^{-\gamma})$ creates physical particle wrapping the A-cycle

2. Algebra of the wrapping operators

Expectation value of N time-wrapping and M space-wrapping operators:





$$\langle \prod_{j=1}^{M} \mathbf{B}(v_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \rangle = \prod_{j=1}^{M} \prod_{k=1}^{N} S(v_j^{\gamma}, w_k) \; \prod_{j=1}^{M} e^{-R\tilde{E}(v_j)} \prod_{k=1}^{N} e^{-LE(w_k)}$$

Fock-space realisation:

$$\mathbf{B}(v)\mathbf{A}(u) = S(v^{\gamma}, u)\, \mathbf{A}(u)\mathbf{B}(v), \quad [\mathbf{B}(u), \mathbf{B}(v)] = [\mathbf{A}(u), \mathbf{A}(v)] = 0$$

$$\langle L \,|\, \mathbf{A}(u) = e^{-LE(u)} \langle L \,|, \quad \mathbf{B}(u) \,|\, R\rangle = e^{-R\tilde{E}(u)} \,|\, R\rangle \qquad \langle L \,|\, R\rangle = 1$$

Fock-space expectation value:

$$\langle \prod_{j=1}^{M} \mathbf{B}(v_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \rangle_{L,R} \equiv \langle L \,|\, \prod_{j=1}^{M} \mathbf{B}(v_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \,|\, R\rangle$$

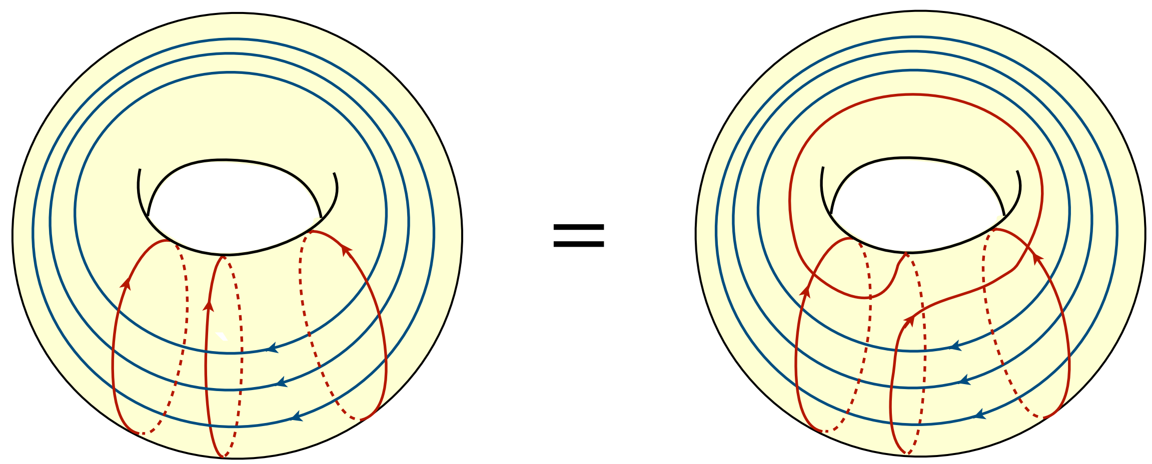
For any operator define

$$\langle \mathcal{O} \rangle_{L,R} = \langle L \,|\, \vdots\, \mathcal{O} \,\vdots \,|\, R\rangle$$

where $\vdots\;\vdots$ is the anti-normal product: all **B**'s are on the left of all **A**'s

3. Operator form of Bethe-Yang equations

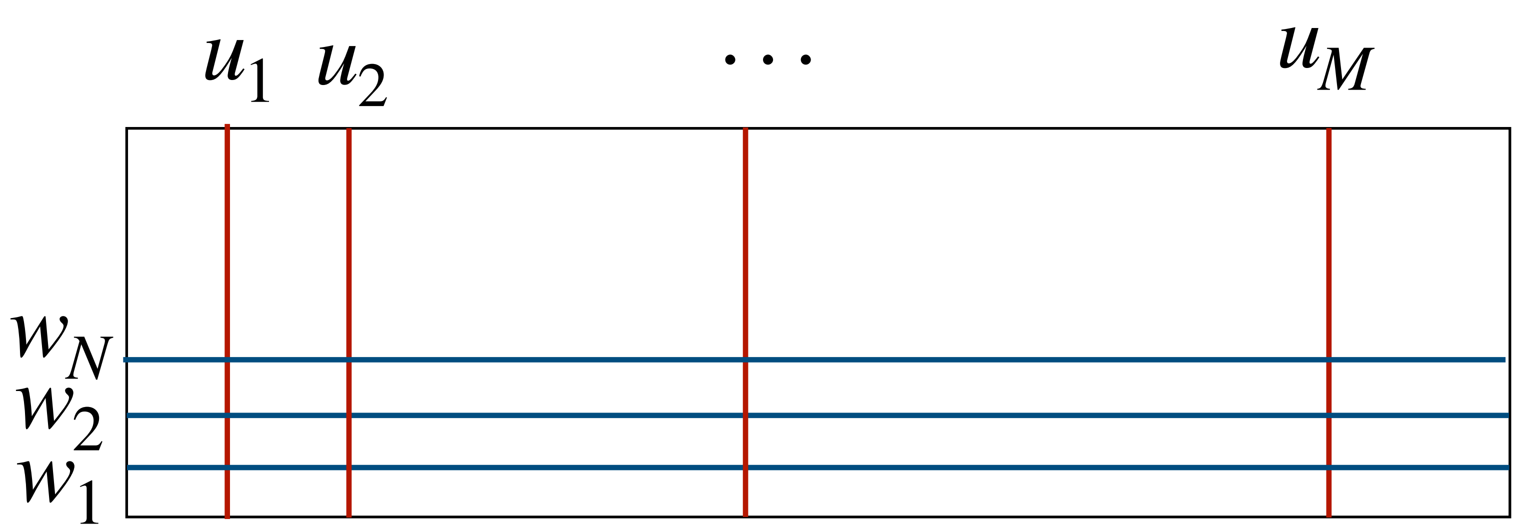
Modular invariance gives the on-shell conditions (Bethe-Yang equations):



A,B

A, B+A

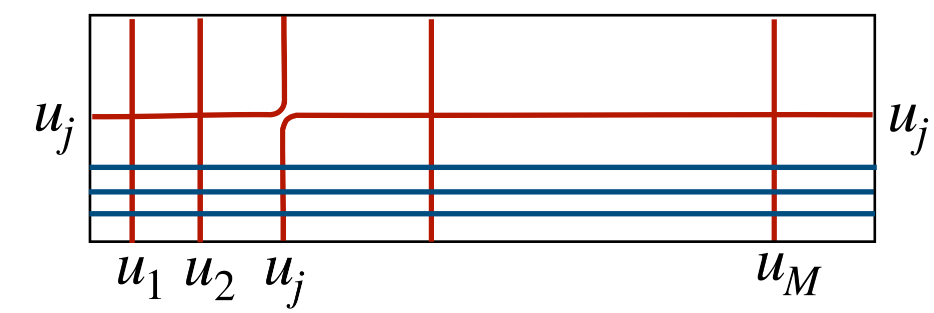
For N physical and M mirror particles: M+N equations for M+N rapidities



3. Operator form of Bethe-Yang equations

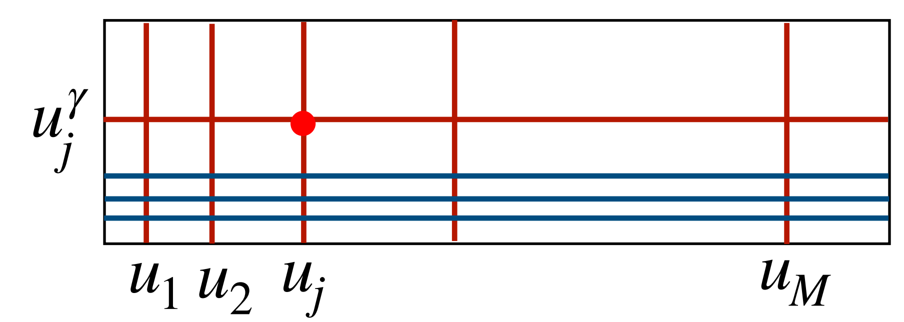
B-Y equations in the mirror channel

$$1 + e^{-iL\tilde{p}(u_j)} \prod_{k=1}^{M} \tilde{S}(u_k, u_j) = 0, \quad j = 1,\ldots,M$$



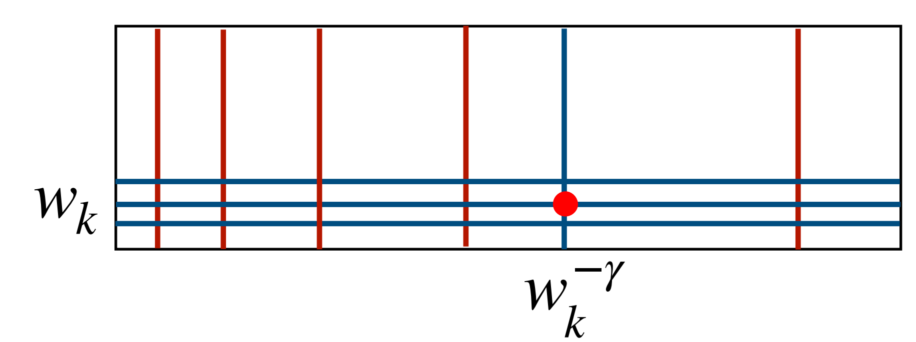
Operator form:

$$\left\langle \prod_{j=1}^{M} \mathbf{B}(u_j) \prod_{k=1}^{N} \mathbf{A}(w_k)\ (1 + \mathbf{A}(u_j^{\gamma})) \right\rangle_{L,R} = 0, \quad j = 1,\ldots,M$$



Similarly, in the physical channel

$$\left\langle \left(1 + \mathbf{B}(w_k^{-\gamma})\right)\ \prod_{j=1}^{M} \mathbf{B}(u_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \right\rangle_{L,R} = 0, \quad k = 1,\ldots,N$$



4. Free-field realisation

Wrapping operators as vertex operators: $\mathbf{B}(u) = e^{-\varphi(u)}, \quad \mathbf{A}(u^\gamma) = e^{-i\bar{\varphi}(u)}$

$\varphi(u), \bar{\varphi}(u)$ - operators for the phases along the B, A cycles in mirror kinemathics

Canonical commutation relations:

$$[\varphi(u), \bar{\varphi}(v)] = i \log \tilde{S}(u, v)$$

$$[\bar{\varphi}(u), \bar{\varphi}(v)] = [\varphi(u), \varphi(v)] = 0$$

Fock space:

$$\langle L | \bar{\varphi}(u) = \bar{\varphi}^\circ(u) \langle L |, \quad \varphi(u) | R \rangle = \varphi^\circ | R \rangle$$

$$\varphi^\circ(u) = R\tilde{E}(u), \qquad \bar{\varphi}^\circ(u) = L\tilde{p}(u), \quad \langle L | R \rangle = 1$$

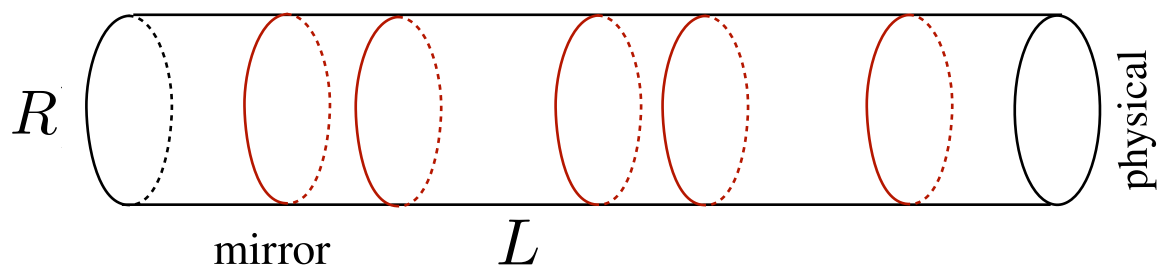
Expectation value:

$$\langle \mathcal{O} \rangle_{L,R} = \langle L | \vdots \mathcal{O} \vdots | R \rangle$$

$$\langle \bar{\varphi}(u)\, \varphi(v) \rangle = i \log \tilde{S}(u, v) + \bar{\varphi}(u)^\circ\, \varphi^\circ(v)$$

$$\langle e^{-i\bar{\varphi}(u)}\, e^{-\varphi(v)} \rangle = \log \tilde{S}(v, u)\, e^{-i\bar{\varphi}^\circ(u) - \varphi^\circ(v)}$$

5. Partition function at finite volume *R in terms of EQFT*



Let us evaluate **exactly** the sum over particles (without assuming that the density is finite)

$$\mathscr{Z}(L,R) = \mathrm{Tr}[e^{-R\tilde{\mathbf{H}}}] = \sum_{\psi} \frac{\langle \psi | e^{-R\tilde{\mathbf{H}}} | \psi \rangle}{\langle \psi | \psi \rangle} = \sum_{M=0}^{\infty} \sum_{n_1 < n_2 < \ldots < n_M} e^{-R(\tilde{E}(u_1)+\ldots+\tilde{E}(u_M))}$$

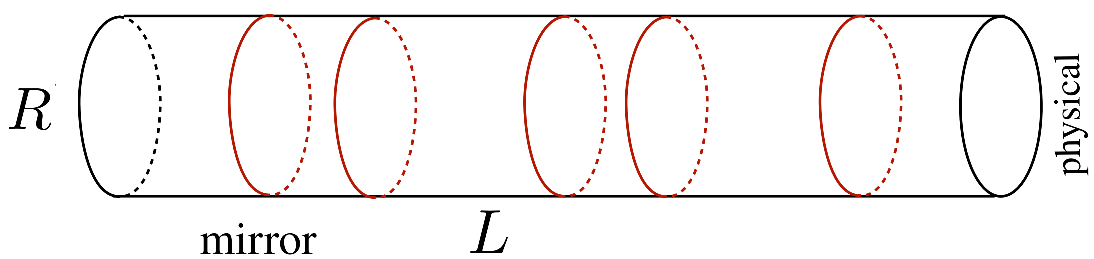
The sum goes over the Bethe quantum numbers which appear in the logarithmic form of the Bethe-Yang equations:

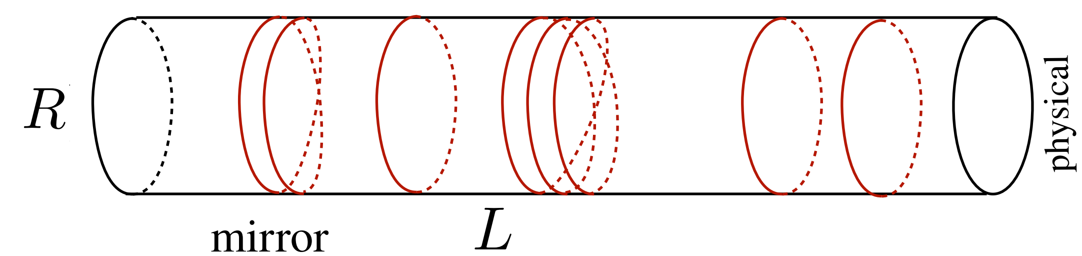
$$-L\tilde{p}(u_j) + i\sum_{k=1}^{M} \log \tilde{S}(u_k, u_j) = 2\pi n_j, \quad j = 1,\ldots,M$$

Our aim is to sum over the solutions of the Bethe-Yang equations (with no approximation) without solving them.

- 1. Relax the constraint $n_1 < n_2 < \ldots < n_M$ by introducing multi-wrapping particles

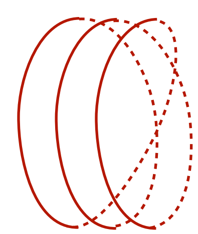
$$\mathcal{Z}(L,R) = \sum_{M=0}^{\infty} \sum_{n_1<n_2<\ldots<n_M} \prod_{j=1}^{M} e^{-R\tilde{E}(u_j)} \rightarrow \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{r_1,\ldots,r_m \geq 1} \sum_{n_1,\ldots,n_m} \prod_{j=1}^{m} \frac{(-1)^{r_j-1}}{r_j} e^{-r_j R\tilde{E}(u_j)}$$





scatters (r-1) times with itself, S(u,u)=-1

r-wrapping particle:



$$\frac{(-1)^{r-1}}{r} e^{-rR\tilde{E}(u)}$$

wraps length r R

cyclic symmetry of order r

Quantization condition:

$$\tilde{\phi}_l \equiv \langle \bar{\varphi}(v_l) \prod_{j=1}^{m} e^{-r_j \varphi(v_j)} \rangle = \tilde{p}(u_l)L + \frac{1}{i} \sum_{k(\neq l)}^{m} r_k \log \tilde{S}(u_l, u_k) + \pi(r_l - 1) = 2\pi n_l$$

- 2. Express the discrete sum over Bethe and wrapping numbers as a contour integral around the real axis

$$\tilde{\phi}_l \equiv \langle \bar{\varphi}(v_l) \prod_{j=1}^{m} e^{-r_j \varphi(v_j)} \rangle = \tilde{p}(u_l)L + \frac{1}{i} \sum_{k(\neq l)}^{m} r_k \log \tilde{S}(u_l, u_k) + \pi(r_l - 1) = 2\pi n_l,\, l = 1,\ldots,m$$

$$\downarrow$$

$$\sum_{r_1,\ldots,r_m} \sum_{n_1,\ldots,n_m} \prod_{j=1}^{M} e^{-r_j R \tilde{E}(u_j)} = \oint_{\mathbb{R}} \cdots \oint_{\mathbb{R}} \prod_{j=1}^{m} \frac{d \log\left(1 + e^{-i\tilde{\phi}(u_j)}\right)}{2\pi i} \sum_{r_1,\ldots,r_m} \prod_{j=1}^{m} \frac{(-1)^{r_j - 1}}{r_j} e^{-R r_j \tilde{E}(u_j)}$$

$$\downarrow$$

$$\prod_{j=1}^{m} \frac{du_j}{2\pi i} \frac{\tilde{G}(u_1, \ldots, u_n; r_1, \ldots, r_m)}{1 + e^{-i\tilde{\phi}(u_j)}}$$

Jacobian
$\tilde{G} = \det[\partial \tilde{\phi}_j / \partial u_k]$
"Gaudin determinant"

- 3. Represent the contour integral as a Fock expectation value. For the we need an operator realization of the Gaudin determinant - the only non-trivial part of this construction

Operator realization of the Gaudin determinant

Introduce a pair of fermionic "ghost" fields $\bar{\psi}(u), \psi(u)$ with commutation relations

and action on the Fock vacua $\langle L | \bar{\psi}(u) = 0, \;\; \psi(u) | R \rangle = 0$

Then

$$\det\left[\frac{\partial \tilde{\phi}_j}{\partial u_k}\right] = \prod_{j=1}^{m} e^{-R\, r_j \tilde{E}(u_j)} = \left\langle \prod_{j=1}^{m} \left[\bar{\varphi}'(u_j) - r_j \bar{\psi}(u_j)\, \psi'(u_j)\right] e^{-r_j \varphi(u_j)} \right\rangle_{L,R}$$

Now we can perform the sum over particle and wrapping numbers:

$$\mathscr{Z}(L,R) = \langle \mathbf{\Omega} \rangle_{L,R}$$

$$\mathbf{\Omega} = \exp \oint_{\mathbb{R}} \frac{du}{2\pi i} \left(-i\bar{\varphi}'(u) \log(1 + e^{-\varphi(u)}) + \frac{i\,\bar{\psi}(u)\psi'(u)}{1 + e^{\varphi(u)}} \right) \frac{1}{1 + e^{i\bar{\varphi}(u)}}$$

$\mathbf{\Omega}$ — operator creating the physical vacuum (finite ***R***) out of the bare vacuum (asymptotically large ***R***)

- 4. Continuum spectrum approximation

The expectation value of $\bar{\varphi}(u) \sim L$ is huge. Up to exponentially small in L terms

$$u$$

$$\frac{1}{1+e^{i\bar{\varphi}(u)}} \to 0$$

$$\frac{1}{1+e^{i\bar{\varphi}(u)}} \to 1$$

$$\boxed{\begin{aligned} &\mathscr{Z}(L,R) = \langle \mathbf{\Omega} \rangle_{L,R} \\ &\mathbf{\Omega} = \exp \oint_{\mathbb{R}} \frac{du}{2\pi i} \left(-i\bar{\varphi}'(u) \log(1+e^{-\varphi(u)}) + \frac{i\,\bar{\psi}(u)\psi'(u)}{1+e^{\varphi(u)}} \right) \end{aligned}}$$

6. Feynman graphs = exact cluster expansion

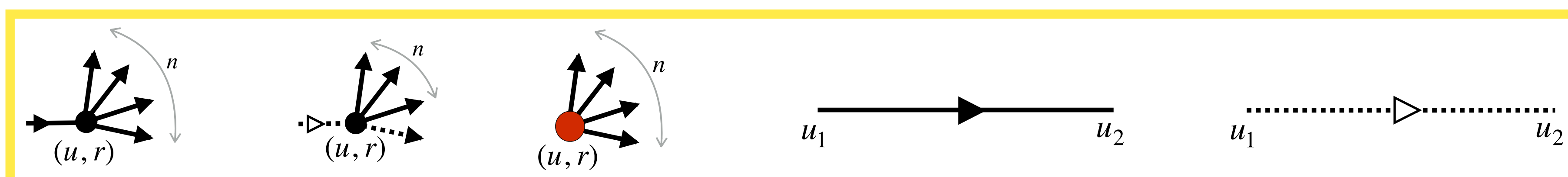
$$\varphi(u) = \varphi^{\circ}(u) + \varphi_q(u), \quad \bar{\varphi}(u) = \bar{\varphi}^{\circ}(u) + \bar{\varphi}_q(u)$$

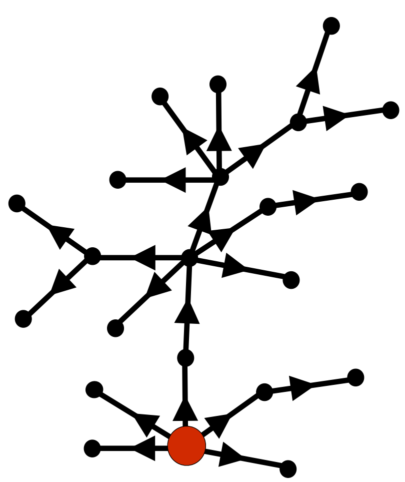
I.K., Didina Serban, D. L. Vu,
arXiv[hep-th]1805.02591,
1809.05705, 1906.01909

Feynman rules

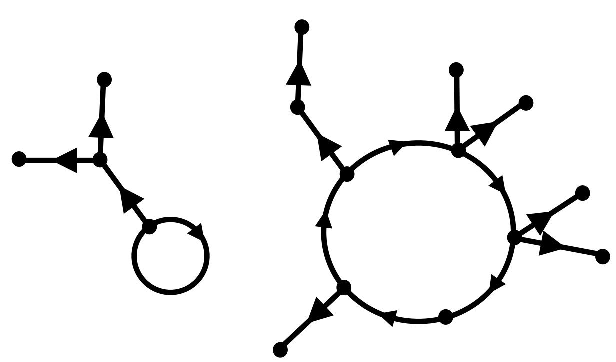
Propagators: $$-\langle \bar{\varphi}_q'(u)\varphi_q(v)\rangle = \langle \psi'(u)\bar{\psi}(v)\rangle = \tilde{K}(u,v) \qquad \tilde{K}(u,v) = \frac{1}{i}\partial_u \log \tilde{S}(u,v)$$

Vertices: $$\bar{\varphi}' \log(1+e^{-\varphi}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, \bar{\varphi}'\, \varphi^n$$

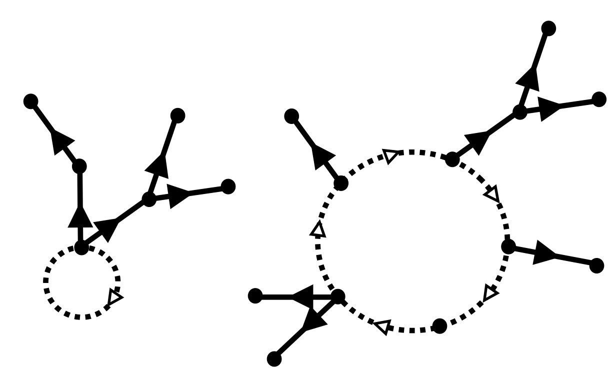




trees



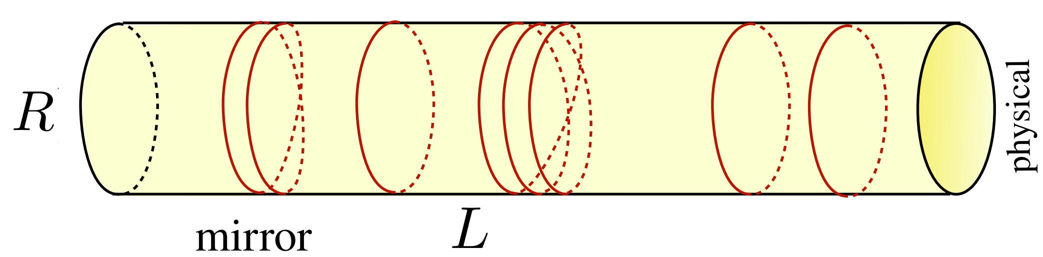
bosonic loops

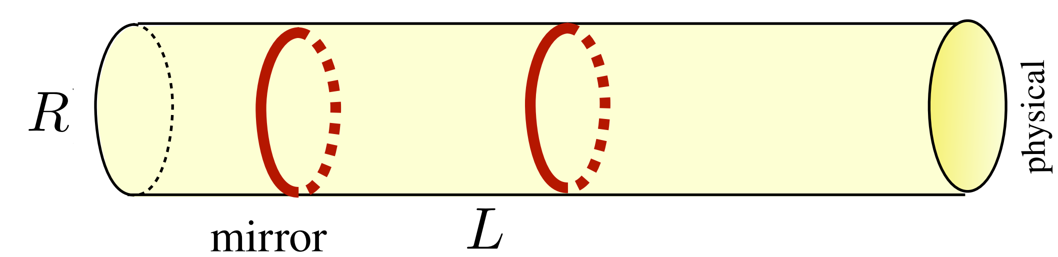


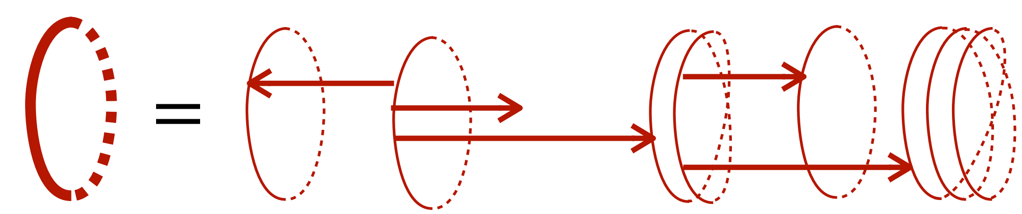
fermionic loops

The bosonic loops and the fermionic loops cancel and the free energy is given by the sum over tree graphs.

The meaning of the tree diagrams in the cluster expansion:







Wrapping particles weakly interacting after being put in a large box

Non-interacting clusters of wrapping particles: behave as free fermions with renormalized energy

7. Path integral and localisation

Impose the correlators by introducing a pair of auxiliary fields $\rho, \bar{\rho}$ $\theta, \bar{\theta}$

$\varphi, \bar{\varphi}$ $\rho, \bar{\rho}$

commutative

$\bar{\psi}, \psi$ $\vartheta, \bar{\vartheta}$

grassmanian

$$\mathscr{Z}(L,R) = \int \mathscr{D}[\text{fields}]\; e^{-\mathscr{A}[\text{fields}]}$$

$$-\mathscr{A}[\text{fields}] = \int \frac{du}{2\pi}\left(\log(1+e^{-\varphi})\,\partial\bar{\varphi} - \frac{\bar{\psi}\,\partial\psi}{1+e^{\varphi}} + (\bar{\varphi}-\bar{\varphi}^{\circ})\rho + (\varphi-\varphi^{\circ})\bar{\rho} + \bar{\vartheta}\psi + \bar{\psi}\vartheta\right)$$

$$-\int \frac{du}{2\pi}\frac{dv}{2\pi}\log\tilde{S}(u,v)\big(\bar{\rho}(u)\rho(v) + \bar{\vartheta}(v)\vartheta(u)\big)$$

the dependence on R and L through the classical fields:

$$\varphi^{\circ}(u) = R\tilde{E}(u), \quad \bar{\varphi}^{\circ}(u) = L\tilde{p}(u)$$

Remark: The bosonic part of the path integral was obtained using different arguments by Jiang, Komatsu and Veskovi [ARXIV:1906.07733]

Localization

Fermionic symmetry: $Q = \int (\bar{\psi}\frac{\delta}{\delta\varphi} + \bar{\varphi}\frac{\delta}{\delta\psi} + \bar{\rho}\frac{\delta}{\delta\vartheta} + \bar{\vartheta}\frac{\delta}{\delta\rho}), \quad \bar{Q} = \int (\psi\frac{\delta}{\delta\bar{\varphi}} + \varphi\frac{\delta}{\delta\bar{\psi}} + \rho\frac{\delta}{\delta\bar{\vartheta}} + \vartheta\frac{\delta}{\delta\bar{\rho}})$

$$Q^2 = \bar{Q}^2 = 0, \quad \bar{Q}Q = Q\bar{Q} = 2$$

$$\mathscr{A} = \mathscr{A}^\circ + \mathrm{Q}(u) \cdot \mathscr{B}$$

Q-exact localisation term

$$\mathscr{A}^\circ = -L\int \frac{d\tilde{p}(u)}{2\pi}\rho(u) \qquad \mathscr{B} = \bar{Q}\mathscr{A}$$

By standard localisation argument integral localises to the critical point:

$$\mathscr{Z} \rightarrow \mathscr{Z}_t = \int e^{-\mathscr{A}_t} \qquad \mathscr{A} \rightarrow \mathscr{A}_t = \mathscr{A}^\circ + t\mathrm{Q}\mathscr{B}, \quad \mathscr{A}^\circ \equiv -L\int \frac{du}{2\pi}p'(u)\rho(u)\,.$$

$$\frac{\partial \mathscr{Z}_t}{\partial t} = \int e^{-\mathscr{A}^\circ - t\mathrm{Q}\mathscr{B}}(-\mathrm{Q}\mathscr{B}) = \int \mathrm{Q}\left(e^{-\mathscr{A}^\circ - t\mathrm{Q}\mathscr{B}}\mathscr{B}\right) = 0$$

Take the limit of infinite perturbation: $t \rightarrow \infty$:

$$\mathscr{Z} = e^{-\mathscr{A}^\circ}\Big|_{\mathrm{Q}\mathscr{B}=0}$$

The critical point:

$$\varphi(u) = R\tilde{E}(u) - i\int \frac{dv}{2\pi} \log \tilde{S}(u,v)\, \rho(v)$$

$$\rho(u) = \partial_u \log(1 + e^{-\varphi(u)})$$

equation for the critical point identical to the TBA integral equation:

$$\epsilon(u) = R\tilde{E}(u) - \int \tilde{K}(v,u)\ \log(1 + e^{-\epsilon(v)})$$

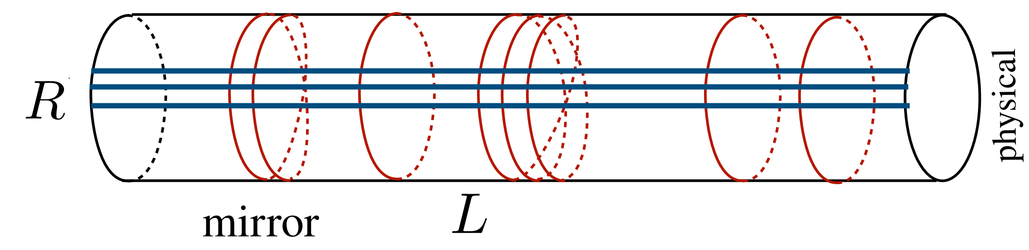
$\epsilon(u) = \varphi^{\rm crit}(u)$ - pseudo energy

The partition function: $$\mathscr{Z}(L,R) = \exp\left(L \int \frac{d\tilde{p}(u)}{2\pi}\ \log\left[1 + e^{-\epsilon(u)}\right]\right)$$

As a consequence of localisation : the theory is one-loop exact and the gaussian fluctuations of the bosons and the fermions cancel => no quantum corrections to the critical action at all

8. Excited states in the physical channel

Naive prescription for the artition function for an excited state with rapidities $\mathbf{w} = \{w_1, \ldots, w_N\}$



(1) $$\mathcal{Z}(L, R, \mathbf{w}) = \left\langle \prod_{k=1}^{N} \mathbf{A}(w_k) \quad \mathbf{\Omega} \right\rangle_{L,R}, \quad \mathbf{A}(w_k) = e^{-i\bar{\varphi}(w_k^{-\gamma})}$$

(2) $$\left\langle \left(1 + \mathbf{B}(w_j^{-\gamma})\right) \prod_{k=1}^{N} \mathbf{A}(w_k) \quad \mathbf{\Omega} \right\rangle = 0, \quad j = 1,2,\ldots,N$$

The correct prescription imposes the on-shell condition as a contour integral: (1)+(2)

$$\prod_{j=1}^{N} \mathbf{A}(w_j) \to \exp\left[\oint_{\mathbf{w}^{-\gamma}} \frac{du}{2\pi i} \, (-i\bar{\varphi}(u)) \, \frac{\varphi'(u) - \bar{\psi}(u)\,\psi'(u)}{1 + e^{\varphi(u)}} \right]$$

Relation to Dorey-Tateo:

$$\boxed{\begin{aligned} &\mathcal{Z}(L, R, \mathbf{w}) = \langle \mathbf{\Omega}_{\mathbf{w}} \rangle_{L,R} \\ &\mathbf{\Omega}_{\mathbf{w}} = \exp \oint_{\mathbb{R} \cup \mathbf{w}^{-\gamma}} \frac{du}{2\pi i} \left(-i\bar{\varphi}'(u) \log(1 + e^{-\varphi(u)}) + \frac{i\,\bar{\psi}(u)\psi'(u)}{1 + e^{\varphi(u)}} \right) \end{aligned}}$$

example: sinh-Gordon model

$$\mathscr{A} = \int d^2x \left[\frac{1}{4\pi} (\partial_\mu \phi)^2 + \frac{2\mu^2}{\sin \pi b^2} \cosh(b\phi) \right]$$

$$\nu = 1 + \frac{1}{b^2}, \quad a = 1 - \frac{2}{\nu} \qquad (0 < b \leq 1)$$

$$\mathbb{D} = \exp\left(\frac{i}{2} \frac{\partial}{\partial u} \right)$$

One particle, no bound states;
relativistic theory: mirror=physical

$$p(u) = m \sinh \pi u, \quad E(u) = m \cosh \pi u$$

$$\log S(u) = (\mathbb{D}^a + \mathbb{D}^{-a}) \log S_0(u)$$

$$S_0(u) = \tanh \frac{\pi(u - i/2)}{2}$$

$$K_0(u, v) = \frac{1}{i} \partial_u \log S_0(u, v)$$ - "universal kernel"

$$(\mathbb{D} + \mathbb{D}^{-1}) K_0(u) = 2\pi \delta(u)$$

$$\mathscr{A}[\text{fields}] = \int \frac{du}{2\pi} \left(\frac{\bar{\varphi}\,\varphi' - \bar{\psi}\,\psi'}{1 + e^{\varphi}} + (\bar{\varphi} - \bar{\varphi}^\circ)\rho + (\varphi - \varphi^\circ)\bar{\rho} + \bar{\theta}\psi + \bar{\psi}\theta \right)$$
$$+ i \int \frac{du}{2\pi} \frac{dv}{2\pi} \left(\bar{\rho}(u)\rho(v) + \theta(u)\bar{\theta}(v) \right) (\mathbb{D}^a + \mathbb{D}^{-a}) \log S_0(u, v)$$

$K \to \mathbb{K} = \frac{\mathbb{D}^a + \mathbb{D}^{-a}}{\mathbb{D} + \mathbb{D}^{-1}}$ $\quad K(u) = K_0(u + ia/2) + K_0(u - ia/2) = (\mathbb{D}^a + \mathbb{D}^{-a}) K_0$ $\quad K_0(u, v) = \frac{\pi}{\cosh \pi (u - v)}$ $\quad K(u, v) = \frac{1}{i} \partial_u \log S(u, v)$ $\quad \mathbb{D} \equiv e^{i\partial_u/2}$

The action can be cast into a quasi-local form by a field redefinition

$$\bar{\varphi} \;\to\; Lp(u) + (\mathbb{D}+\mathbb{D}^{-1})\bar{\varphi} \qquad \bar{\psi} \;\to\; (\mathbb{D}+\mathbb{D}^{-1})\bar{\psi}$$

$$\mathscr{A} = \int \frac{du}{2\pi}\left[\varphi(\mathbb{D}+\mathbb{D}^{-1})\partial\bar{\varphi} - \frac{(\mathbb{D}^{a}+\mathbb{D}^{-a})\partial\bar{\varphi}}{1+e^{-\varphi}}\right]$$

$$+\int \frac{du}{2\pi}\left[\psi(\mathbb{D}+\mathbb{D}^{-1})\bar{\partial}\psi - \frac{\bar{\psi}(\mathbb{D}^{a}+\mathbb{D}^{-a})\partial\psi}{1+e^{\varphi}}\right]$$

Critical point: $\left[\mathbb{D}+\mathbb{D}^{-1}\right]\varphi = -\,(\mathbb{D}^{a}+\mathbb{D}^{-a})\log(1+e^{-\varphi})$ [Zamolodchikov, Lukyanov]

“Discrete Liouville equation”

Baxter Q-function: $\varphi = (\mathbb{D}^{a}+\mathbb{D}^{-a})\log Q$

$$Q^{\mathbb{D}+\mathbb{D}^{-1}} = 1 + Q^{\mathbb{D}^{a}+\mathbb{D}^{-a}} \quad \Rightarrow \quad Q(u+i/2)Q(u-i/2) - Q\,(u+ia/2)\,Q\,(u-ia/2) = 1$$

$$T(u) = \frac{(\mathbb{D}^{1-a}+\mathbb{D}^{a-1})Q}{Q}, \quad \tilde{T}(u) = \frac{(\mathbb{D}^{1+a}+\mathbb{D}^{-1-a})Q}{Q} \qquad \begin{array}{c}(\mathbb{D}^{a}-\mathbb{D}^{-1})\,T = 0\\ (\mathbb{D}^{a}-\mathbb{D})\tilde{T} = 0\end{array}$$

Summary

Path integral formulation of the Thermodynamic Bethe Ansatz

The theory is one-loop exact. Explains why there are only exponential corrections to the free energy

Works also for scattering matrices not of difference type, as in AdS/CFT

Generalizations?

— EQFT be generalised easily to

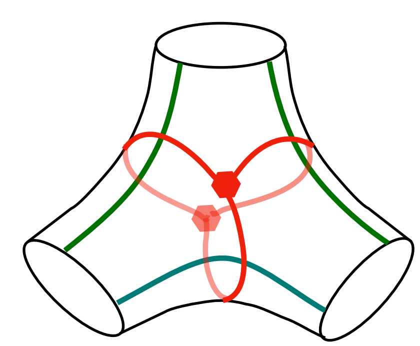
1) open boundary conditions

2) the case of diagonal scattering theories as ADE

3) the case of non-diagonal scattering (nested Bethe Ansatz) and bound states ***assuming the string hypothesis*.**

Do we really need this assumption? - not clear at the moment

— Hopefully can be adapted to other geometries with application to AdS/CFT



Thank you!