One more construction for Bethe vectors

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Contents

R-matrix

$\mathfrak{gl}_2$ R-matrix

Let's start with the most basic R-matrix

$$R(u) = \begin{pmatrix} u+c & 0 & 0 & 0 \\ 0 & u & c & 0 \\ 0 & c & u & 0 \\ 0 & 0 & 0 & u+c \end{pmatrix}$$

Monodromy matrix of $Y(\mathfrak{gl}_2)$

Monodromy matrix

Monodromy matrix is 2×2 matrix with nocommutative elements

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

RTT

that satisfy RTT relation

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v),$$

where $T_1(u) = T(u) \otimes 1$ and $T_2(v) = 1 \otimes T(v)$.

Bethe vector

Vacuum

Let's suppose existence of special vector called *pseudovacuum vector* Ω, such that

$$C(u)\Omega = 0,$$
$$A(u)\Omega = \lambda_1(u)\Omega,$$
$$D(u)\Omega = \lambda_2(u)\Omega,$$

where $\lambda_i(u)$ are some scalar functions.

Standart construction

Then, usual construction for Bethe vector is

$$\mathbb{B}(\bar{u}) = B(u_1) \cdot \ldots \cdot B(u_n)\Omega$$

Eigenvector

Bethe vector becomes an eigenvector

$$t(z)\mathbb{B}(\bar{u}) = \tau(z|\bar{u})\mathbb{B}(\bar{u})$$

with eigenvalue

$$\tau(z|\bar{u}) = \lambda_1(z)\prod_{i=1}^{a}\frac{u_i - z + c}{u_i - z} + \lambda_2(z)\prod_{i=1}^{a}\frac{z - u_i + c}{z - u_i}$$

Bethe equations

if Bethe equations are satisfied

$$\frac{\lambda_1(u_i)}{\lambda_2(u_i)} = \prod_{k\neq i}\frac{u_i - u_k + c}{u_i - u_k - c}$$

Gauss decomposition of $Y(\mathfrak{gl}_2)$

Monodromy matrix

Let's consider change coordinates on monodromy matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} k_1 & Fk_1 \\ k_1 E & k_2 + Fk_1 E \end{pmatrix}.$$

We call operators k_i, E, F by *semicurrent*.

This reparametrisation can be write in compact form

$$T(u)^t = \mathbb{F}(u)^t \cdot \mathbb{K}(u)^t \cdot \mathbb{E}(u)^t,$$

where t is transposition by the second diagonal, and

$$\mathbb{F} = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{K} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \qquad \mathbb{E} = \begin{pmatrix} 1 & 0 \\ E & 1 \end{pmatrix}$$

Current representation

New commutation relations

We can rewrite RTT-relation for new coordinates

$$
\begin{aligned}
[k_i(u), k_j(v)] &= 0, \qquad\qquad i,j = 1,2,\\
F(u)k_1(v) &= \frac{v-u+c}{v-u}k_1(v)F(u) - \frac{c}{v-u}k_1(v)F(v),\\
F(u)k_2(v) &= \frac{v-u-c}{v-u}k_2(v)F(u) + \frac{c}{v-u}k_2(v)F(v),\\
k_1(v)E(u) &= \frac{v-u+c}{v-u}E(u)k_1(v) - \frac{c}{v-u}E(v)k_1(v),\\
k_2(v)E(u) &= \frac{v-u-c}{v-u}E(u)k_2(v) + \frac{c}{v-u}E(v)k_2(v),\\
[E(u), F(v)] &= \frac{c}{u-v}\left(\frac{k_2(u)}{k_1(u)} - \frac{k_2(v)}{k_1(v)}\right).
\end{aligned}
$$

And two more relations

$$(u-v-c)E(u)E(v) = (u-v+c)E(u)E(v) - c\,\left(E(u)^2 + E(v)^2\right),$$

$$(u-v-c)F(v)F(u) = (u-v+c)F(v)F(u) - c\,\left(F(u)^2 + F(v)^2\right).$$

Semicurrent representation of Bethe vectors

In terms of semicurrents Bethe vector can be rewrite as

$$\mathbb{B}(\bar{u}) = \prod_i \lambda_1(u_i) \prod_{i<j} \frac{1}{f(u_i, u_j)}$$
$$F(u_1) \cdot F(u_2; u_1) \cdot \ldots \cdot F(u_n; u_1, \ldots, u_{n-1})\Omega,$$

where

$$F(u_k; u_1, \ldots, u_{k-1}) = F(u_n) - \sum_{i=1}^{k-1} \frac{1}{h(u_i, u_k)} \prod_{j \neq i} \frac{f(u_j, u_i)}{f(u_j, u_k)} F(u_i).$$

We used functions

$$f(u, v) = \frac{u - v + c}{u - v}, \quad h(u, v) = \frac{u - v + c}{c}.$$

Double Yangian

Two copies of RTT

Let us consider two monodromy matrices $T^{\pm}(u)$ that satisfy 4 (four!) sets RTT relations

$$R_{12}(u,v)T_1^{\mu}(u)T_2^{\nu}(v) = T_2^{\nu}(v)T_1^{\mu}(u)R_{12}(u,v), \qquad \mu,\nu = \pm.$$

Full currents

For two monodromy matrices $T^{\pm}(u)$ there are two sets of Gauss coordinates $k_i^{\pm}, E^{\pm}, F^{\pm}$. Then we define *full currents*

$$\mathcal{F}(u) = F^{+}(u) - F^{-}(u), \qquad \mathcal{E}(u) = E^{+}(u) - E^{-}(u).$$

Current construction of Bethe vector

Projection

Let's define projection P_f^+. The projection is a linear operator. On the normal ordered term the projection acts as

$$P_f^+\left(F^-...F^- \cdot F^+ \dots F^+\right) = 0,$$
$$P_f^+\left(F^+ \dots F^+\right) = F^+ \dots F^+.$$

Normal ordering

We call term ordered if term has form $F^-...F^- \cdot F^+ \dots F^+$. One can make normal ordering using FF-commutation relation

$$(u-v-c)F^+(v)F^-(u) = \\ (u-v+c)F^-(v)F^+(u) - c\ \left(F^-(u)^2 + F^+(v)^2\right)$$

Projection formula

$$\mathbb{B}(\bar{u}) = \prod_i \lambda_1(u_i) \prod_{i<j} \frac{1}{f(u_i, u_j)} \times P_f^+\Big(\mathcal{F}(u_1) \cdot \ldots \cdot \mathcal{F}(u_n)\Big)\Omega.$$

And dropping the prefactor we can write

$$\mathbb{B}(\bar{u}) \sim P_f^+\Big(\mathcal{F}(u_1) \cdot \ldots \cdot \mathcal{F}(u_n)\Big)\Omega.$$

RTT and RRR

Monodromy matrix

The monodromy matrix $T(u)$ is $N \times N$ matrix which satisfies bilinear relations called RTT-relation

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v),$$

where R_{12} acts in (graded) tensor product of two spaces, $T_1(u) = T(u) \otimes 1$ and $T_2(v) = 1 \otimes T(v)$.

Yang-Baxter equation

Consistency condition of this algebra is Yang-Baxter equation

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v).$$

Transfer matrix

The transfer matrix is defined as the (super)trace of the monodromy matrix

$$t(u) = \operatorname{tr} T(u) = \sum_i (-1)^{p[i]} T_{ii}(u).$$

It defines an integrable system, due to the relation $[t(u)\,,\, t(v)] = 0$.

Hamiltonian

Usually, Hamiltonian is one of the coefficient of series expansion of the transfer matrix $t(u)$ or some combination of them.
For example, local Hamiltonian for spin chains

$$H = t(u)^{-1} \frac{\mathrm{d}}{\mathrm{d}u} t(u) \Big|_{u=0} .$$

Other coefficients are called symmetries or higher Hamiltonians.

Algebraic Bethe ansatz framework

Vacuum

Using this approach requires the existence of special vector called *pseudovacuum vector* Ω, such that

$$T_{i,j}(u)\Omega = 0, \qquad i > j\,,$$
$$T_{i,i}(u)\Omega = \lambda_i(u)\Omega,$$

where $\lambda_i(u)$ are some scalar functions.

Bethe vectors belong to the space $\mathcal{H}$ in which the monodromy matrix entries act. We do not specify this space. However, we assume that it contains the *pseudovacuum vector* Ω.

Bethe vectors

Bethe vector

Typically, the Bethe vector can be represented as a polynomial in the elements of the monodromy matrix with different spectral parameters acting on the vacuum.

$$\mathbb{B}(u_1, \ldots, u_n) = Pol(T_{ij}(u_1), \ldots, T_{ij}(u_n))\Omega.$$

Eigenvector

The most important property of the Bethe vector is that it becomes an eigenvector

$$t(z)\mathbb{B}(\bar{u}) = \tau(z|\bar{u})\mathbb{B}(\bar{u})$$

when the spectral parameters satisfy certain constraints, called the *Bethe equations*.

R-matrix

$\mathfrak{gl}_n$ R-matrix

Let's consider the most typical R-matrix

$$R(u,v) = I + \frac{c}{u-v}P,$$

where P is the permutation operator

$$P = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}.$$

RTT-relation

For this R-matrix one can RTT relation in components

$$[T_{ij}(u), T_{kl}(v)] = \frac{c}{u-v}\left(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\right).$$

Gauss decomposition

$\mathfrak{gl}_n - Yangian$

For $n \times n$ monodromy matrix the Gauss decomposition is

$$T(u)^t = \mathbb{F}(u)^t \cdot \mathbb{K}(u)^t \cdot \mathbb{E}(u)^t,$$

where t is transposition by the second diagonal, $\mathbb{F}$ is uppertriangular matrix, $\mathbb{K}$ is diagonal matrix, and $\mathbb{E}$ is lowerdiagonal matrix.

Ding-Frenkel isomorphism

Gauss decomposition describes isomorphism between RTT-algebra and *new realisation* of Yangian discovered by Drinfeld.

Ding-Frenkel isomorphism

New commutation relations

We can rewrite RTT-relation for new coordinates

$$[k_i(u), k_j(v)] = 0, \qquad i, j = 1, \dots n,$$

$$F_i(u)k_i(v) = \frac{v-u+c}{v-u}k_i(v)F_i(u) - \frac{c}{v-u}k_i(v)F_i(v),$$

$$F_i(u)k_{i+1}(v) = \frac{v-u-c}{v-u}k_{i+1}(v)F_i(u) + \frac{c}{v-u}k_{i+1}(v)F_i(v),$$

$$k_i(v)E_i(u) = \frac{v-u+c}{v-u}E_i(u)k_i(v) - \frac{c}{v-u}E_i(v)k_i(v),$$

$$k_{i+1}(v)E_i(u) = \frac{v-u-c}{v-u}E_i(u)k_{i+1}(v) + \frac{c}{v-u}E_i(v)k_{i+1}(v),$$

$$[E_i(u), F_j(v)] = \frac{c\,\delta_{i,j}}{u-v}\left(\frac{k_{i+1}(u)}{k_i(u)} - \frac{k_{i+1}(v)}{k_i(v)}\right).$$

Here we use notation $F_i = F_{i+1,i}$ and $E_i = E_{i,i+1}$.

And few more relations

$$(u-v-c)E_i(u)E_i(v) = (u-v+c)E_i(u)E_i(v) - c\ \left(E_i(u)^2 + E_i(v)^2\right),$$

$$(u-v+c)E_i(v)E_{i+1}(u) = (u-v)E_{i+1}(u)E_i(v) + c\ \left(E_{i+1}(u)E_i(u) + E_{i,i+2}(u) - E_{i,i+2}(v)\right),$$

$$(u-v-c)F_i(v)F_i(u) = (u-v+c)F_i(v)F_i(u) - c\ \left(F_i(u)^2 + F_i(v)^2\right),$$

$$(u-v+c)F_{i+1}(u)F_i(v) = (u-v)F_i(v)F_{i+1}(u) + c\ \left(F_i(u)F_{i+1}(u) + F_{i+2,i}(u) - F_{i+2,i}(v)\right).$$

Projection formula

$$\mathbb{B}(\bar{u}^1,\dots,\bar{u}^{n-1}) \sim P_f^+\Big(\mathcal{F}_1(u_1^1)\cdot\ldots\cdot\mathcal{F}_1(u_{r_1}^1)\cdot\ldots\cdot \mathcal{F}_{n-1}(u_1^{n-1})\cdot\ldots\cdot\mathcal{F}_{n-1}(u_{r_{n-1}}^{n-1})\Big)\Omega.$$

Here we also use notation for *full current* $\mathcal{F}_i = F^+_{i,i+1} - F^-_{i.i+1}$.

R-matrix

$\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}, \mathfrak{sp}_{2n}$ R-matrices

Let's consider the most typical R-matrix

$$R(u,v) = I + \frac{c}{u-v}P - \frac{c}{u-v+c\kappa}Q,$$

where P is the permutation operator

$$P = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}, \qquad Q = P^{t_1} = \sum_{i,j=1}^{n} \epsilon_i \epsilon_j \; e_{j'i'} \otimes e_{ji},$$

where $i' = N + 1 - i$.

Center

Quantum orthogonality condition

Considering the residue at the point of the equation, we can prove the existence of the center

$$z(u) = T(u + c\kappa)^t T(u)$$

Constrains

This center implies constraints

$$F_i(u) = -F_{i'}(u + c(\kappa - i)).$$

Projection formula

$$\mathbb{B}(\bar{u}^1, \ldots, \bar{u}^n) \sim P_f^+ \Big(\mathcal{F}_1(u_1^1) \cdot \ldots \cdot \mathcal{F}_1(u_{r_1}^1) \cdot \ldots \cdot \mathcal{F}_n(u_1^n) \cdot \ldots \cdot \mathcal{F}_n(u_{r_n}^n) \Big) \Omega.$$

R-matrix

Beisert representation of Shastry R-matrix for Hubbard model

$$R(u,v) = I(u,v) + P(u,v) + Q(u,v).$$

Notation

We use notation

$$U(z) = \sqrt{\frac{x^+(z)}{x^-(z)}}, \qquad \gamma(z) = \sqrt{U(z)(x^+(z) - x^-(z)}),$$

and coordinates $x^\pm$ live on the two sheets covering of complex plane

$$x^\pm(z) + \frac{1}{x^\pm(z)} = z \pm \frac{\eta}{2}.$$

This R-matrix has graded tensor product

$$(e_{ij} \otimes e_{kl}) \cdot (e_{ab} \otimes e_{cd}) = (-1)^{(p[k]+p[l])(p[a]+p[b])} e_{ij} e_{ab} \otimes e_{kl} e_{cd},$$

with the following grading

$$p[1] = p[4] = 0, \qquad p[2] = p[3] = 1.$$

I-operator

$$I(u,v) = \sum_{i,j=1}^{4} id_{i,j}(u,v) e_{ii} \otimes e_{jj},$$

with

$$id_{11}(u,v) = id_{44}(u,v) = 1, \quad id_{22}(u,v) = id_{33}(u,v) = \frac{U(v)}{U(u)},$$

$$id_{14}(u,v) = id_{41}(u,v) = \frac{1}{U(u)^2}, \quad id_{23}(u,v) = id_{32}(u,v) = \frac{1}{U(u)U(v)},$$

$$id_{12}(u,v) = id_{13}(u,v) = id_{42}(u,v) = id_{43}(u,v) = \frac{1}{U(u)},$$

$$id_{21}(u,v) = id_{31}(u,v) = id_{24}(u,v) = id_{34}(u,v) = U(v)\frac{x^-(u) - x^-(v)}{x^+(u) - x^+(v)}.$$

P-operator

$$P(u,v) = \sum_{i,j=1}^{4} p_{i,j}(u,v) e_{ij} \otimes e_{ji},$$

with

$$p_{11}(u,v) = p_{44}(u,v) = p_{14}(u,v) = p_{41}(u,v) = \frac{x^-(u) - x^+(u)}{x^+(u) - x^+(v)},$$

$$p_{21}(u,v) = p_{31}(u,v) = p_{24}(u,v) = p_{34}(u,v) = \frac{\gamma(v)}{\gamma(u)} \frac{x^-(u) - x^+(u)}{x^+(u) - x^+(v)},$$

$$p_{22}(u,v) = p_{33}(u,v) = p_{23}(u,v) = p_{32}(u,v) = \frac{U(v)}{U(u)} \frac{x^+(v) - x^-(v)}{x^+(u) - x^+(v)},$$

$$p_{12}(u,v) = p_{13}(u,v) = p_{42}(u,v) = p_{43}(u,v) = \frac{U(v)\gamma(u)}{U(u)\gamma(v)} \frac{x^+(v) - x^-(v)}{x^+(u) - x^+(v)}$$

Q-operator

$$Q(u,v) = \sum_{i,j=1}^{4} q_{i,j}(u,v) e_{j'i'} \otimes e_{ji},$$

with

$$q_{11}(u,v) = q_{44}(u,v) = \frac{x^-(v)(x^+(u) - x^-(u))}{U(u)^2(x^-(u)x^-(v) - 1)},$$

$$q_{14}(u,v) = q_{41}(u,v) = \frac{(x^+(u) - x^-(u))}{x^+(u)(x^-(u)x^-(v) - 1)},$$

$$q_{22}(u,v) = q_{33}(u,v) = \frac{x^-(u)(x^+(v) - x^-(v))}{U(u)U(v)(x^-(u)x^-(v) - 1)},$$

$$q_{23}(u,v) = q_{32}(u,v) = \frac{U(v)(x^-(v) - x^+(v))}{U(u)x^+(v)(x^-(u)x^-(v) - 1)},$$

$$-q_{12}(u,v) = q_{13}(u,v) = q_{42}(u,v) = -q_{43}(u,v) = \frac{(x^+(u)-x^-(u))(x^+(v)-x^-(v))}{U(u)\gamma(u)\gamma(v)(x^-(u)x^-(v)-1)},$$

$$-q_{21}(u,v) = q_{31}(u,v) = q_{24}(u,v) = -q_{34}(u,v) = \frac{\gamma(u)\gamma(v)}{U(u)^2U(v)(x^-(u)x^-(v)-1)}.$$

Operator $I(u,v)$ looks like unity and

$$\lim_{v\to\infty} \underset{u=v}{res}\, I(u,v) \sim I = \sum_{ij} e_{ii} \otimes e_{jj}.$$

Operator $P(u,v)$ has pole at the point $u=v$

$$\underset{u=v}{res}\, P(u,v) \sim P = \sum_{ij} (-1)^{p[j]} e_{ij} \otimes e_{ji}.$$

Operator $Q(u,v)$ has pole at the pole on another sheet

$$\underset{u=v}{res}\, Q(u,v)^{C_u} \sim Q = \sum_{ij} sgn_{ij}\ e_{j'i'} \otimes e_{ji},$$

where $C_u : x^{\pm}(u) \to 1/x^{\pm}(u)$ and then at point $u=v$.
Operator has properties: $Q^2 = 0$, $\mathrm{rank}\, Q = 1$, $Q = P^{\tau_1}$.

Schematic view of the R-matrix

The following simplified representation of R-matrix

$$R(u,v) \approx I + \frac{P}{x^+(u) - x^+(v)} + \frac{Q}{1 - x^-(u)x^-(v)}$$

helps a lot to understand structure of Gauss decomposition.

Center

For this algebra here we have the "orthogonal" center

$$z(u) = T(u)\left(T(u)^{C_u}\right)^{\tau},$$

here τ is graded transposition by the second diagonal with extra signs

$$\tau: \quad e_{ab} \to (-1)^{p[a](p[b]+1)} \varepsilon^{a,a'} \varepsilon^{b,b'} \; e_{b',a'}.$$

Constrains

The center imposes constrains

- $E_{34}(x^{\pm}(z)) = -\frac{\gamma(z)U(z+\eta)}{\gamma(z+\eta)} \quad E_{12}\left(\frac{1}{x^{\pm}(z+\eta)}\right)$
- $F_{43}(x^{\pm}(z)) = -\frac{\gamma(z)U(z+\eta)}{\gamma(z+\eta)} \quad F_{21}\left(\frac{1}{x^{\pm}(z+\eta)}\right)$
- $k_4(x^{\pm}(z))k_3(x^{\pm}(z))^{-1} = k_2\left(\frac{1}{x^{\pm}(z+\eta)}\right)k_1\left(\frac{1}{x^{\pm}(z+\eta)}\right)^{-1}$

Qunatum algebra

E_{12} with k_1, k_2 - $su(1|1)$ (à la fermion)

$$E_{12}(z)k_{1,2}(w) = \frac{x^+(w) - x^-(z)}{(x^-(w) - x^-(z))U(w)} k_{1,2}(w)E_{12}(z) - \frac{(x^+(z) - x^-(z))\gamma(w)}{(x^-(w) - x^-(z))U(w)\gamma(z)} k_{1,2}(w)E_{12}(w),$$

E_{23} with k_2, k_3 - $su(2)$ (à la boson)

$$E_{23}(z)k_2(w) = \frac{w - z - \eta}{w - z} k_2(w)E_{23}(z) + \frac{\eta}{w - z} k_2(w)E_{23}(w),$$

$$E_{23}(z)k_3(w) = \frac{w - z + \eta}{w - z} k_3(w)E_{23}(z) - \frac{\eta}{w - z} k_3(w)E_{23}(w).$$

F_{21} with k_1, k_2 - $su(1|1)$ (à la fermion)

$$k_{1,2}(w)F_{12}(z) = \frac{x^+(w) - x^-(z)}{(x^-(w) - x^-(z))U(w)} F_{21}(z)k_{1,2}(w) - \frac{(x^+(w) - x^-(w))\gamma(z)}{(x^-(w) - x^-(z))U(z)\gamma(w)} F_{12}(w)k_{1,2}(w),$$

F_{32} with k_2, k_3 - $su(2)$ (à la boson)

$$k_2(w)F_{32}(z) = \frac{w - z - \eta}{w - z} F_{32}(z)k_2(w) + \frac{\eta}{w - z} F_{32}(w)k_2(w),$$

$$k_3(w)F_{32}(z) = \frac{w - z + \eta}{w - z} F_{32}(z)k_3(w) - \frac{\eta}{w - z} F_{32}(w)k_3(w).$$

EF-commutation relations

$E_{12}(z), F_{21}(z)$ - $su(1|1)$ (à la fermion)

$$\{E_{12}(w), F_{21}(z)\} = \frac{\gamma(w)U(w)\gamma(z)U(z)^{-1}}{x^-(w) - x^-(z)} \left(\frac{k_2(w)}{k_1(w)} - \frac{k_2(z)}{k_1(z)} \right)$$

$E_{23}(z), F_{32}(z)$ - $su(2)$ (à la boson)

$$[E_{23}(w), F_{32}(z)] = \frac{\eta}{w - z} \left(\frac{k_3(w)}{k_2(w)} - \frac{k_3(z)}{k_2(z)} \right)$$

EE and FF-commutation relations

$E_{12}(z), F_{21}(z)$ - $su(1|1)$ (à la fermion)

$$\{E_{12}(w), E_{12}(z)\} = \{F_{21}(w), F_{21}(z)\} = 0.$$

$E_{23}(z), F_{32}(z)$ - $su(2)$ (à la boson)

$$(w - z + \eta)E_{23}(w)E_{23}(z) = (z - w - \eta)E_{23}(z)E_{23}(w) + \eta\left(E_{23}(z)^2 + E_{23}(w)^2\right)$$

$$(w - z + \eta)F_{32}(z)F_{32}(w) = (z - w - \eta)F_{32}(w)F_{32}(z) + \eta\left(F_{32}(z)^2 + F_{32}(w)^2\right)$$

$E_{12} - E_{23}$ commutation relations

$$E_{23}(z)E_{12}(w) = \frac{z-w-\eta}{z-w}E_{12}(w)E_{23}(z) + \frac{\eta}{w-z}E_{13}(w) + \frac{(x^-(w)x^+(z)-1)\gamma(z)\gamma(w)U(w)}{(w-z)x^+(z)x^-(w)}\left(E_{12}(z)E_{23}(z) - E_{13}(z)\right) - \frac{\gamma(z)\gamma(w)U(z)U(w)}{x^-(z)x^-(w)-1}E_{24}(z)$$

$F_{21} - F_{32}$ commutation relations

$$F_{21}(w)F_{32}(z) = \frac{z-w-\eta}{z-w}F_{32}(z)F_{21}(w)E_{23}(z) + \frac{\eta}{w-z}F_{31}(w) + \frac{(x^-(w)x^+(z)-1)\gamma(w)\gamma(z)U(w)}{(w-z)x^+(z)x^+(w)}\left(F_{32}(z)F_{21}(z) - F_{13}(z)\right) - \frac{\gamma(w)\gamma(z)U(w)^{-1}U(z)^{-1}}{x^-(z)x^-(w)-1}F_{24}(z)$$

Projection formula

$$\mathbb{B}_{a,b}(\bar{u},\bar{v}) \sim P_f^+\Big(\mathcal{F}_1(u_1)\cdot\ldots\cdot\mathcal{F}_1(u_a)\cdot\mathcal{F}_2(v_1)\cdot\ldots\cdot\mathcal{F}_2(v_b)\Big)\Omega.$$

On-shell Bethe vector

Bethe vector becomes eigenvalue for transfer matrix $t(u)$

$$t(z)\mathbb{B}_{a,b}(\bar{u},\bar{v}) = \tau(z|\bar{u},\bar{v})\mathbb{B}_{a,b}(\bar{u},\bar{v})$$

with eigenvalue

$$\tau(z|\bar{u},\bar{v}) = \prod_{i=1}^{a} \frac{z-u_i}{(x^-(z)-x^-(u_i))(x^+(z)-x^-(u_i)^{-1})}$$
$$\left(\lambda_1(z)\prod_{i=1}^{a}\frac{z-u_i+\eta}{z-u_i} - \lambda_2(z)\prod_{i=1}^{a}\frac{z-u_i+\eta}{z-u_i}\prod_{j=1}^{b}\frac{v_j-z+\eta}{v_j-z}\right.$$
$$\left. - \lambda_3(z)\prod_{j=1}^{b}\frac{z-v_j+\eta}{z-v_j} + \lambda_4(z)\right)$$

Bethe equation

if Bethe equations are satisfied

$$\frac{\lambda_1(u_i)}{\lambda_2(u_i)} = \prod_{j=1}^{b} \frac{v_j - u_i + \eta}{v_j - u_i}, \qquad i = 1, \ldots, a,$$

$$\frac{\lambda_2(v_j)}{\lambda_3(v_j)} = \prod_{k=1, k\neq j}^{b} \frac{v_k - v_j - \eta}{v_k - v_j + \eta} \prod_{i=1}^{a} \frac{v_j - u_i}{v_j - u_i + \eta}, \qquad j = 1, \ldots, b.$$

Thank you for your attention!