Form Factors: From Integrability to Bootstrap and a New Duality

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IGST, Budapest, Hungary
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[2009.11297],[2105.13367],[2112.10569] with A. Sever and A. Tumanov
[2012.12286] with L. Dixon and A. McLeod
[2112.06243],[2204.11901] with L. Dixon, Ö. Gurdogan and A. McLeod

The Niels Bohr International Academy

VILLUM FONDEN

Motivation to study form factors

$\mathcal{A}(1,\ldots,n)$
$= \langle 1,\ldots,n|0\rangle$

$\mathcal{C}_{\mathcal{O}_1\ldots\mathcal{O}_n}(x_1,\ldots,x_n)$
$= \langle 0|\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)|0\rangle$

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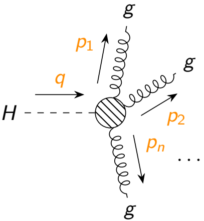
$\mathcal{F}_{\mathcal{O}}(1,\ldots,n;x)$
$= \langle 1,\ldots,n|\mathcal{O}(x)|0\rangle$

Bridge between on-shell and off-shell quantities:

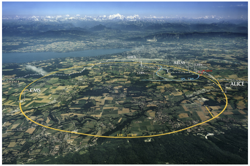
$$\begin{aligned}\mathcal{F}_{\mathcal{O}}(1,\ldots,n;q) &= \int \frac{\mathrm{d}^4x}{(2\pi)^4}\,\mathrm{e}^{-iqx}\langle 1,\ldots,n|\mathcal{O}(x)|0\rangle \\ &= \delta^4\left(q-\sum_{i=1}^{n}p_i\right)\langle 1,\ldots,n|\mathcal{O}(0)|0\rangle\end{aligned}$$

with $p_i^2 = 0$ but generically $q^2 \neq 0$

Motivation to study form factors of $\mathrm{tr}(F_{SD}^2)$



relevant for



$p_i{}^2 = 0$, $q^2 \neq 0$

- Limit $m_t \gg m_H$: Described by form factor of $\mathcal{O} = \mathrm{tr}(F_{SD}^2)$

  [Wilczek (1977)], [Shifman, Vainshtein, Zakharov (1978)]

- Tree-level MHV formula similar to Parke-Taylor amplitude:

$$\mathcal{F}_{\mathrm{tr}(F_{SD}^2)}(1^-, 2^-, 3^+, \dots, n^+; q) = \delta^4(q - \textstyle\sum_{i=1}^n p_i)\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}$$

  [Dixon, Glover, Khoze (2004)], [Brandhuber, Spence, Travaglini, Yang (2010)]

- Maximal transcendentality principle: Planar three-gluon two-loop remainder in $\mathcal{N} = 4$ SYM theory agrees with the maximally transcendental part of its counterpart in QCD

  [Brandhuber, Travaglini, Yang (2012)]

$\to$ $\mathcal{N} = 4$ SYM theory gives part of the full answer in QCD!

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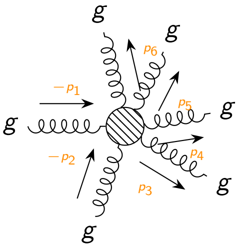
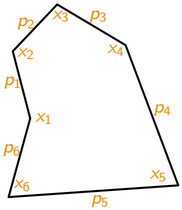
Dual description via Wilson loops

Dual coordinates x_i: $x_{i+1} - x_i = p_i$

Amplitudes: $\sum_{i=1}^{n} p_i = 0 \Rightarrow$ closed null polygonal Wilson loop



$\Leftrightarrow$



[Alday, Maldacena (2007)], [Drummond, Korchemsky, Sokatchev (2007)], [Brandhuber, Heslop, Travaglini (2007)], [Drummond, Henn, Korchemsky, Sokatchev (2007)], ...

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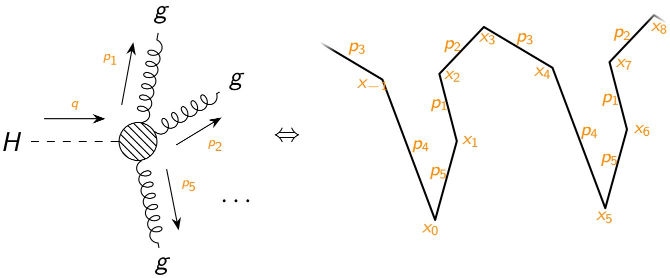

$$x_{i+n} = x_i + q$$

[Alday, Maldacena (2007)], [Maldacena, Zhiboedov (2010)], [Brandhuber, Spence, Travaglini, Yang (2010)], [Ben-Israel, Sever, Tumanov (2018)], [Bianchi, Brandhuber, Panerai, Travaglini (2018)]

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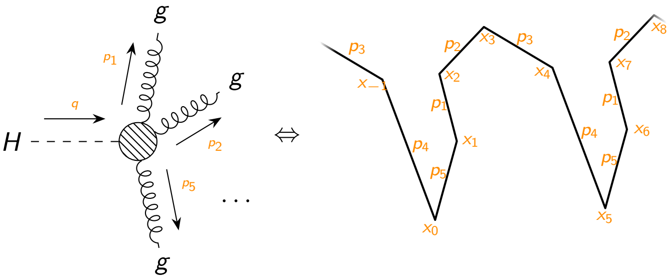
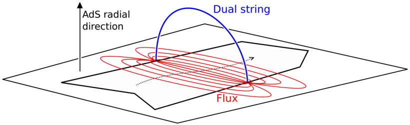


Manifest dual conformal symmetry: $x_{i+n} = x_i + q \to P(x_i)$

[Alday, Maldacena (2007)], [Maldacena, Zhiboedov (2010)], [Brandhuber, Spence, Travaglini, Yang (2010)], [Ben-Israel, Sever, Tumanov (2018)], [Bianchi, Brandhuber, Panerai, Travaglini (2018)]

Idea of the Form Factor Operator Product Expansion



[Alday, Gaiotto, Maldacena, Sever, Vieira (2009)] (figure: talk by B. Basso)

Amplitudes: Pentagon Operator Product Expansion
$\rightarrow$ Closed Wilson loops (in near-colinear limit) at any 't Hooft coupling
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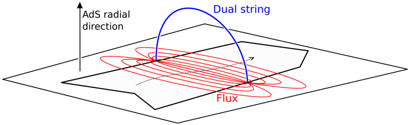

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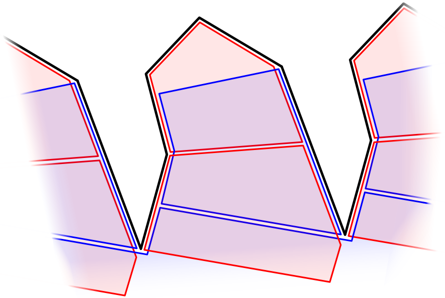
Form factors: Operator Product Expansion based on periodic Wilson loops! [Sever, Tumanov, **MW** (2020–2021)]

Idea of the Form Factor Operator Product Expansion

Decompose n-sided periodic Wilson loop into $n-2$ pentagons and a 2-sided periodic Wilson loop



[Sever, Tumanov, **MW** (2020)]

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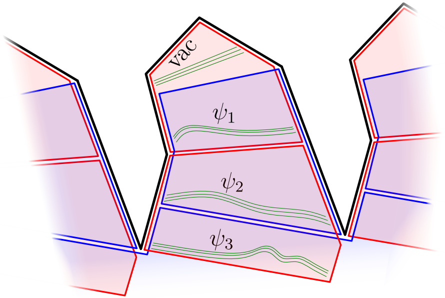

Idea behind Form Factor Operator Product Expansion:

$$\text{vacuum} \xrightarrow{\mathcal{P}} \Psi_1 \xrightarrow{\mathcal{P}} \Psi_2 \xrightarrow{\mathcal{P}} \dots \xrightarrow{\mathcal{P}} \Psi_{n-2} \dashv \mathcal{F}$$

$\mathcal{P} =$ pentagon transition
$\mathcal{F} =$ form factor transition

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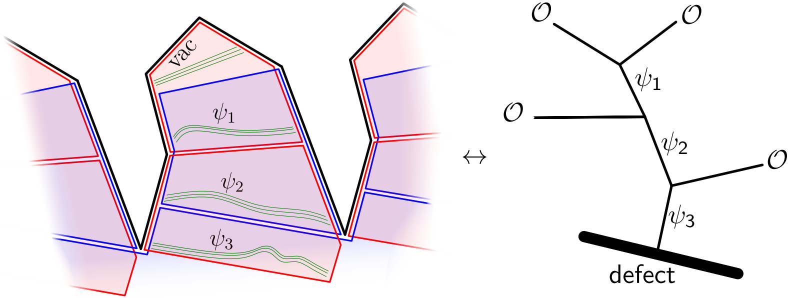

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$\mathcal{P}$ = pentagon transition $\sim$ three-point function
$\mathcal{F}$ = form factor transition $\sim$ one-point function

[Sever, Tumanov, **MW** (2020)]

Regularization

UV divergences at cusps $\Rightarrow$ Consider UV finite ratio

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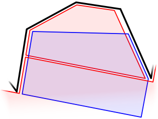
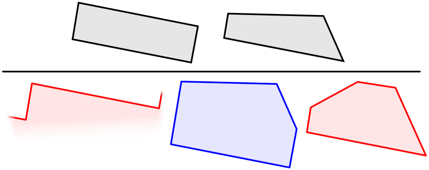
$\mathcal{W}_4 =$  $\times$ 

$$\mathcal{W}_n = \frac{\langle W_{n\text{-pt ff}}\rangle \times \langle W_{\text{2nd square}}\rangle\langle W_{\text{3rd square}}\rangle \cdots}{\langle W_{\text{2-pt ff}}\rangle \times \langle W_{\text{1st pentagon}}\rangle\,\langle W_{\text{2nd pentagon}}\rangle\,\cdots}$$

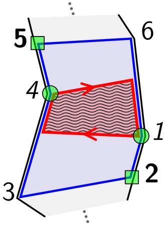
[Sever, Tumanov, **MW** (2020)]

Parametrization

Each null square has three Abelian symmetries parametrized by τ_i, σ_i, ϕ_i

[Alday,Gaiotto,Maldacena,Sever,Vieira (2009)]

Overlap pentagon with pentagons: geometry parametrized as



$$u_i = u_{i+3} \equiv \frac{{x_{i-1,i+1}}^2 {x_{i-2,i+2}}^2}{{x_{i-1,i+2}}^2 {x_{i+1,i-2}}^2}$$

$$\frac{1}{u_2} = 1 + e^{2\tau}$$
$$\frac{1}{u_3} = 1 + (e^{-\tau} + e^{\sigma + i\phi})(e^{-\tau} + e^{\sigma - i\phi})$$
$$\frac{u_1}{u_2 u_3} = e^{2\sigma + 2\tau}$$

[Basso, Sever, Viera (2013)]

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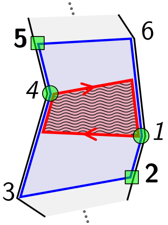

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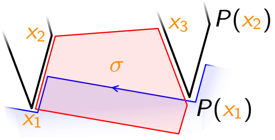
[Basso, Sever, Viera (2013)]

Overlap pentagon with two-sided periodic WL: geometry parametrized as



$$\frac{(x_1-x_3)^2(P(x_1)-P(x_3))^2}{(x_1-P(x_1))^2(x_3-P(x_3))^2} = \frac{1}{(1+e^{-2\tau}+e^{2\sigma})^2} = \left(\frac{s_{12}}{q^2}\right)^2$$

$$\frac{(x_2-P(x_1))^2(P(x_2)-P(P(x_1)))^2}{(x_2-P(x_2))^2(P(x_1)-P(P(x_1)))^2} = \frac{1}{(1+e^{2\tau})^2} = \left(\frac{s_{23}}{q^2}\right)^2$$

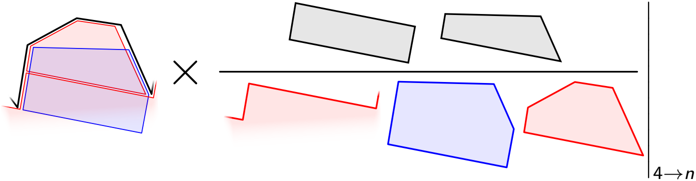
and $\phi = 0$

[Sever, Tumanov, **MW** (2020)]

Statement of the Form Factor Operator Product Expansion

Main statement:

$\mathcal{W}_n =$



$$= \sum_{\Psi_1,\ldots,\Psi_{n-2}} e^{\sum_j(-E_j\tau_j+ip_j\sigma_j+im_j\phi_j)}\mathcal{P}(0|\Psi_1)\ldots\mathcal{P}(\Psi_{n-3}|\Psi_{n-2})\,\mathcal{F}(\Psi_{n-2})$$

- GKP energy E_i, momentum p_i and angular momentum m_i known at any coupling via integrability of GKP flux tube [Basso (2010)]
- Pentagon transition $\mathcal{P}$ known at any coupling [Basso, Sever, Viera (2013–2015)], [Basso, Caetano, Cordova, Sever, Vieira (2014–2015)], [Belitsky (2014–2016)]
- New universal building block: Form factor transition $\mathcal{F}$

[Sever, Tumanov, **MW** (2020)]

Reminder: Integrability of the GKP flux tube

Integrability for gauge-invariant local composite operators

$$\text{e.g.} \qquad \mathrm{tr}(\phi^{12}\psi^{1}D\bar{\psi}^{1}F)$$

Scaling dimension $\Delta \leftrightarrow$ energy E of integrable spin chain

$\rightarrow$ Bethe ansatz yields $\Delta(g^2)$ at any 't Hooft coupling $g^2 = \frac{g_{\mathrm{YM}}^2 N}{16\pi^2}$

[Minahan, Zarembo (2002)], ..., [Beisert, Eden, Staudacher (2006)], ..., [Gromov, Kazakov, Leurent, Volin (2013)]

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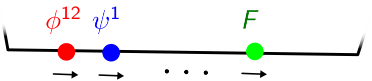
Integrability for the Gubser-Klebanov-Polyakov (GKP) flux tube

$$\mathrm{tr}(D^{k_1}\phi^{12}D^{k_2}\psi^{1}D^{k_3}\bar{\psi}^{1}D^{k_4}F) \xrightarrow{k_i \to \infty}$$



Twist $\Delta - S \leftrightarrow$ flux-tube energy E
$\rightarrow$ Bethe ansatz yields $(\Delta - S)(g^2)$ at any coupling [Basso (2010)]

Energy of state Ψ with N excitations: $E_\Psi = \sum_{i=1}^{N} E_i$

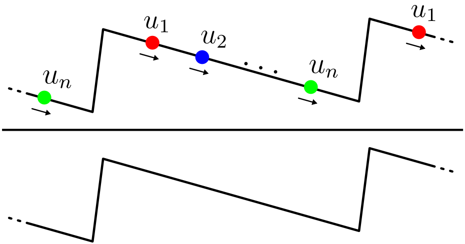
Form Factor Transition

GKP eigenstate Ψ parametrized by

- the number of excitations N
- their species $\mathbf{a} = \{a_1, \dots, a_N\}$
- their Bethe rapidities $\mathbf{u} = \{u_1, \dots, u_N\}$ ($\sim$ momenta)

Form factor transition

$F_{\mathbf{a}}(\mathbf{u}) \equiv$



Satisfies axioms $\Rightarrow$ Bootstrap at finite coupling

Bootstrap Axiom I: Singlet

Two-sided periodic Wilson loop is invariant under:

- $U(1)_\phi$: rotations in the two-dimensional transverse plane
- $SU(4)_R$: R-symmetry group

[Sever, Tumanov, **MW** (2020)]

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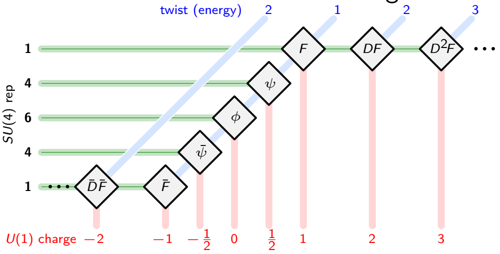
Fundamental GKP excitations all charged under $U(1)_\phi \times SU(4)_R$



picture:

[Basso, Sever, Viera (2014)]

[Sever, Tumanov, **MW** (2020)]

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Fundamental GKP excitations all charged under $U(1)_\phi \times SU(4)_R$

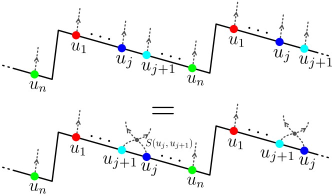
$\Rightarrow$ FF transition cannot absorb a fundamental single-particle state

$\Rightarrow$ Only states with conjugate pairs $\phi^i \bar{\phi}_i$, $\psi^i \bar{\psi}_i$, $D^{n-1} F \bar{D}^{n-1} \bar{F}$ and their products can have non-vanishing FF transition

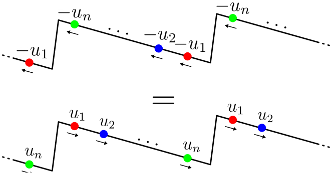
[Sever, Tumanov, **MW** (2020)]

Bootstrap Axioms II-V

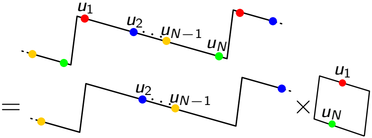
Watson



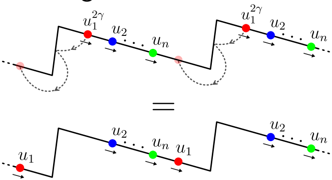
Reflection



Square limit



Crossing



[Sever, Tumanov, **MW** (2020)]

Bootstrapping form factor transitions at finite coupling

Simplest solution to axioms for fermions and gluons:

$$F_{\psi^A\bar{\psi}^B}(u,v) = \delta^{AB}\frac{2\pi}{\mu_\psi(u)}\delta(u-v)$$

$$F_{D^{n-1}F\bar{D}^{n-1}\bar{F}}(u,v) = \frac{2\pi}{\mu_{D^{n-1}F}(u)}\delta(u-v)$$

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Solution to axioms for scalars?

[Sever, Tumanov, **MW** (2021)]

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Solution to axioms for scalars?

$\Rightarrow$ Start solving order by order in coupling

$\Rightarrow$ Fruitful interplay with symbol bootstrap!

FFOPE

Symbol bootstrap

[Sever, Tumanov, **MW** (2021)]

Scalar form factor transition at finite coupling

Pentagon transition [Basso, Viera, Sever (2013)]

$$P_{0|\phi\bar{\phi}} = \texttt{tree} \times \exp(\texttt{source terms evolved with BES kernel})$$

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Octagon kernel familiar from octagon [Coronado (2018)], [Kostov, Petkova, Serban (2019)], [Belitsky, Korchemsky (2019)], [Bargheer, Coronado, Vieira (2019)],... and origin of six-point amplitude [Basso, Dixon, Papathanasiou (2020)]

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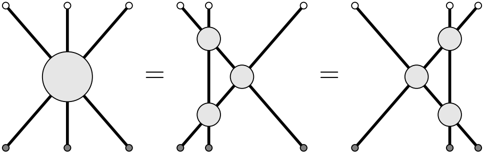
$\Rightarrow$ Match up to 8-loop order and with minimal solution to the axioms at strong coupling

Multi-particle form factor transitions and factorization

Factorized scattering:



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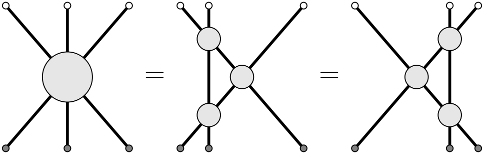

Factorized ansatz for form factor transitions: e.g.

$$F_{F\psi\bar{\psi}\bar{F}}(u_1, u_2, u_3, u_4) = F_{F\bar{F}}(u_1, u_4) F_{\psi\bar{\psi}}(u_2, u_3)$$

[Sever, Tumanov, **MW** (2021)]

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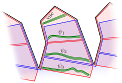
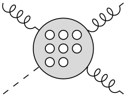
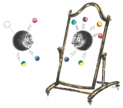
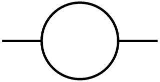




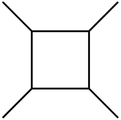


Functions in Quantum Field Theory

At one loop: Polylogarithms



$$\overset{D=2}{\sim} \log\left(\frac{1+2\frac{m^2}{s}+\sqrt{1+4\frac{m^2}{s}}}{1+2\frac{m^2}{s}-\sqrt{1+4\frac{m^2}{s}}}\right),$$



$$\overset{D=4}{\sim} \mathrm{Li}_2(\dots)+\dots$$

Recall: $\mathrm{Li}_n(x) \equiv \int_0^x \frac{dt}{t}\,\mathrm{Li}_{n-1}(t)$ and $\mathrm{Li}_1(x) \equiv -\log(1-x) = \int_0^x \frac{dt}{1-t}$

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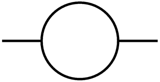
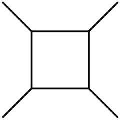
 $\overset{D=2}{\sim} \log\left(\frac{1+2\frac{m^2}{s}+\sqrt{1+4\frac{m^2}{s}}}{1+2\frac{m^2}{s}-\sqrt{1+4\frac{m^2}{s}}}\right)$,  $\overset{D=4}{\sim} \mathrm{Li}_2(\dots)+\dots$

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More generally: Multiple polylogarithms [Chen (1977)], [Goncharov (1995)], ...

$$G(a_1,\dots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\dots,a_n,t) \qquad G(;z) = 1$$

weight $= n$

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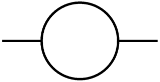

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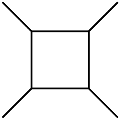

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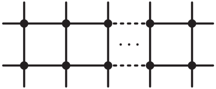
Beyond: Generalizations of elliptic integrals and integrals over Calabi-Yau manifolds even in planar $\mathcal{N}=4$ SYM theory!



$=$

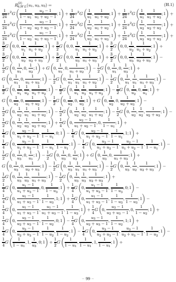
Calabi-Yau $(L-1)$-fold

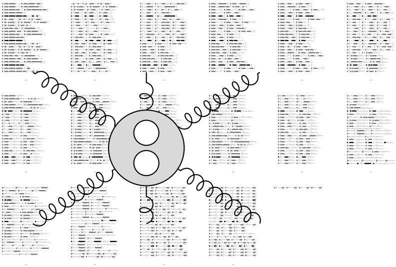
[Bourjaily, He, McLeod, von Hippel, **MW** (2018)], [Vergu, Volk (2020)]

Hidden simplicity in multiple polylogarithms

Two-loop six-gluon reminder function in $\mathcal{N} = 4$ SYM theory

[Del Duca, Duhr, Smirnov (2010)]





$$\hat{u}_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad \hat{u}_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}} \qquad \hat{u}_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

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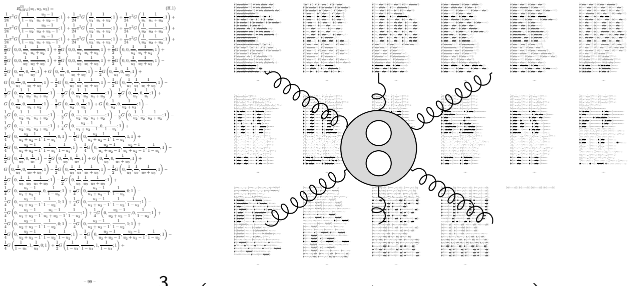

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[Goncharov, Spradlin, Vergu, Volovich (2010)]

The Symbol

Symbol [Goncharov, Spradlin, Vergu, Volovich (2010)]

$$\mathrm{d}G(\boldsymbol{a};z) = \sum_i F_i \,\mathrm{dlog}(\phi_i) \quad \Rightarrow \quad \mathcal{S}(G(\boldsymbol{a};z)) = \sum_i \mathcal{S}(F_i) \otimes \log(\phi_i)$$

E.g.

$$\mathrm{dLi}_2(x) = -\log(1-x)\,\mathrm{dlog}(x) \quad \Rightarrow \quad \mathcal{S}(\mathrm{Li}_2(x)) = -\log(1-x) \otimes \log(x)$$

- Tensor product
- $\log(ab) = \log(a) + \log(b)$

$\Rightarrow$ Easy to manipulate and well understood

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Idea: Decompose complicated functions to simple functions!

$\rightarrow$

$= \log(\phi_i)$

Reverse idea: Use simple functions to build complicated functions!

$\log(\phi_i) =$  $\rightarrow$ 

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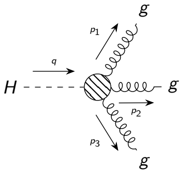
$\log(\phi_i) =$  $\rightarrow$ 

Bootstrapping

- $=$ Ansatz for result from polylogarithms
- $+$ Fix coefficients via physical constraints
- $\Rightarrow$ Avoid Feynman diagrams and Feynman integrals altogether!

[Dixon, Drummond, Henn (2011)], ...

$\mathcal{F}_{\mathrm{tr}(F_{SD}^2)}(1^-, 2^-, 3^+) \Leftrightarrow$ Higgs $\to$ 3 gluons



Symbol alphabet

$\mathcal{L}$ =

$$= \{\log(u_1), \log(u_2), \log(u_3), \log(1-\underbrace{u_1}_{\frac{s_{12}}{q^2}}), \log(1-\underbrace{u_2}_{\frac{s_{23}}{q^2}}), \log(1-\underbrace{u_3}_{\frac{s_{31}}{q^2}})\}$$

Momentum conservation: $u_1 + u_2 + u_3 = 1$

BDS-like normalization:

$$F_3 = \mathcal{W}_3 \exp\left[-\frac{1}{4}\Gamma_{\mathrm{cusp}}\left(\log^2\left(\frac{u_1}{u_3}\right) + \log^2 u_2 + 2\log(1-u_2)\log\left(\frac{u_1(1-u_2)}{u_2 u_3}\right) + 4\zeta_2\right)\right]$$

The Symbol Bootstrap of the three-gluon form factor

Starting point at L loops: Tensor product in letters $\in \mathcal{L}$ of length $2L$

[Dixon, McLeod, **MW** (2020)], [Dixon, Gurdogan, McLeod, **MW** (2022)]

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Constraints:

- Invariant under permutations of u_1, u_2, u_3
- $[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}] = 0$
- Branch cuts start at $s_{ij} = 0$
  (= threshold for particle production in massless theories)
  $\Rightarrow$ first entry is $\log(u_i)$
- Last entry is $\log \frac{u_i}{1-u_i}$
- ...
- Near-collinear limit given by FFOPE [Sever, Tumanov, **MW** (2020–2021)]

$\Rightarrow$ Unique solution

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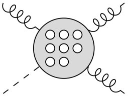
New results at 3,4,5,6,7 and 8-loop order! **New record!**



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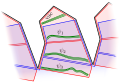
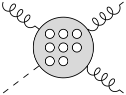
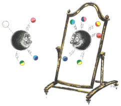
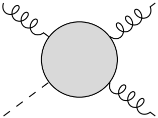




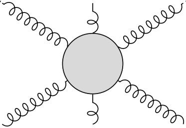


A new and surprising duality

Duality between cosmically and BDS-like normalized three-gluon MHV form factor and six-gluon MHV amplitude:



$= S \Bigg($



$\Bigg)\Bigg|_{\substack{\hat{u}_1 \to \frac{u_2 u_3}{(1-u_2)(1-u_3)} \\ \hat{u}_2 \to \frac{u_1 u_3}{(1-u_1)(1-u_3)} \\ \hat{u}_3 \to \frac{u_1 u_2}{(1-u_1)(1-u_2)}}}$

at each loop order, with antipode S

Properties:

- At symbol level $S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m x_m \otimes \cdots \otimes x_2 \otimes x_1$
  Antipode also defined beyond symbol but mod $i\pi$
- Relates discontinuities $\leftrightarrow$ derivatives $\Rightarrow$ **Completely unexpected**
- Implies duality between periodic and closed Wilson loops
- Checked up to 7 loops. Derivation?

[Dixon, Gurdogan, McLeod, **MW** (2021)]

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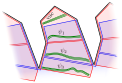

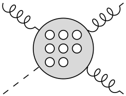

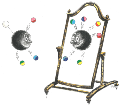

Conclusions and outlook

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- Bootstrapped form factor transition at finite coupling
  $\Rightarrow$ Form factors in near-collinear expansion at any coupling
  [Sever, Tumanov, **MW** (2020–2021)]
- Bootstrapped three-gluon form factor for general kinematics at 3,4,5,6,7 and 8-loop order
  $\Rightarrow$ Maximally transcendental part of $gg \to Hg$ at LHC up to N^8LO
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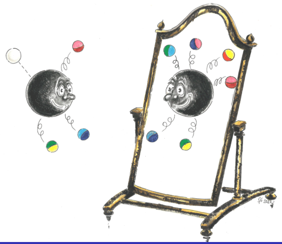

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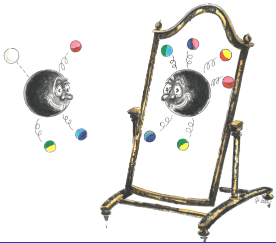
- Matrix part
- N^kMHV form factors
- Form factors of other operators
- Derivation and generalization of duality



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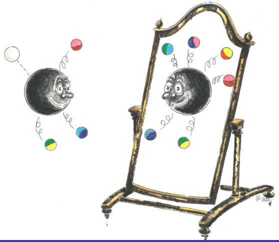

Thank you!

Back-up slide: Hidden simplicity

$$18\ \text{pages} = \sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

[Gancharov, Spradlin, Vergu, Volovich (2010)]

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3}, \qquad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

$$L_4(x^+, x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+) + \ell_{4-m}(x^-))$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right), \qquad J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-))$$