Resurgence and renormalons in integrable QFTs

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mostly based on work with Ramon Miravitllas and Tomás Reis

P and NP

Integrability often provides exact, non-perturbative answers for observables in QFT. However, in generic situations we still use perturbative series to obtain asymptotic expansions:

$$E(g) \sim \sum_{n \geq 0} a_n g^n$$

On top of this perturbative piece, observables often involve non-perturbative corrections. They are typically exponentially small in the coupling constant:

$$\mathrm{e}^{-A/g}$$

We then expect something like

$$\boxed{\begin{matrix}\text{physical} \\ \text{observable}\end{matrix} \quad = \quad \begin{matrix}\text{perturbative} \\ \text{series}\end{matrix} \quad + \quad \begin{matrix}\text{exponentially} \\ \text{small corrections}\end{matrix}}$$

This is in fact the answer suggested by the saddle-point method applied to the path integral

$$E(g) \sim \sum_{n\geq 0} a_n g^n + C g^{-b} \mathrm{e}^{-A/g} \left(1 + \sum_{n\geq 1} c_n g^n \right) + \cdots$$

This formal object combining perturbative series with exponentially small terms is called a **trans-series**.

Making sense of a trans-series is however very subtle, since all the series in g appearing there are factorially divergent

$$E(g) \sim \underbrace{\sum_{n\geq 0} a_n g^n}_{a_n \sim n!} + C g^{-b} \mathrm{e}^{-A/g} \left(1 + \underbrace{\sum_{n \geq 1} c_n g^n}_{c_n \sim n!} \right) + \cdots$$

There are however resummation methods, developed in the so-called **theory of resurgence**, which make it possible to resum trans-series into actual numbers, under mild conditions.

A conjecture

We conjecture that, in a quantum theory, observables with an asymptotic expansion can be computed **exactly** by an appropriate resummation of a trans-series. The trans-series is obtained by adding appropriate exponentially small terms (non-perturbative effects) to the asymptotic expansion.

This was called the (weak) **resurgence conjecture** in [Di Pietro-M.M.-Sberveglieri-Serone]. It gives a generalized semiclassical picture of observables in quantum theories, and it holds in e.g. one-dimensional quantum mechanics.

To test this conjecture, we need the ingredients. We know how to obtain perturbative series, but what about exponentially small corrections?

The origin of non-perturbative corrections

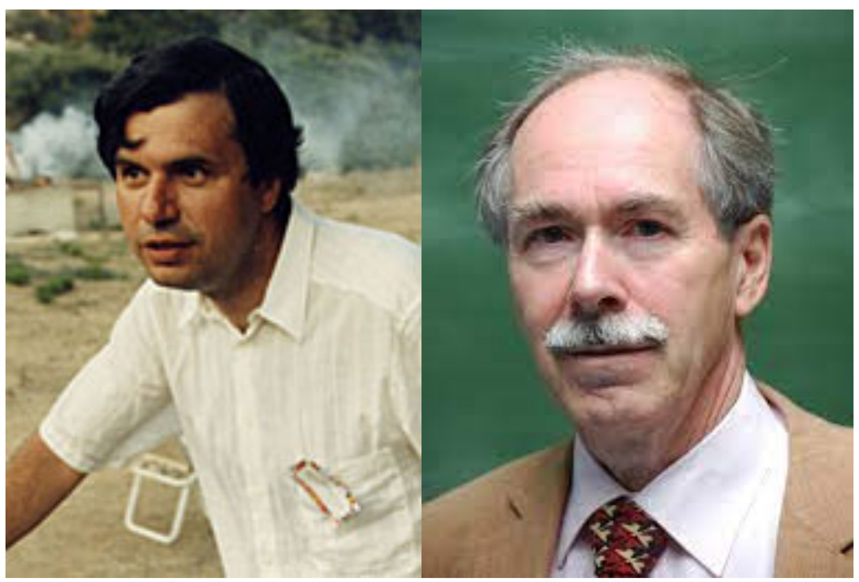
What is the physical origin of exponentially small corrections in quantum theories?

One obvious source are **instantons**, i.e. non-trivial classical configurations leading to additional saddle-points in the path integral.

Historically, instantons were regarded as the natural building blocks of trans-series, and they actually do the job in quantum mechanics and in some SUSY theories.

Renormalons

However, in the 1970s-1980s a new and mysterious source of exponentially small corrections was found in many QFTs: **renormalons**



Parisi and ’t Hooft conjectured that in asymptotically free (AF) theories at infinite volume, renormalons lead to exponentially small corrections of the form

$$\exp\left(-\frac{d}{2|\beta_0|g(\mu)}\right) \sim \left(\frac{\Lambda}{\mu}\right)^d \qquad d = 1, 2, \ldots$$

Renormalons vs. instantons

Exponential corrections due to renormalons are generically more important than would-be instanton corrections, and they survive in the large *N* limit (in contrast to instanton corrections)

$$|\beta_0| \sim N \qquad\qquad \mathrm{e}^{-\frac{1}{|\beta_0|g}} \sim \mathrm{e}^{-\frac{1}{\lambda}} \longleftarrow \text{'t Hooft parameter}$$

In addition, renormalons do not have a known semi-classical realization. They are corrections to the trivial saddle-point in the path integral with no saddle-point description!

How do you detect a renormalon?

Since they do not have a semiclassical description, how do you detect renormalons?

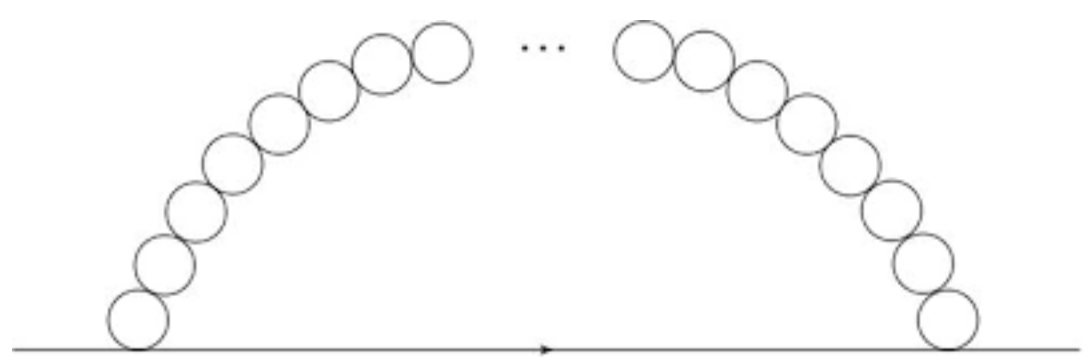
The key idea comes from the theory of resurgence: the **large order behavior** of the perturbative series **predicts** some of the exponentially small corrections in the trans-series

$$\boxed{\begin{array}{c}\displaystyle\sum_{n\geq 0} a_n g^n \\[2em] a_n \sim A^{-n}\Gamma(n+b)\end{array}} \longrightarrow \boxed{g^{-b} \mathrm{e}^{-A/g}}$$

How do you obtain information on the perturbative series at very large number of loops?

You can try to produce a finite but large number of coefficients and extract *A*, *b* numerically [Bauer-Bali-Pineda]

Or you can work at large *N*, where the perturbative series is dominated by "bubble-like" diagrams



Both techniques have been used to produce some (limited) information on renormalon corrections

Enter integrability

Many asymptotically free QFTs in 2d are integrable, i.e. their *S*-matrix is exactly known (*O(n)* sigma model, Gross-Neveu, …). Can we use these models to obtain information on renormalons?

One should "decode" the exact answers provided by integrability in terms of a trans-series. This is not easy! Even perturbative series are hard to extract from the Bethe ansatz.

Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State

Elliott H. Lieb and Werner Liniger

It is interesting to note that the exact equation for $e(\gamma)$ is so pathological at $\gamma=0$ that it was an effort to find even the zeroth-order term for $e(\gamma)$, while perturbation theory gives the first two terms by elementary quadrature.

Choosing the observable

We will focus on the ground state energy of 2d integrable models, once they are coupled to a "chemical potential" *h* through a global conserved charge *Q*

$$\mathcal{F}(h) = -\lim_{V,\beta\to\infty} \frac{1}{V\beta} \log \mathrm{Tr}\, \mathrm{e}^{-\beta(\mathsf{H}-h\mathsf{Q})}$$

Thanks to asymptotic freedom, this observable can be computed in perturbation theory when $h \gg \Lambda$ and leads to a perturbative series in the running coupling constant *g=g(h)* at the scale set by *h:*

$$\mathcal{F}(h) \sim \sum_n a_n g^n$$

Exact solution

This free energy can be also calculated by using the exact S-matrix and the Bethe ansatz [Polyakov-Wiegmann]. The Fermi density of Bethe roots satisfies an integral equation

$$\epsilon(\theta) - \int_{-B}^{B} \mathrm{d}\theta' \, K(\theta - \theta')\epsilon(\theta') = h - \underset{\substack{\uparrow \\ \text{mass gap}}}{m} \cosh\theta$$

$$\epsilon(\pm B) = 0$$

The kernel can be computed from the *S*-matrix, plus an ansatz on the types of particles which appear by turning on *h*. Then, one has

$$\mathcal{F}(h) = -\frac{m}{2\pi} \int_{-B}^{B} \mathrm{d}\theta \, \cosh\theta \epsilon(\theta)$$

How do you detect renormalons in *F(h)*?

Thanks to the work of D. Volin, it is now possible to extract very long perturbative series for *F(h)* directly from the Bethe ansatz. This leads to numerical results on renormalons [M.M.-Reis, Bajnok-Balog-Hegedus et al.] which confirm a first correction

$$\mathrm{e}^{-\frac{1}{|\beta_0|g}} \sim \left(\frac{\Lambda}{h}\right)^2$$

However, higher order exponential corrections are harder to obtain in this way

One can also work at large *N*, either diagramatically [M.M.-Miravitllas-Reis] or in the Bethe ansatz [Fateev-Kazakov-Wiegmann, Zarembo, Di Pietro-M.M.-Sberveglieri-Serone]

Analytic results at finite *N*

However, we can do better [M.M.-Miravitallas-Reis] : we analyze the integral equations with the Wiener-Hopf method, as in old work [Wiegmann, Hasenfratz, Niedermayer, Balog, Weisz…], but incorporating exponentially small corrections that were previously neglected (generalizing work of [Al. B. Zamolodchikov] on sine-Gordon).

This leads to fully **analytic results** for the trans-series. For the *O(3)* sigma model we find for example

$$-\frac{1}{h^2}\mathcal{F}(h) \sim -\frac{1}{h^2}\mathcal{F}_{\text{pert}}(h) + \frac{16\pi^2}{\mathrm{e}^2 g^4}\mathrm{e}^{-\frac{1}{\beta_0 g^2}}\left(\pm\mathrm{i} - \frac{8}{g^2} + \cdots\right)$$

This result has been confirmed and extended by the Budapest group [Bajnok-Balog-Hegedus-Vona]

This new approach gives **all** exponentially small corrections. Perhaps the most striking consequence is the existence of corrections which are **not** of the form predicted by Parisi and 't Hooft:

$$\exp\left(-\frac{d}{(1-\mathfrak{r})|\beta_0|g}\right) \qquad d = 1, 2, \cdots$$

$\mathfrak{r} = \dfrac{2}{N-2}$ Gross-Neveu $\qquad$ $\mathfrak{r} = \dfrac{1}{N}$ PCF

The Parisi-'t Hooft conjecture is recovered in the large *N* limit, but fails at finite *N*

We obtain as well **instanton-like corrections**. For example, in the O(*N*) non-linear sigma model we find, on top of the leading renormalon singularity, corrections of the form

$$\exp\left(-\frac{d(N-2)}{|\beta_0|g}\right) \qquad d = 1, 2, \cdots$$

One can verify that the corrections obtained with this method control the large order behavior of the perturbative series, as expected from resurgence. This is a strong check that the result is correct.

In addition, all available evidence indicates that exact results from integrability can be recovered by resumming the resulting trans-series, in agreement with the resurgence conjecture.

These are very likely the only available analytic results on exponentially small corrections at finite N in non-SUSY QFTs.

Conclusions

Conventional perturbation theory should be replaced by "resurgent perturbation theory", in which perturbative series are replaced by trans-series. This framework seems to be powerful enough to include realistic QFTs.

We **do not really know** how to compute exponential corrections to perturbative series in QFT, in general. Instanton calculus was developed to supply this, but it is difficult and subtle, and some times it breaks down due to IR problems.

Renormalons are even more subtle since they are not described by saddles of the path integral. There is no "renormalon calculus" for $F(h)$, for example.

Integrability makes a difference. We have now explicit and analytic results for exponentially small corrections in asymptotically free theories (and in some integrable condensed matter models). They vindicate the renormalon picture…

… but they also make it more complicated: the Parisi-'t Hooft conjecture turns out to be a large N approximation. Is there a more general conjecture describing all exponentially small corrections?

Most important open problem: the calculation from first principles of trans-series associated to renormalons. These are largely unexplored new sectors of physical theories! The results from integrability give a signpost for any future theory of renormalon calculus.

Thank you for your attention