Yangian Symmetry, Fishnet Integrals and Geometry

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Based On

[to be published] with C. Duhr, A. Klemm, C. Nega, F. Porkert

2112.06928 with L. Corcoran, J. Miczajka

2012.07852 with L. Corcoran, J. Miczajka, M. Staudacher

2008.11739 with J. Miczajka

2005.01735, 2010.08552 with J. Miczajka, D. Müller, H. Münkler

1912.05561 with D. Müller, H. Münkler

1704.01967, 1708.00007 with D. Chicherin, V. Kazakov, D. Müller, D. Zhong

Truth vs Beauty

"My work always tried to unite the true with the beautiful, but when I had to choose one or the other, I usually chose the beautiful." — Hermann Weyl

Picture ETH Zürich - CC

From a particle physics perspective, bringing integrability to 4D quantum field theory, the AdS/CFT duality has certainly reduced the gap between truth and beauty.

Phenomenology vs Integrability?

Where can integrability help particle physics?

Feynman integrals are still a bottle neck for phenomenological predictions

Picture John Kannenberg - CC

- Lower loops/legs: Multiple polylogarithms (under good control)

$$G_{a_1,\ldots,a_n}(z) = \int_0^z \frac{\mathrm{d}t}{t-a_1} G_{a_2,\ldots,a_n}(t), \qquad G_\emptyset = 1,$$

- Generically: elliptic integrals and worse (Calabi–Yau geometries) with roots of higher order polynomials, e.g.

$$\int \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-zt^2)}}$$

- Currently very active field, see e.g. recent review 2203.07088 by

[Bourjaily, Broedel, Chaubey, Duhr, Frellesvig, Hidding, Marzucca, McLeod
Spradlin, Tancredi, Vergu, Volk, Volovich, von Hippel, Weinzierl, Wilhelm, Zhang]

This Talk: Integrability for Feynman Integrals

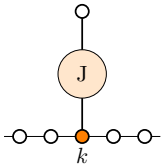
1. Brief intro on Yangian symmetry
2. Origin of Yangian symmetry for massless integrals from AdS/CFT
3. Generalization to massive Feynman integrals
4. Application to fishnet integrals and connection to geometry

Integrability and the Yangian

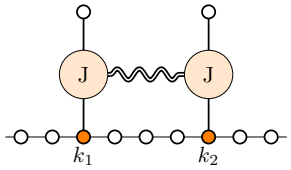
The Yangian is an infinite dimensional extension of a Lie algebra $\mathfrak{g}$.

Yangian algebra $Y[\mathfrak{g}]$ (first realization): [Drinfeld 1985]

Level 0 : $$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}^a_k =$$



Level 1 : $$\widehat{\mathrm{J}}^a = f^a{}_{bc} \sum_{k_1<k_2=1}^{n} \mathrm{J}^c_{k_1}\, \mathrm{J}^b_{k_2} =$$

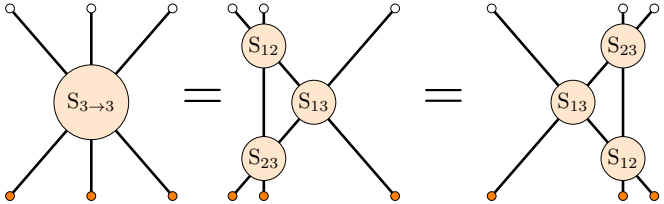


Serre relations: $$[\widehat{\mathrm{J}}_a, [\widehat{\mathrm{J}}_b, \mathrm{J}_c]] - [\mathrm{J}_a, [\widehat{\mathrm{J}}_b, \widehat{\mathrm{J}}_c]] = \mathcal{O}(\mathrm{J}^3).$$

Examples:

- Heisenberg chain: $\mathfrak{g} = \mathfrak{su}(2)$
- AdS/CFT: $\mathfrak{g} = \mathfrak{psu}(2,2|4)$

Consequence in 2D: Factorized Scattering

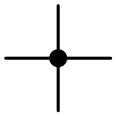
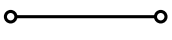


(Yang–Baxter equation)

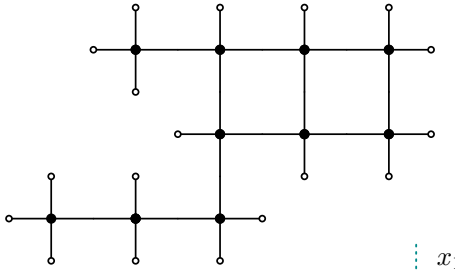
From AdS/CFT to Feynman Graphs

Massless Fishnet Feynman Graphs

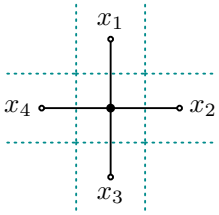
Feynman graphs made from scalar four-point vertices in 4D ($x^2 = x^\mu x_\mu$):

- Vertex  : $\int \mathrm{d}^4 x$
- Propagator j  k : $\frac{1}{x_{jk}^2} \equiv \frac{1}{(x_j - x_k)^2}$

e.g.



$p_j^\mu = x_j^\mu - x_{j+1}^\mu$

Mostly unsolved, cross under control [Ussyukina '93, Davydychev]: 

$= \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$

Remarkable properties:

- conformal symmetry (e.g. momentum $\mathrm{P}^\mu = -i \sum_j \partial_{x_j}^\mu$)
- finite integrals (generic kinematics)
- Related to AdS/CFT integrability ...

Feynman Graphs as Fishnet Correlators

Double-scaling limit:

$\mathcal{N}=4$ SYM		γ-Deformation		Fishnets
$\mathcal{L}_{\mathcal{N}=4}$	$\xrightarrow{X\,Y\to e^{i\gamma_j(\cdots)}XY}$	$\mathcal{L}^{\gamma}_{\mathcal{N}=4}$	$\xrightarrow[\xi=ge^{-i\gamma_3/2}\text{ fix}]{g\to 0,\ \gamma_3\to i\infty}$	$\mathcal{L}_{\text{bi}}$

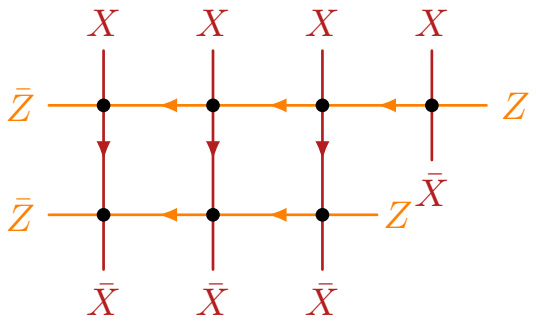
Resulting non-unitary bi-scalar fishnet theory: [Gürdoğan Kazakov 2015]

$$\mathcal{L}_{\text{bi}} = N_{\text{c}}\,\text{tr}\big(-\partial_\mu\bar{X}\partial^\mu X - \partial_\mu\bar{Z}\partial^\mu Z + \xi^2\bar{X}\bar{Z}XZ\big)$$

- Correlators given by single fishnet Feynman graphs.
- Fishnet integrals inherit a conformal Yangian symmetry $Y[\mathfrak{so}(1,5)]$:

differential operator

$\widehat{\mathrm{J}}^a$



$= 0.$

[Chicherin, Kazakov, FL Müller, Zhong 2017]

What does this imply?

Yangian Constraints on Feynman Integrals

Level 0:

$$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}^a_k \qquad \text{with} \qquad \mathrm{J}^a \in \begin{cases} \mathrm{D} = -ix_\mu \partial^\mu - i\Delta, \\ \mathrm{L}_{\mu\nu} = ix_\mu \partial_\nu - ix_\nu \partial_\mu, \\ \mathrm{P}_\mu = -i\partial_\mu, \\ \mathrm{K}_\mu = ix^2 \partial_\mu - 2ix_\mu x^\nu \partial_\nu - 2i\Delta x_\mu. \end{cases}$$

$\Rightarrow I_n = V_n\, \phi$ with $\phi(z_1, z_2, \dots)$ function of conformal cross ratios.

Level 1: additional non-local generators $\widehat{\mathrm{J}}^a = f^a{}_{bc} \sum_{j<k} \mathrm{J}^c_j\, \mathrm{J}^b_k$ e.g.

$$\widehat{\mathrm{P}}^\mu = \sum_{j<k=1}^{n} \left[\left(\mathrm{L}^{\mu\nu}_j + \eta^{\mu\nu} \mathrm{D}_j \right) \mathrm{P}_{k,\nu} - (j \leftrightarrow k) \right]$$

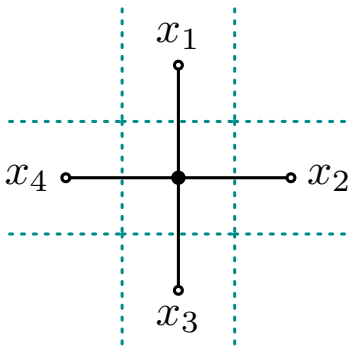
Yangian invariance: $$0 = \widehat{\mathrm{P}}^\mu I_n = V_n \sum_{j<k=1}^{n} \frac{x^\mu_{jk}}{x^2_{jk}} \mathrm{PDE}_{jk} \phi$$

Leads to **system of Yangian PDEs** in the cross ratios:

$$\mathrm{PDE}_{jk}\, \phi = 0, \qquad 1 \le j < k \le n.$$

Example: Yangian Bootstrap for Box Integral

Consider Euclidean cross (or box) integral in 4D:



$$= \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} = \int \frac{\mathrm{d}^4 \ell}{\ell^2 (\ell + p_1)^2 (\ell + p_1 + p_2)^2 (\ell - p_4)^2}$$

- x_j^μ interpreted as positions or dual momenta via $p_j^\mu = x_j^\mu - x_{j+1}^\mu$
- (Dual) conformal symmetry implies

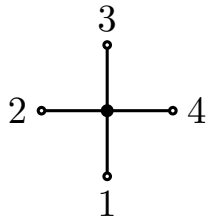
$$I_4 = \frac{1}{x_{13}^2 x_{24}^2} \phi(z, \bar{z})$$

with

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

Example: Yangian Bootstrap for Box Integral

Yangian symmetry implies differential equations:



$$[D_j(z) - D_j(\bar{z})]\phi(z,\bar{z}) = 0, \qquad j = 1,2,$$

with
$$\begin{cases} D_1(z) = z(z-1)^2\partial_z^2 + (3z-1)(z-1)\partial_z + z, \\ D_2(z) = z^2(z-1)\partial_z^2 + (3z-2)z\partial_z + z. \end{cases}$$

Find four solutions $f_j(z,\bar{z})/(z-\bar{z})$ by elementary methods:

$$\begin{aligned} f_1 &= 1, \\ f_2 &= \log(\bar{z}) - \log(z), \\ f_3 &= \log(1-\bar{z}) - \log(1-z), \\ f_4 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log\tfrac{1-z}{1-\bar{z}}\log(\bar{z}z). \end{aligned}$$

Permutation symmetry singles out Bloch-Wigner dilogarithm. [Usyukina Davydychev '93]

- Euclidean box integral fixed by symmetries [FL, Müller Münkler 2019]
- Further solutions required in Minkowski space [Corcoran, FL, Miczajka Staudacher 2020]

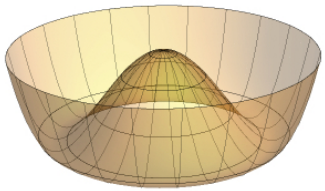
Yangian Symmetry for the Masses

Massive Generalization?

In the massless case, $\mathcal{N}=4$ SYM theory was the starting point:

$\mathcal{N}=4$ SYM $\rightarrow$ fishnet theory $\rightarrow$ Feynman graphs

- Introduce masses via VEV: $\hat{\Phi} = \langle \Phi \rangle + \Phi$



- Leads to massive propagators

$$1/x_{jk}^2 \quad \rightarrow \quad 1/\hat{x}_{jk}^2 = 1/(x_{jk}^2 + (m_j - m_k)^2)$$

- Well known massive dual conformal symmetry: [Alday, Henn 2010 / Plefka, Schuster]

$$\begin{aligned}
\mathrm{P}_j^\mu &= -i\partial_{x_j}^\mu, \qquad\qquad \mathrm{L}_j^{\mu\nu} = i x_j^\mu \partial_{x_j}^\nu - i x_j^\nu \partial_{x_j}^\mu, \\
\mathrm{D}_j &= -i\big(x_{j\mu}\partial_{x_j}^\mu + m_j \partial_{m_j} + \Delta_j\big), \\
\mathrm{K}_j^\mu &= -2i x_j^\mu \big(x_{j\nu}\partial_{x_j}^\nu + m_j \partial_{m_j} + \Delta_j\big) + i(x_j^2 + m_j^2)\partial_{x_j}^\mu .
\end{aligned}$$

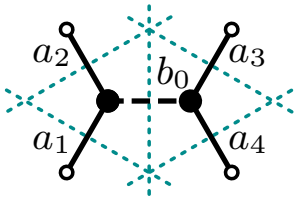
- Mass interpreted as $(D+1)$th dimension: $x^{D+1} = m_j$.

. . . but no massive Yangian symmetry (integrability) known.

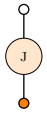
Feynman Integrals with Massive Propagators

Consider massive Feynman integrals directly, e.g.

$p_j^\mu = x_j^\mu - x_{j+1}^\mu$

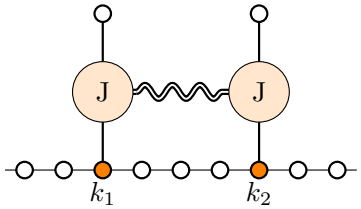


$$= \int \frac{\mathrm{d}^D x_0 \mathrm{d}^D x_{\bar{0}}}{\hat{x}_{01}^{2a_1} \hat{x}_{02}^{2a_2} x_{0\bar{0}}^{2b_0} \hat{x}_{\bar{0}3}^{2a_3} \hat{x}_{\bar{0}4}^{2a_4}},$$

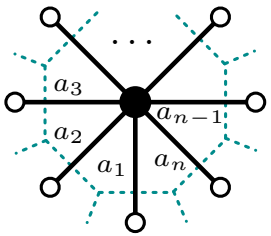
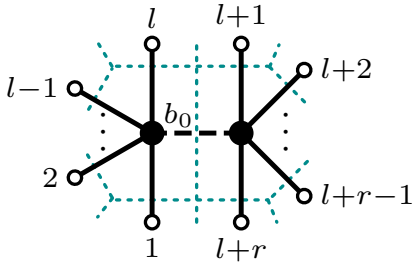
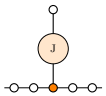
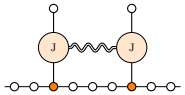
- Level Zero: Known massive dual conformal generators $\mathrm{J}^a =$ 

- Leve One: Build Yangian generators:

$$\widehat{\mathrm{J}}^a = \tfrac{1}{2} f^a{}_{bc} \sum_{k_1 < k_2 = 1}^{n} \mathrm{J}^c_{k_1} \mathrm{J}^b_{k_2} =$$



Massive Yangian at 1 and 2 Loops

	1 loop	2 loops
	$I_n =$ 	$I_{lr} =$ 
Level 0 	✓	✓
Level 1 	✓	✓

- Symmetries hold if dual conformal condition satisfied: [FL, Miczajka Müller, Münkler 2020]

$$\sum_{k \in \text{vertex}} a_k = D.$$

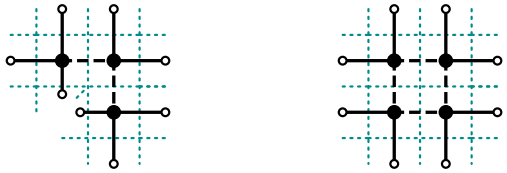
Conjecture at higher loops:
Graphs cut from regular tilings of the plane with massive propagators on the boundary have Yangian symmetry.

Massive Fishnet Theory

- Massive fishnet theory from Coulomb-branch $\mathcal{N}=4$ SYM: [FL, Miczajka 2020]

$$\mathcal{L}_{\mathsf{MF}} = N_{\mathsf{c}}\,\mathrm{tr}\big(-\partial_\mu \bar{X}\partial^\mu X - \partial_\mu \bar{Z}\partial^\mu Z + \xi^2 \bar{X}\bar{Z}XZ\big) - N_{\mathsf{c}}(m_a - m_b)^2 X^a{}_b \bar{X}^b_a - N_{\mathsf{c}}(m_a - m_b)^2 Z^a{}_b \bar{Z}^b_a.$$

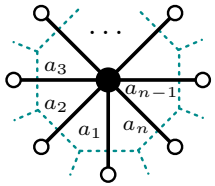
- Differs from spontaneously broken fishnet theory of [Karananas, Kazakov Shaposhnikov 2019]
- Planar amplitudes in 1-to-1 correspondence with Yangian-invariant integrals, e.g.:



First finding of quantum integrability in massive QFT in 4D

Example: Massive Yangian Bootstrap

Consider one-loop integrals with massive propagators for $D = \sum_j a_j$

$$I_n =$$ 

$$= V_n\, \phi(u_{ij})$$

Note: Recently pointed out that I_n equal Witten contact diagrams in AdS which thus have the same Yangian symmetry [Rigatos Zhou 2022]:

Yangian Symmetry in Holographic Correlators

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Department of Physics and Center for Theory of Quantum Matter, 390 UCB University of Colorado Boulder, CO 80309, USA

Xinan Zhou (周稀楠)
Kavli Institute for Theoretical Sciences, University of Chinese Academy of Sciences, Beijing 100190, China.

We point out that an infinite class of Witten diagrams is invariant under a Yangian symmetry. These diagrams are building blocks of holographic correlators and are related by a web of differential recursion relations. We show that Yangian invariance is equivalent to the consistency conditions of the recursion relations.

Level 0: Dependence on conformal variables $u_{ij} = \frac{x_{ij}^2 + (m_i - m_j)^2}{-4 m_i m_j}$

Level 1: Yangian PDEs for conformal function $\phi(u_{ij})$

Example: Massive Yangian Bootstrap

Evaluate Yangian constraints:

$n = 2$: fixed by symmetries

Gauß' hypergeometric function

$$I_2 = \quad = \frac{\pi^{D/2}\Gamma_{D/2}}{\Gamma_D m_1^{a_1} m_2^{a_2}}\, {}_2F_1\left[\begin{smallmatrix} a_1, a_2 \\ (D+1)/2 \end{smallmatrix}; u\right].$$

$n = 3$: fixed by symmetries

Srivastava's hypergeometric function

$$I_3 = \quad = \frac{\pi^{D/2}\Gamma_{D/2}}{\Gamma_D m_1^{a_1} m_2^{a_2} m_3^{a_3}} H_C(u, v, w).$$

$n =$generic: natural conjecture [FL, Miczajka Müller,Münkler 2020]

$$I_n = \quad = \frac{\pi^{D/2}\Gamma_{D/2}}{\Gamma_D \prod_{j=1}^n m_j^{a_j}} \sum_{k_{ij}=0}^{\infty} \frac{\prod_{j=1}^n (a_j)_{\sum_{\alpha\in B_{n|j}} k_\alpha}}{\left(\frac{D+1}{2}\right)_{\sum_{\alpha\in B_n} k_\alpha}} \prod_{\alpha\in B_n} \frac{u_\alpha^{k_\alpha}}{k_\alpha!}.$$

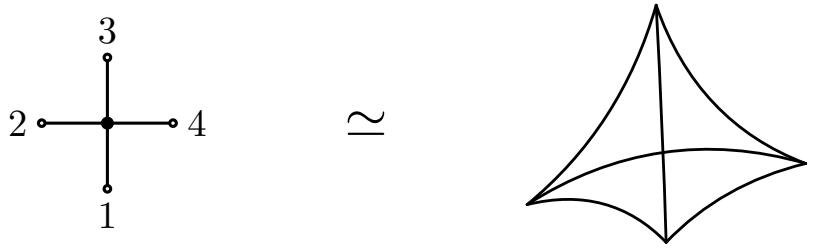
confirmed in [Ananthanarayan, Banik Friot, Ghosh 2020]

Feynman Integrals and Geometry

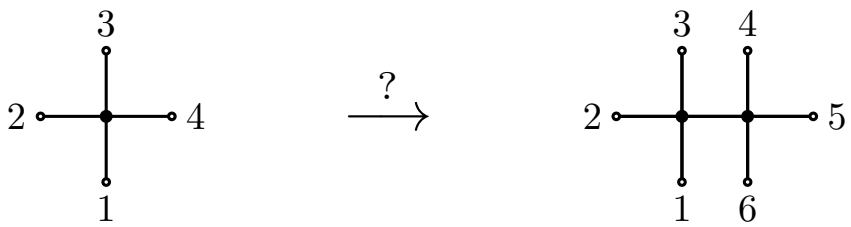
Above one-loop integrals relate to volumes of simplices with angles
[Davydychev Delbourgo 1997] [Mason Skinner 2010] [Schnetz 2010] [Bourjaily, Gardi McLeod, Vergu 2019]

$$\varphi_{jk} = \frac{m_j^2 + m_k^2 + x_{jk}^2}{2 m_j m_k}$$

E.g. in the massless limit, box integral in 4D computes volume of ideal hyperbolic tetrahedron:



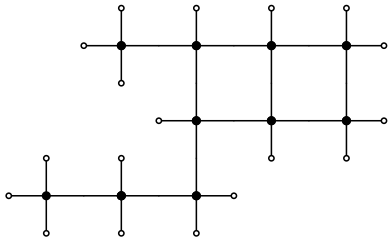
Similar geometric interpretation for integrals at higher loops?



Geometry of Fishnet Integrals

Conformal Fishnets in 1D and 2D

Fishnet integrals in lower dimensions with conformal choice of powers in propagators $|x|^{-2a_j}$: [Duhr, Klemm, FL Nega, Porkert, in progress]

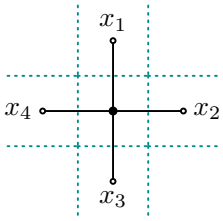


$$\text{1D}: a_j = \tfrac{1}{4}, \qquad \text{2D}: a_j = \tfrac{1}{2}, \qquad \sum_{j=1}^{4} a_j = D.$$

Integrals are correlators in D-dimensional fishnet theory: [Kazakov Olivucci 2018]

$$\mathcal{L} = N_c \,\mathrm{tr}\Big[X(-\partial_\mu\partial^\mu)^{\frac{D}{4}}\bar{X} + Z(-\partial_\mu\partial^\mu)^{\frac{D}{4}}\bar{Z} + \xi^2 X Z \bar{X}\bar{Z}\Big]$$

Simplest example: Box integral in 1D (here $x_1 < x_2 < x_3 < x_4$):

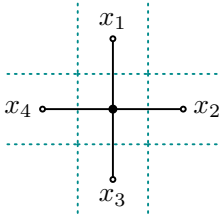
$I_4^{\text{1D}} =$ 

$$= \frac{4}{\sqrt{x_{13}x_{24}}}\big[K(z) + K(1-z)\big], \qquad z = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

with elliptic K integral: $K(z) = \int_0^{\pi/2} \mathrm{d}\theta \frac{1}{\sqrt{1 - z\sin^2\theta}}$.

1D Box from Yangian Bootstrap

Consider Yangian over 1D conformal algebra $Y[\mathfrak{sl}(2,\mathbb{R})]$ on one-loop box:

$I_4 =$ 

$= \dfrac{1}{\sqrt{x_{13}x_{24}}}\phi(z)$

Yangian differential equation (= Legendre equation):

$$0 = \phi(z) + 4(2z-1)\phi'(z) + 4(z-1)z\phi''(z), \qquad z = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

Two solutions:

1 power series	1 single-log solution
$K(z) = \sum_j c_j z^j$	$K(1-z) = \log(z)\sum_j c_j z^j + \dots$

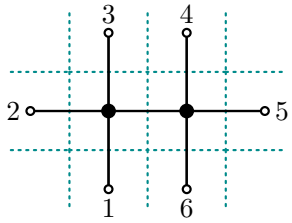
For $\vec{\Pi} = (K(z), K(1-z))$ integral must be given by

$$\phi(z) = \vec{v}\cdot\vec{\Pi}.$$

Fix linear combination using e.g. numerics to find $\vec{v} = (4,4)$.

1D Double Box from Yangian Bootstrap

Two loops:

$I_6 =$ 

$= \frac{1}{\sqrt{x_{14}x_{26}x_{35}}}\phi(z_1, z_2, z_3)$

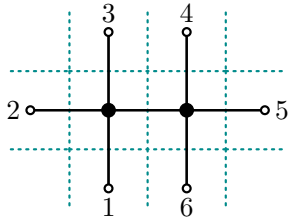
Set of PDEs generated by Yangian symmetry $\widehat{\mathrm{P}} I_6 = 0$, e.g.

$$
\begin{aligned}
0 =& z_2\phi + 2\left(z_2 - 1\right)\left(z_1 z_2\left(z_3 - 1\right) + 1\right) z_2^2 \phi^{(0,2,0)} \\
& - \left(z_3 - 1\right)\left(5z_3 + z_1 z_2\left(3z_3^2 - 5z_3 + 2\right) - 2\right) z_2 \phi^{(0,0,1)} \\
& - 2\left(z_1 z_2\left(z_3 - 1\right) + 1\right)\left(z_3 - 1\right)^2 z_3 z_2 \phi^{(0,0,2)} \\
& + \left(3z_1\left(z_3 - 1\right) z_2^2 + \left(5 - z_1\left(z_3 - 1\right)\right) z_2 - 3\right) z_2 \phi^{(0,1,0)} \\
& + \left(z_2\left(z_3 - 1\right) z_1^2 + 3z_1 - 2\right) \phi^{(1,0,0)} \\
& + 2\left(z_1 - 1\right) z_1\left(z_1 z_2\left(z_3 - 1\right) + 1\right) \phi^{(2,0,0)}
\end{aligned}
$$

Note: In 1D, $\widehat{\mathrm{P}}$ yields less PDEs than in higher dimensions, cf. [FL, Müller Münkler 2019]

1D Double Box from Yangian Bootstrap

Two loops:

$I_6 =$ 

$= \frac{1}{\sqrt{x_{14}x_{26}x_{35}}}\phi(z_1, z_2, z_3)$

New trick: Get additional PDEs from permutations σ

$$\widehat{\mathrm{P}} I_6 = 0, \qquad (\sigma \circ \widehat{\mathrm{P}}) I_6 = 0$$

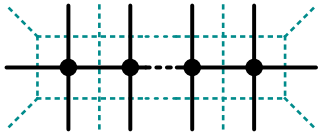
Frobenius Method: Ansatz yields 5-dimensional solution vector $\vec{\Pi}$

1 power series	3 single-log	1 double-log solution
$\sum_{jkl} c_{jkl} z_1^j z_2^k z_3^l$	$\log(z_a) \sum_{jkl} c_{jkl} z_1^j z_2^k z_3^l$	$\log(z_a)\log(z_b) \sum_{jkl} c_{jkl} z_1^j z_2^k z_3^l$

Fix linear combination e.g. by using numerics:

$$\phi(z_1, z_2, z_3) = \vec{v} \cdot \vec{\Pi}.$$

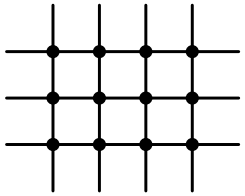
General Fishnets in 1D

- Train track integrals  [Duhr, Klemm, FL Nega, Porkert, in progress]

  are Calabi–Yau ℓ-folds with known solution pattern:

Loops	1	2	3	...
Geometry	Elliptic Curve	K3 surface	CY 3-fold	...
Solutions	1|1	1|3|1	1|5|5|1	...

- Cf. train tracks in 4D: Calabi–Yau $(\ell - 1)$-folds [Bourjaily, He, Mcleod Von Hippel, Wilhelm, 2018]

- Generic fishnets in 1D/2D have Calabi–Yau structure! 

Relation to Integrability:

Yangian+permutations generates Picard–Fuchs ideal of differential operators with Calabi–Yau periods as solutions!

From 1 to 2 Dimensions

Split 2D Yangian into holomorphic and anti-holomorphic part:

$$Y[\mathfrak{sl}(2,\mathbb{R})] \oplus \overline{Y[\mathfrak{sl}(2,\mathbb{R})]}$$

Double Copy Structure: Use same Yangian invariants $\vec{\Pi}$ as in 1D:

$$\phi(z) = \vec{\Pi}^T(z) \cdot \Sigma \cdot \vec{\Pi}(\bar{z})$$

Indeed: Box integral in 2D given by linear combination of two factorized Yangian invariants [Derkachov, Kazakov Olivucci 2018] [Corcoran, FL Miczajka 2021]:

$$\begin{aligned}\phi(z,\bar{z}) &= 4\big[K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})\big] \\ &= \begin{pmatrix} K(z) & K(1-z) \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} K(\bar{z}) \\ K(1-\bar{z}) \end{pmatrix}\end{aligned}$$

What is the role of the matrix Σ?

Kähler Potential and Calabi–Yau Volume

Fishnet integral should be single-valued since it represents a correlator in the fishnet theory.

Natural single-valued function for Calabi–Yau is the Kähler potential V. Indeed we observe [Duhr, Klemm, FL / Nega, Porkert, in progress]

$$\phi(z) = \vec{\Pi}^T \cdot \Sigma \cdot \vec{\Pi} = e^{-V} = \text{Volume}(\text{CY}_\ell)$$

with Σ the Calabi–Yau intersection matrix.

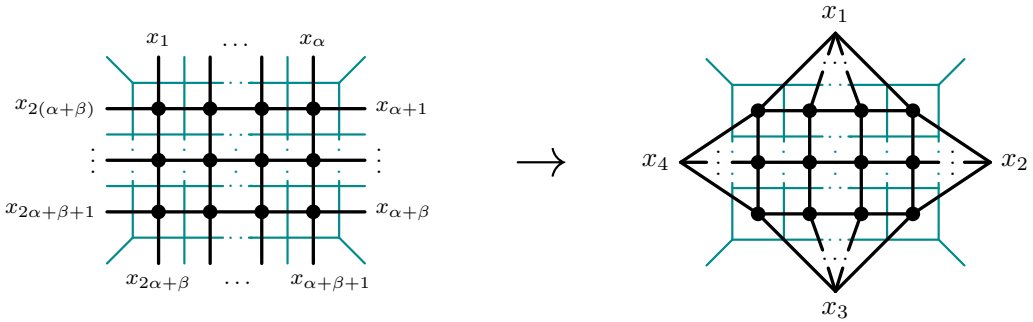
This structure persists for higher loop integrals!

Relation to Geometry:

Fishnet integrals compute volumes of Calabi–Yau ℓ-folds!

Connection with 2D Basso–Dixon Formula

Basso–Dixon found determinant representation for 4D integrals in four-point coincidence limit of $\alpha \times \beta$ fishnet [Basso Dixon 2017]



2D generalization of [Derkachov, Kazakov Olivucci 2018] has double copy form :

explicit check up to 5-loop ladder integrals

$$\phi_{\alpha\beta} \simeq \det_{1\leq j,k\leq\beta}\left[(z\partial_z)^{j-1}(\bar{z}\partial_{\bar{z}})^{k-1}\partial_\varepsilon^{\alpha+\beta-1}\big|_{\alpha+\beta+1}F_{\alpha+\beta}(\varepsilon,z)\big|^2\right]_{\varepsilon=0} \simeq \vec{\Pi}^T\cdot\Sigma\cdot\vec{\Pi}$$

Yangian induces dimension-shift relations on BD integrals [Corcoran, FL Miczajka 2021]

$$D_{uv}\,\phi^D_{\alpha\beta} \simeq \phi^{D+2}_{\alpha\beta}(a_j \to a_j+1).$$

Extension of BD-Formula to graphs with shifted dimensions/powers? Cf.

$$\frac{\partial}{\partial z}\,{}_2F_1(a,b,c|z) = \frac{ab}{c}\,{}_2F_1(a+1,b+1,c+1|z)$$

Conclusions

Integrability

- is present in 4D QFT beyond planar $\mathcal{N} = 4$ SYM theory
- extends beyond the massless phase
- relates to theory of Calabi–Yau manifolds

Geometry

- Feynman integrals compute volumes beyond one loop and beyond polytopes
- Geometry used to compute Feynman integrals via $\vec{\Pi} \cdot \Sigma \cdot \vec{\Pi}$
- Yangian generates Picard–Fuchs ideal of differential operators

General Lesson

- 1D/2D fishnets provide instructive models for involved function classes in the theory of Feynman integrals

Outlook

Integrability for Feynman Integrals

- Generalize findings on 2D fishnets to massive situation
- Apply insights to Yangian approach for 4D fishnets, cf. [FL, Müller Münkler 2019]
- Use higher symmetries in dimensional regularization, cf. [FL, Plefka Shi, Wang 2020]
- Extend to other fishnet structures (triangular/hexagonal)

Massive Integrability

- Chaos vs integrability for massive fishnet theory [McLoughlin, Pereira Spiering 2020]
- Massive analogue of dual fish chain? [Gromov Sever 2019]

Mathematics

- Yangian PDEs vs Picard–Fuchs ideal for Calabi–Yau periods
- Find more integrals that compute volumes of geometric objects

Further Extensions

- Correlators in AdS? See recent relation to Witten diagrams [Rigatos Zhou 2022]

Extra Slides

Proof with Lasso Method

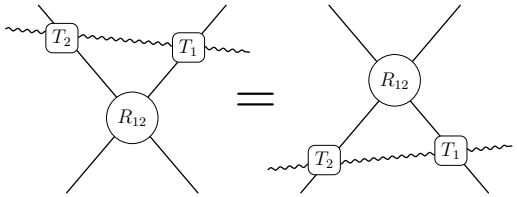
Package Yangian generators in monodromy: [Chicherin, Kazakov, FL Müller, Zhong 2017]

$$T(u) \simeq \mathbb{1} + \frac{1}{u}\mathrm{J} + \frac{1}{u^2}\widehat{\mathrm{J}} + \ldots,$$

Yangian algebra encoded in "RTT-relations"

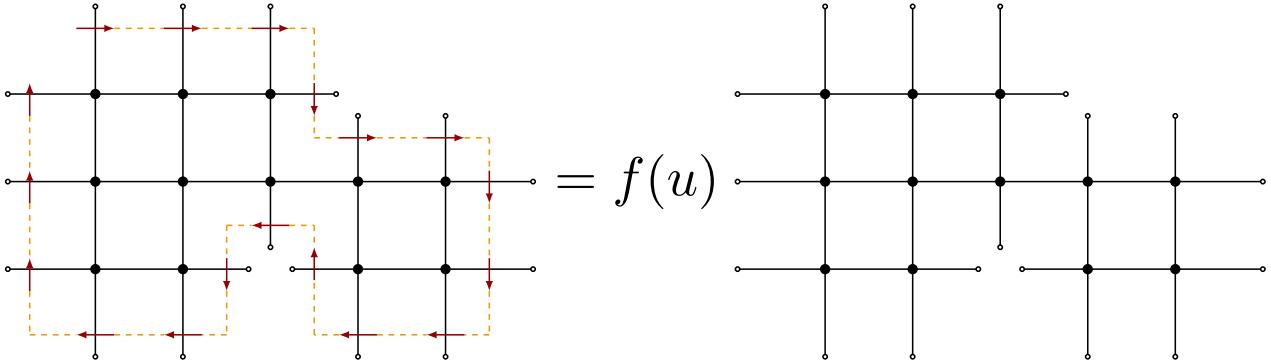
$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

with Yang's R-matrix $R_{12}(u-v) = \mathbb{1}_{12} + u\,\mathbb{P}_{12}$.



Yangian invariance maps to eigenvalue equation for the invariant I_n:

$$T(u)I_n = f(u)I_n \cdot \mathbb{1}, \qquad T(u) \simeq f(u)\big(\mathbb{1} + \frac{1}{u}\mathrm{J} + \frac{1}{u^2}\widehat{\mathrm{J}} + \ldots.$$



Evaluation Parameters

Parameters s_j enter definition of level-one generators:

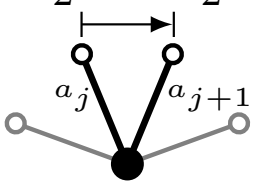
$$\widehat{\mathrm{J}}^a = \tfrac{1}{2} f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}^c_j \mathrm{J}^b_k + \sum_{j=1}^{n} s_j \mathrm{J}^a_j ,$$

Prescription:

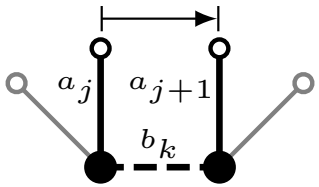
[FL, Miczajka Müller, Münkler 2020]

- choose (arbitrary) leg 1 and set $s_1 = 0$,
- get s_{j+1} from s_j by adding e.g.

$$-\frac{a_j}{2} - \frac{a_{j+1}}{2}$$



$$-\frac{a_j}{2} - \frac{a_{j+1}}{2} - b_k + \frac{D}{2}$$



Momentum Space Symmetry

Note: Generator $\widehat{\mathrm{P}}$ is symmetry even if level-zero symmetry is broken.

Translate level-one momentum generators in x-space into dual momentum space ($p_j^\mu = x_j^\mu - x_{j+1}^\mu$):

$$\widehat{\mathrm{P}}^\mu \to \bar{\mathrm{K}}^\mu$$

Massive generalization of momentum space symmetry: [FL, Miczajka '20 / Müller, Münkler]

$$\boxed{\begin{aligned}
\bar{\mathrm{P}}_j^\mu &= p_j^\mu\,, \qquad \bar{\mathrm{L}}_j^{\mu\nu} = p_j^\mu \partial_{p_j}^\nu - p_j^\nu \partial_{p_j}^\mu\,,\\
\bar{\mathrm{D}}_j &= p_{j\nu}\partial_{p_j}^\nu + \frac{m_j\partial_{m_j} + m_{j+1}\partial_{m_{j+1}}}{2} + \bar{\Delta}_j\,,\\
\bar{\mathrm{K}}_j^\mu &= p_j^\mu \partial_{p_j}^2 - 2\left[p_{j\nu}\partial_{p_j}^\nu + \frac{m_j\partial_{m_j} + m_{j+1}\partial_{m_{j+1}}}{2} + \bar{\Delta}_j\right]\partial_{p_j}^\mu\,.
\end{aligned}}$$

. . . in addition to known massive *dual* conformal symmetry [Alday, Henn 2010 / Plefka, Schuster]

Double-Scaling Limit of Coulomb-Branch $\mathcal{N} = 4$ SYM

$\mathcal{L}_{\text{Coul}} \xrightarrow{e^{i\gamma}} \mathcal{L}^{\gamma}_{\text{Coul}} \to$ double-scaling limit $\to$ massive Feynman graphs

Critical step: γ-deformation

- Phase operator only well-defined on R-symmetry singlets:

$$\mathcal{P}_\gamma = \exp\left(\tfrac{i}{2}\gamma_A \epsilon_{ABC}\, q^B \wedge q^C\right)$$

- Ad hoc solution: average first over ways to break up trace:

$$Q : \mathrm{tr}(\Phi_1\Phi_2\ldots\Phi_n) \mapsto \tfrac{1}{n}\left(\Phi_1\Phi_2\ldots\Phi_n + \Phi_2\Phi_3\ldots\Phi_1 + \ldots\right)$$

- and define deformation as

$$\mathcal{L}^{\gamma}_{\text{Coul}} = Q^{-1}\mathcal{P}_\gamma Q \mathcal{L}_{\text{Coul}}.$$

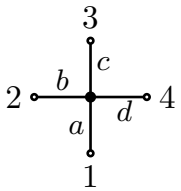
Then proceed with double-scaling limit as in massless case ...

Yangian Bootstrap

Example: Euclidean Box in D Dimensions

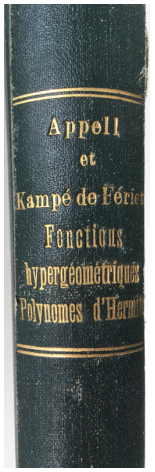
Deformed cross integral (simplest "fishnet"): [FL, Müller Münkler 2019]

$$\int \frac{\mathrm{d}^D x_0}{x_{10}^{2a} x_{20}^{2b} x_{30}^{2c} x_{40}^{2d}} =$$



$$a+b+c+d=D$$

Yangian constraints in $u = x_{12}^2 x_{34}^2 / x_{13}^2 x_{24}^2$ and $v = x_{14}^2 x_{23}^2 / x_{13}^2 x_{24}^2$:



$$0=\Big[\alpha\beta+(\alpha+\beta+1)u\partial_u+\big((\alpha+\beta+1)v-\gamma'\big)\partial_v+u^2\partial_u^2+(v-1)v\partial_v^2+2vu\partial_u\partial_v\Big]\phi(u,v),$$

$$0=\Big[\alpha\beta+(\alpha+\beta+1)v\partial_v+((\alpha+\beta+1)u-\gamma)\partial_u+v^2\partial_v^2+(u-1)u\partial_u^2+2vu\partial_v\partial_u\Big]\phi(u,v)\,.$$

with parameters

$$\alpha=b, \qquad \beta=\frac{D}{2}-d, \qquad \gamma=+\frac{D}{2}-c-d+1, \qquad \gamma'=-\frac{D}{2}+b+c+1.$$

Defining PDEs for Appell hypergeometric function F_4:

$$F_4\left[{}^{\alpha,\beta}_{\gamma,\gamma'};u,v\right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n(1)_m(1)_n}\, u^m v^n\,,$$

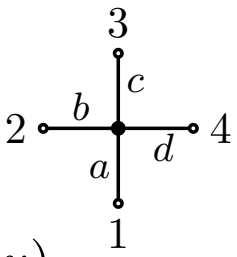
Pochhammer symbol: $(\lambda)_k = \Gamma(\lambda+k)/\Gamma(\lambda)$.

Permutation Symmetries

Algorithm yields four-dimensional solution space:

$$
\begin{aligned}
g_1 &= F_4\left[{}^{\alpha,\beta}_{\gamma,\gamma'};u,v\right],\\
g_2 &= u^{1-\gamma}F_4\left[{}^{\alpha+1-\gamma,\beta+1-\gamma}_{2-\gamma,\gamma'};u,v\right],\\
g_3 &= v^{1-\gamma'}F_4\left[{}^{\alpha+1-\gamma',\beta+1-\gamma'}_{\gamma,2-\gamma'};u,v\right],\\
g_4 &= u^{1-\gamma}v^{1-\gamma'}F_4\left[{}^{\alpha+2-\gamma-\gamma',\beta+2-\gamma-\gamma'}_{2-\gamma,2-\gamma'};u,v\right].
\end{aligned}
$$

Fix linear coefficients from permutation symmetries



$$\phi(u,v) = c_1\, g_1(u,v) + c_2\, g_2(u,v) + c_3\, g_3(u,v) + c_4\, g_4(u,v)$$

- Limit $a,b,c,d \to 1$ reproduces Bloch–Wigner Function.
- Three-point limit agrees with result of [Boos '90, Davydychev].

$\Rightarrow$ D-dimensional box integral fixed by symmetries! [FL, Müller, Münkler 2019]

Example: Minkowski Box

$$I_4 = \frac{1}{i\pi^2} \int_{\mathbb{R}^{1,3}} \frac{\mathrm{d}^4 x_0}{(x_{01}^2 + i\epsilon)(x_{02}^2 + i\epsilon)(x_{03}^2 + i\epsilon)(x_{04}^2 + i\epsilon)}$$

Remember: Yangian PDEs have four solutions: [FL, Müller Münkler 2019]

$$\begin{aligned}
f_1 &= 1, \\
f_2 &= \log(\bar{z}) - \log(z), \\
f_3 &= \log(1-\bar{z}) - \log(1-z), \\
f_4 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log\tfrac{1-z}{1-\bar{z}} \log(\bar{z}z).
\end{aligned}$$

Kinematic regions characterized by sign block:

$$\text{sign} \begin{bmatrix} x_{12}^2, x_{34}^2 \\ x_{23}^2, x_{14}^2 \\ x_{13}^2, x_{24}^2 \end{bmatrix}$$

Only f_4 needed in Euclidean region: $\phi\left(\begin{smallmatrix} - & - \\ - & - \\ - & - \end{smallmatrix}\right) = f_4(z, \bar{z})$.

How about rest of the 64 regions? And the remaining Yangian invariants?

Compact Formula for Minkowski Box

Discrete symmetries reduce $256 = 4 \times 64$ constants to 12 unkowns.

Remaining parameters fixed by analytic continuation to find [Corcoran, FL 2020 / Miczajka, Staudacher]

$$\begin{aligned}\phi(R) = &+ g_1 \\ &+ 2\pi i g_2 \big(+\theta^{--}_{--}{}^{-+}_{-+} + \theta^{-+}_{+-}{}^{+-}_{-+} - \theta^{++}_{++}{}^{-+}_{++} + \theta^{--}_{--}{}^{++}_{--} - \theta^{-+}_{++}{}^{-+}_{++} + \theta^{--}_{-+}{}^{++}_{-+}\big) \\ &+ 2\pi i g_3 \big(-\theta^{-+}_{--}{}^{--}_{--} - \theta^{++}_{-+}{}^{++}_{-+} + \theta^{--}_{++}{}^{--}_{++} - \theta^{++}_{--}{}^{++}_{--} - \theta^{++}_{-+}{}^{--}_{-+} + \theta^{--}_{++}{}^{++}_{++}\big) \\ &+ 2\pi i g_4 \big(-\theta^{++}_{-+}{}^{++}_{-+} - \theta^{-+}_{--}{}^{-+}_{--} - \theta^{++}_{--}{}^{++}_{--} + \theta^{--}_{++}{}^{--}_{++} - \theta^{-+}_{--}{}^{-+}_{--} - \theta^{++}_{--}{}^{++}_{--}\big).\end{aligned}$$

Here we use a slightly different basis of Yangian invariants:

$$\{f_1, f_2, f_3, f_4\} \quad \rightarrow \quad \{g_1, g_2, g_3, g_4\}$$

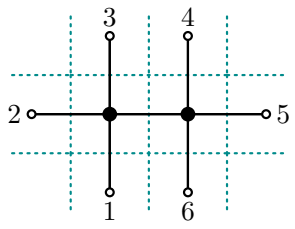
Theta-functions are only non-vanishing in specific regions R, e.g.

$$\theta^{++}_{++}{}^{++}_{++} = 1, \qquad \theta^{++}_{++}{}^{-+}_{++} = 0.$$

Example: Double Box

Yangian PDEs

[FL, Müller Münkler 2019]



$$D_I\,\phi(u_1,\ldots,u_9)=0,\qquad I\in\{A,B,C,D,E,F\},$$

with differential operators

$$\begin{aligned}
D_A=&-\theta_6^2+u_6(D_{168}+1)D_{365}+u_5u_6(D_{168}+1)(D_{2534}+1)-u_6u_8D_{365}D_{1928}\\
&+u_4u_5u_6D_{142}(D_{168}+1)+u_6u_8u_9D_{365}D_{392}-u_5u_6u_7u_8D_{1928}(D_{2534}+1),\\
D_B=&\theta_8(D_{168}+1)-u_8D_{1928}D_{5867}-u_7u_8D_{475}D_{1928}+u_8u_9D_{392}D_{5867},\\
D_C=&(\theta_1-\theta_9)D_{1928}-u_1D_{142}(D_{168}+1)+u_9D_{392}(D_{798}+1),\\
D_D=&-\theta_2D_{392}+u_2D_{1928}(D_{2534}+1)-u_1u_2(D_{168}+1)(D_{2534}+1)+u_2u_4D_{475}D_{1928},\\
D_E=&(\theta_3-1)\theta_3-u_3D_{365}D_{392}+u_2u_3D_{365}D_{1928}-u_3u_5D_{392}D_{5867}\\
&-u_1u_2u_3(D_{168}+1)D_{365}-u_3u_5u_7D_{392}(D_{798}+1)+u_2u_3u_4u_5D_{1928}D_{5867},\\
D_F=&(\theta_3-\theta_4)(D_{2534}+1)+u_3D_{365}D_{392}-u_4D_{142}D_{475}.
\end{aligned}$$

Here Euler operators $\theta_j=u_j\partial_{u_j}$ are packaged into

$$D_{ijk}=\theta_i+\theta_j-\theta_k,\qquad D_{ijkl}=\theta_i+\theta_j-\theta_k-\theta_l.$$

Big step from box (2 cross ratios) to double box (9 cross ratios)!

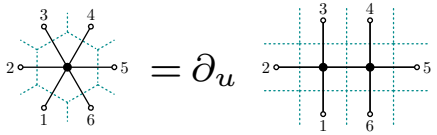
Double Box and Hexagon

Solution of Yangian constraints yields basis of 4834 series (double box) or 2530 series (hexagon) compared to 12 for the box integral:

$$H_k = \sum_{n_j \in x_j + \mathbb{Z}} h_{n_1,\ldots,n_9} w_1^{n_1} \ldots w_9^{n_9} .$$

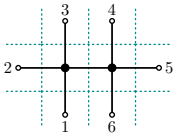
Mellin–Barnes approach yields linear combinations [Ananthanarayan, Banik Friot, Ghosh 2020]

Hexagon (26 Series):



$$\begin{aligned}\phi_6 = \frac{Q_6}{\Gamma(D/2-f)\Gamma(f)}(&H_1 + H_4 + H_9 + H_{15} + H_{22} + H_{35} + H_{58} + H_{94} + H_{103} + H_{123} \\ &+ H_{133} + H_{199} + H_{210} + H_{270} + H_{331} + H_{409} + H_{637} + H_{653} \\ &+ H_{838} + H_{925} + H_{960} + H_{1375} + H_{1664} + H_{1675} + H_{2062} + H_{2442})\end{aligned}$$

Double Box (44 Series):



$$\begin{aligned}\phi_{3,3} = \frac{Q_{3,3}\, u_8^{D/2-\ell}}{\Gamma(D/2-f)\Gamma(f)}(&D_1 + D_4 + D_{10} + D_{18} + D_{27} + D_{45} + D_{76} + D_{140} + D_{158} + D_{190} \\ &+ D_{208} + D_{318} + D_{340} + D_{440} + D_{542} + D_{674} + D_{1063} + D_{1091} + D_{1435} + D_{1581} + D_{1646} \\ &+ D_{2382} + D_{3047} + D_{3068} + D_{3786} + D_{4580} + D_5 + D_{19} + D_{28} + D_{46} \\ &+ D_{142} + D_{210} + D_{826} + D_{926} + D_{942} + D_{988} + D_{1094} + D_{1112} + D_{1330} + D_{1449} \\ &+ D_{1647} + D_{2436} + D_{3069} + D_{3806})\end{aligned}$$

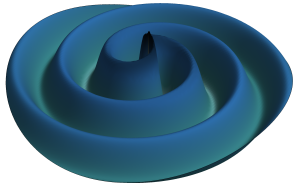
What's the best choice of cross ratios? Increase loops but not legs?

Example: Three-Point Integrals in Dim-Reg

Consider family of Euclidean three-point integrals with half integer propagator powers a_j in $D = 3 - 2\epsilon$:

$$I_3^D[a_1, a_2, a_3] := \int \frac{\mathrm{d}^D x_0}{(x_{01}^2)^{a_1}(x_{02}^2)^{a_2}(x_{03}^2)^{a_3}},$$

with $a_1 + a_2 + a_3 \leq \frac{D}{2}$.



Motivation:

Post-Newtonian ($v \ll c$) expansion of 3-body potential in GR [FL, Plefka 2020 / Shi, Wang]

$$S^{\text{3PN}} = \cdots + \sum_{\substack{j\neq i\\ k\neq i,j}} \frac{G^2 m_i m_j m_k}{4\pi} \Big\{ \Big[(6\mathbf{v}_i^2 + \mathbf{v}_k^2 - 8\mathbf{v}_i\cdot\mathbf{v}_j)(\mathbf{v}_{ki}\cdot\partial_{x_i})(\mathbf{v}_{kj}\cdot\partial_{x_j}) + (8\mathbf{v}_{ik}^2 - 4\mathbf{v}_k^2)(\mathbf{v}_{ji}\cdot\partial_{x_i})(\mathbf{v}_{ij}\cdot\partial_{x_j}) \Big] I_3[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}] + (\mathbf{v}_k\cdot\partial_{x_k})^2 \Big[(\mathbf{v}_{ki}\cdot\partial_{x_i})(\mathbf{v}_{kj}\cdot\partial_{x_j}) + 2(\mathbf{v}_{ik}\cdot\partial_{x_k})(\mathbf{v}_{ij}\cdot\partial_{x_j}) + 4(\mathbf{v}_{ji}\cdot\partial_{x_i})(\mathbf{v}_{ij}\cdot\partial_{x_j}) + 8(\mathbf{v}_{jk}\cdot\partial_{x_k})(\mathbf{v}_{kj}\cdot\partial_{x_j}) \Big] I_3[\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}] \Big\}.$$

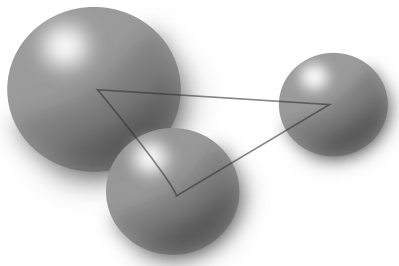
Three-Body Potential in General Relativity

Higher symmetry implies second order PDEs in $r_{jk} = |x_{jk}|$:

$$D_1 I_3 = 0, \qquad\qquad D_2 I_3 = 0.$$

Solved by [FL, Plefka 2020 Shi, Wang]

$$\mu^{-2\epsilon} I_3^{3-2\epsilon}[a_1, a_2, a_3] = \frac{A}{2\epsilon} + B + C \log\left(\tfrac{r_{12}+r_{13}+r_{23}}{\mu}\right) + \mathcal{O}(\epsilon),$$

with A, B, C given polynomials.

- New $v^{2n} G^2$ contributions to Post-Newtonian expansion of 3-body effective potential.