

Integrable D-branes

and Lpt functions

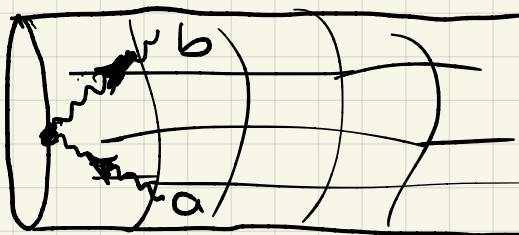
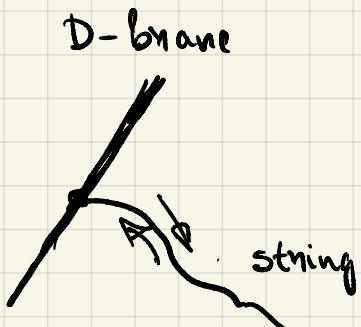
R. Zanelli (Nordita)

W. Charlotte Christjanson

&

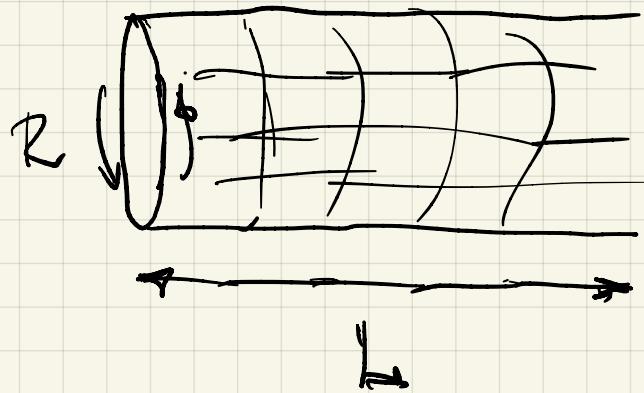
Dinh - Long Vu

Integrable D-branes



Reflection amplitude: $K_{ab}(p)$

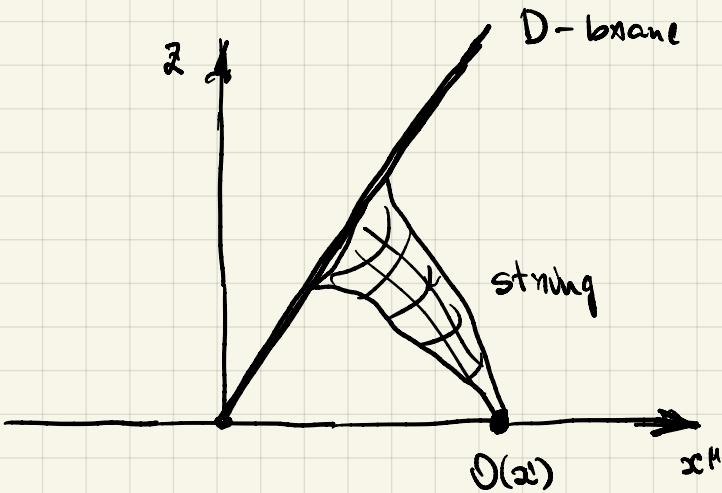
Elastic reflection \iff Integrability



$$Z(R, L) \xrightarrow{L \rightarrow \infty} g(R) - e^{-E_0 L}$$

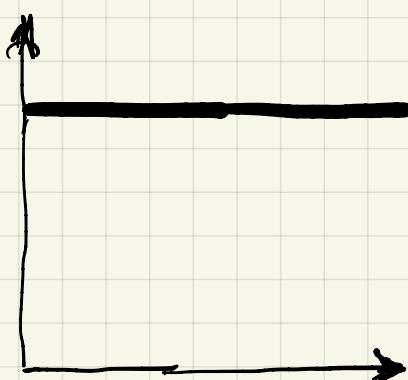
g -function
- calculable by TBA?

Affleck, Ludwig '81
Dorey, Fioravanti, Kim, Tateo '94
Pozsgay '10



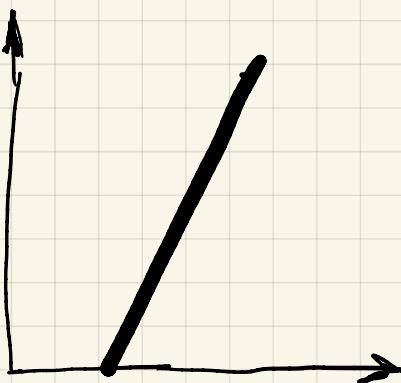
$$g\text{-function} = \langle O(x) \rangle$$

Jiang, Komatsu, Vesovi '19

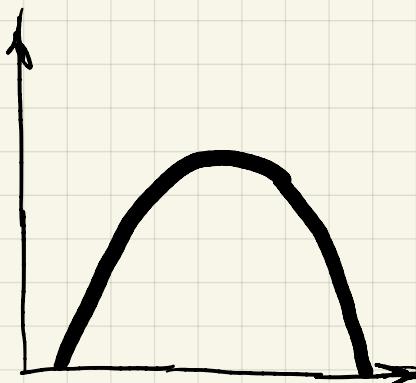


Coulomb branch:

$$\langle \Phi^i \rangle = n^i v$$



defect CFT



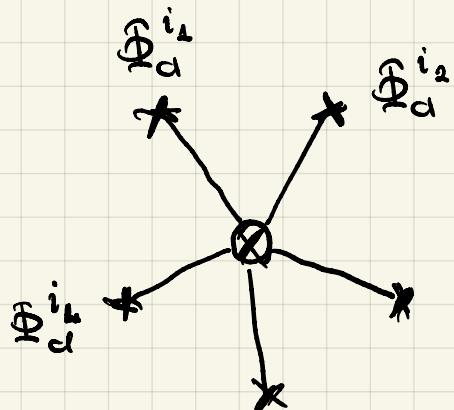
Heavy operation insertion:

$$\langle HH O \rangle$$

$\left. \right\}$
this talk

At weak coupling:

D-brane \leadsto Classical field configuration
(e.g. domain wall)



$$\mathcal{O} = \Phi_{i_1 \dots i_n} + \bar{\Phi}^{i_1 \dots i_n}$$

$$\langle \mathcal{O} \rangle = \langle B | \Psi \rangle$$

de Heue, Kristjansen, Z'LS

Boundary state:

$$B^{i_1 \dots i_n} = \text{tr } \Phi_a^{i_1} \dots \Phi_a^{i_n}$$

ABJM theory

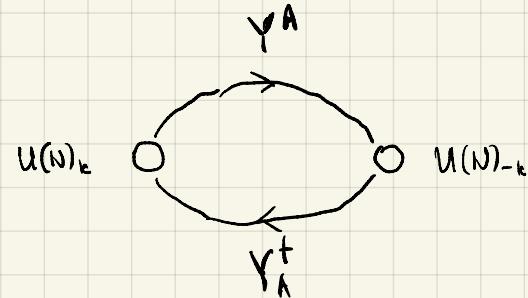
$$\mathcal{L} = \frac{k}{4\pi} \text{tr} \left[\epsilon^{\mu\nu\lambda} (A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2}{3} A_\mu A_\nu \hat{A}_\lambda \right. \\ + D_\mu Y_A^\dagger D^\mu Y^A + \frac{1}{12} Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + \frac{1}{12} Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C Y_A^\dagger \\ \left. - \frac{1}{2} Y^A Y_A^\dagger Y^B Y_C^\dagger Y^C Y_B^\dagger + \frac{1}{3} Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger + \text{fermions} \right]$$

Perturbative regime: $k \rightarrow \infty$.

Expansion parameter:

't Hooft coupling

$$\boxed{\mathcal{I} = \frac{N}{k}}$$



BPS domain walls

$$Y_\alpha(x) \quad \alpha = 1, 2 \quad x = x_2 \quad Y_{3+} = 0$$

Static energy:

$$\frac{E}{A_{\parallel}} = \frac{k}{4\pi} \int d\omega + \chi \left| \frac{dY_\alpha}{d\omega} - \frac{1}{2} Y^\alpha Y_\beta^+ Y^\beta + \frac{1}{2} Y^\beta Y_\beta^+ Y^\alpha \right|^2 + \text{total derivatives}$$

BPS eqs:

$$\frac{dY^\alpha}{d\omega} = \frac{1}{2} Y^\alpha Y_\beta^+ Y^\beta - \frac{1}{2} Y^\beta Y_\beta^+ Y^\alpha$$

Nahm's equations

$$\frac{dY^\alpha}{dx} = \frac{1}{2} Y^\alpha Y_\beta^+ Y^\beta - \frac{1}{2} Y^\beta Y_\beta^+ Y^\alpha$$

Composite field:

$$Y^\alpha Y_\beta^+ = \Phi^i \sigma_{i\beta}^\alpha + \Phi \epsilon_\beta^\alpha$$

$$\frac{d\Phi^i}{dx} = \frac{i}{2} \epsilon^{ijk} [\Phi^j, \Phi^k]$$

$$\frac{d\Phi}{dx} = \Phi^i \Phi_i - \Phi^2$$

same as in N=4 SYM?

Nahm-pole

solution

$$Y^\alpha Y_\beta^t = \frac{t^i \sigma_{i\beta}^\alpha - \frac{1}{2} \mathbb{1} \delta_\beta^\alpha}{x}$$

$$[t^i, t^j] = i \epsilon^{ijk} t^k$$

tⁱ define (q-1)-dim. rep. of su(2)

$$Y_\alpha^t Y_\beta^p = \frac{-\hat{t}_i \sigma_\alpha^i \delta_\beta^p + \frac{q-1}{2} \mathbb{1} \delta_\alpha^p}{x}$$

\hat{t}_i - q-dim. su(2) rep.

$$Y^\alpha = \frac{\partial}{\partial Y^\alpha}$$

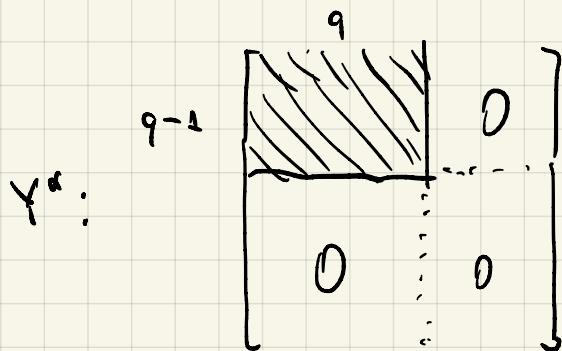
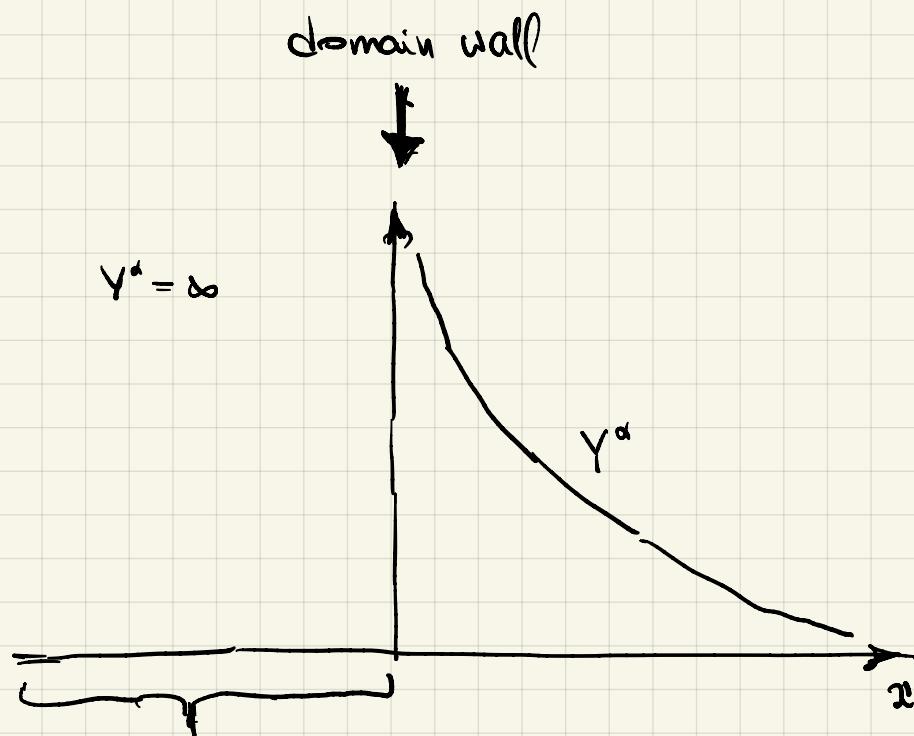
$$\delta_{ij}^1 = \delta_{i,j-1} \sqrt{i}$$

$$\tilde{\delta}_{ij}^2 = \delta_{ij} \sqrt{q-i}$$

$$i = 1 \dots q-1$$

$$j = 1 \dots q$$

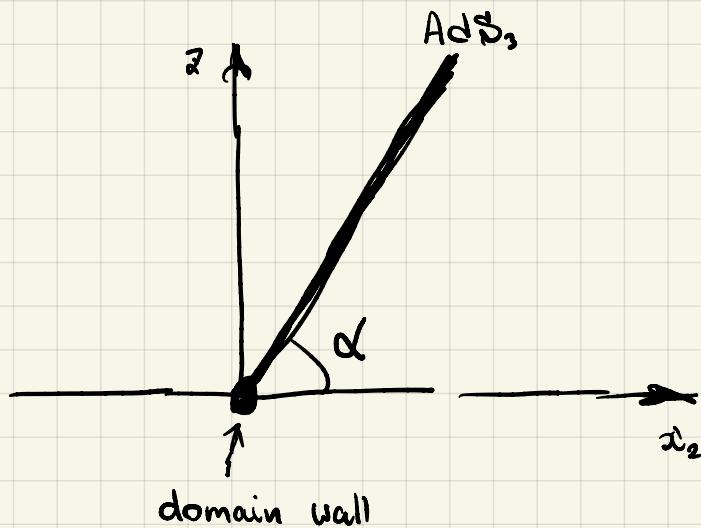
Terashima'08



$$U(N) \times U(N) \longrightarrow U(N-q+1) \times U(N-q)$$

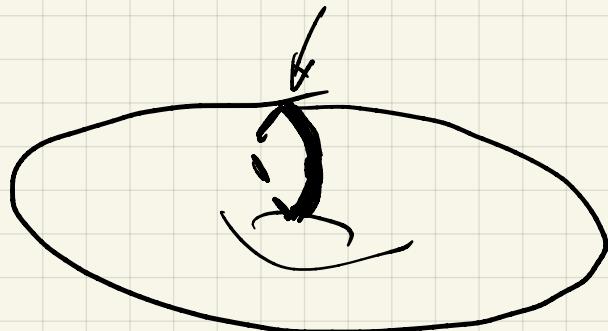
D2 - D4 dCFT

AdS_4 :



\mathbb{CP}^3 :

non-contractible \mathbb{CP}^1



D4 embedding:

$$\cot \alpha = \frac{q}{\sqrt{s}}$$

world-volume gauge field

$$\oint_{\mathbb{CP}^1} F = 2\pi q$$

$q = 2, 3, \dots$

- defines integrable boundary cond. in the σ -model

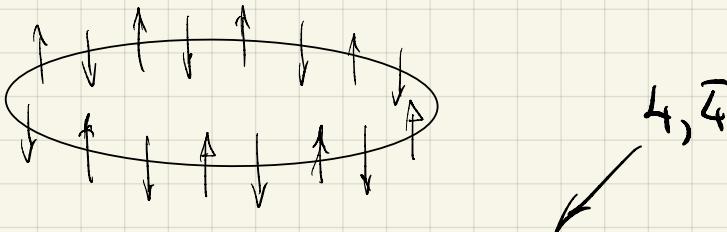
Linardopoulos'22

Local operators and the spin chain

$$\mathcal{O} = \Psi_{A_1}^{A_2} \dots \Psi_{A_{2L-1}}^{A_{2L}} \text{tr } Y^{A_1} Y_{A_2}^+ \dots Y^{A_{2L-1}} Y_{A_{2L}}^+$$

$$Y^A \leftrightarrow \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

$$Y_A^+ \leftrightarrow \begin{array}{c} \downarrow \\ \uparrow \end{array}$$



- Alternating $SU(4)$ spin chain of length $2L$

$$H = 2 \left(\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \right) - 2 \left(\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|} \hline & \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|} \hline & \\ \hline \end{array} \right)$$

- integrable à la de Vega - Woynarovich

de Vega, Woynarovich'92

Minahan, Z'08

One-point functions

$$\mathcal{O} = \Psi_{A_1 \dots A_L}^{B_1 \dots B_L} \text{tr } Y^{A_1} Y_{B_1}^+ \dots Y^{A_L} Y_{B_L}^+$$

Leading order in λ : $Y^A \rightarrow Y_A^A = \begin{cases} \frac{\mathcal{S}^A}{\sqrt{x_2}} & A=1,2 \\ 0 & A=3,4 \end{cases}$

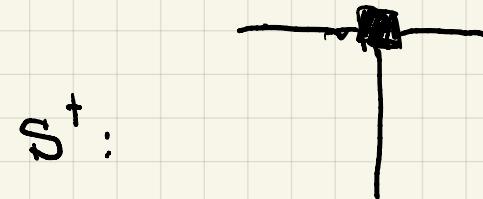
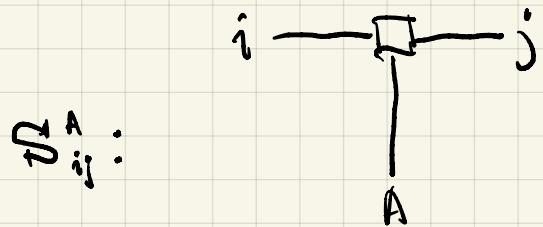
$$\langle \mathcal{O}(x) \rangle = \frac{1}{x_2^L} \cdot \frac{1}{2^L L^z} \cdot \frac{\langle \text{MPS} | \phi \rangle}{\langle \psi | \psi \rangle^{1/2}}$$

accounts for unit operator normalization: $\langle \mathcal{O} \mathcal{O} \rangle = \frac{1}{x^{2A}}$

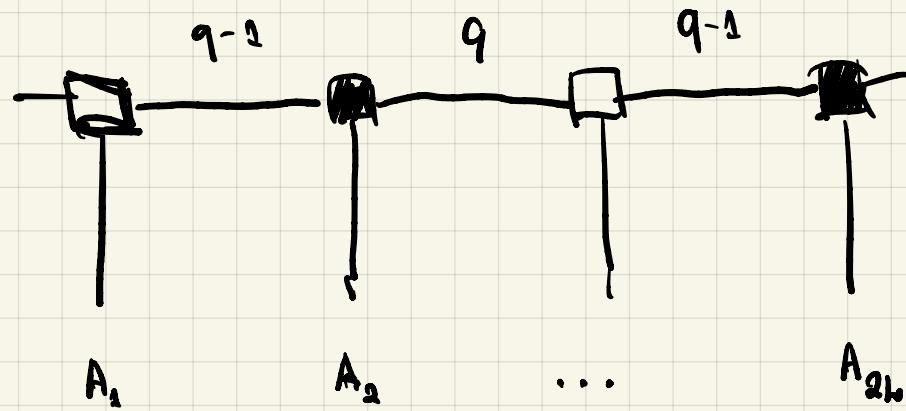
$$\text{MPS}_{B_1 \dots B_L}^{A_1 \dots A_L} = \text{tr } S^{A_1} S_{B_1}^+ \dots S^{A_L} S_{B_L}^+$$

 presence integrability!

Structure of the boundary state



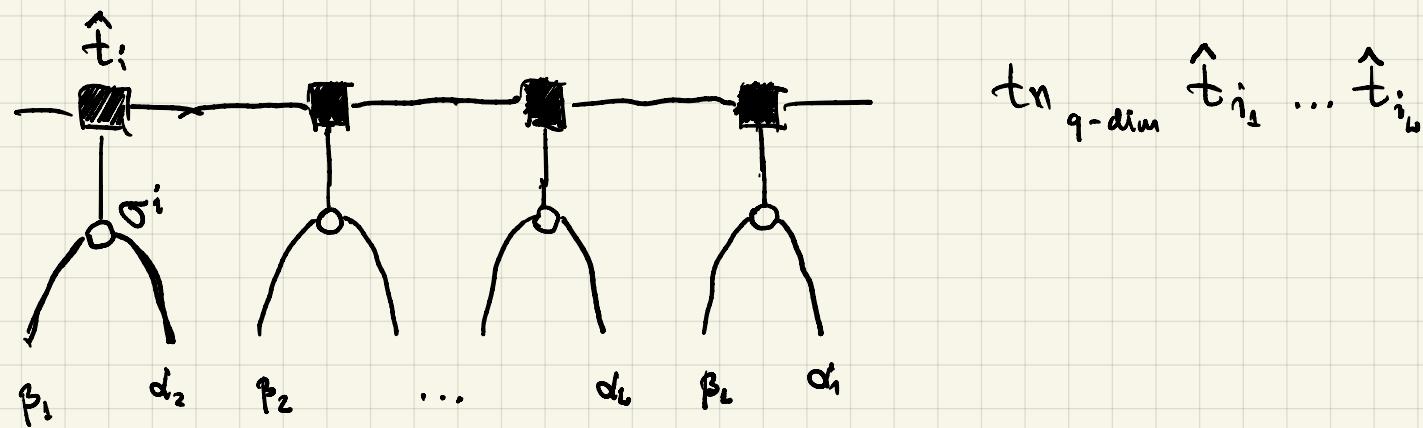
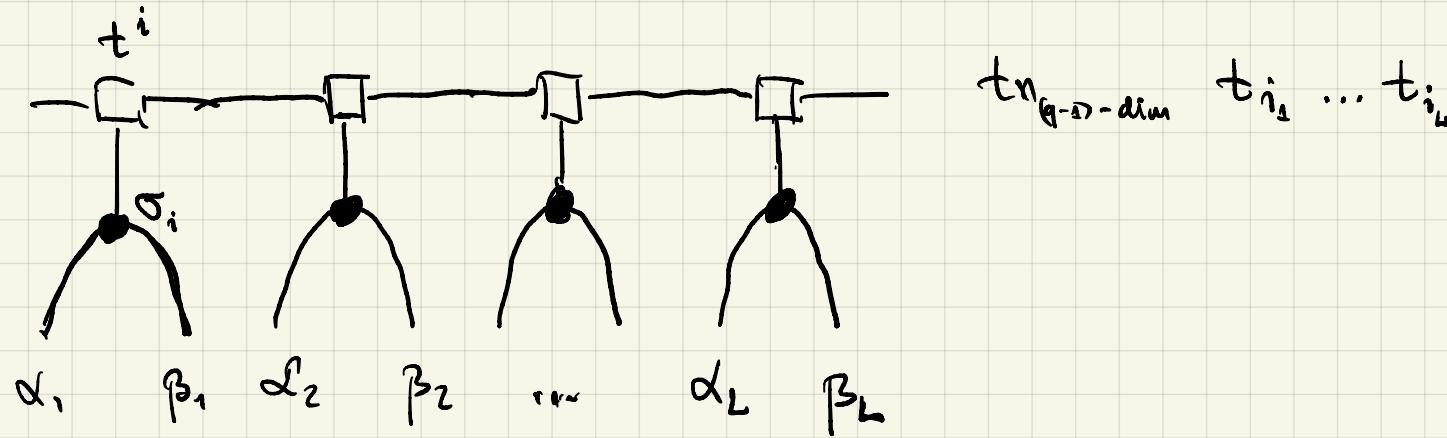
bond dimensions:



- Matrix product state

$$S_\alpha^\dagger S_\beta^\dagger = t^i \sigma_{i,\beta}^\alpha - \frac{g}{2} \delta_{\alpha\beta}$$

$$S_\alpha^\dagger S_\beta^\dagger = - \hat{t}_i \sigma_{\alpha i}^\beta + \frac{g-1}{2} \delta_{\alpha\beta}$$



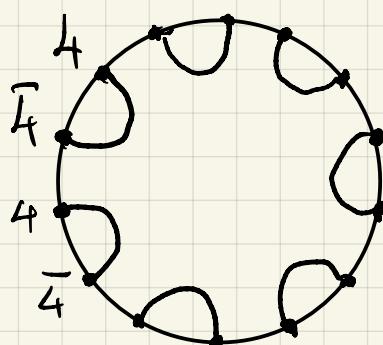
The simplest case: $\boxed{q=2} \Rightarrow$ rep. is trivial (bond dim = 1)

$$S^a S_p^t = \delta^a_p$$

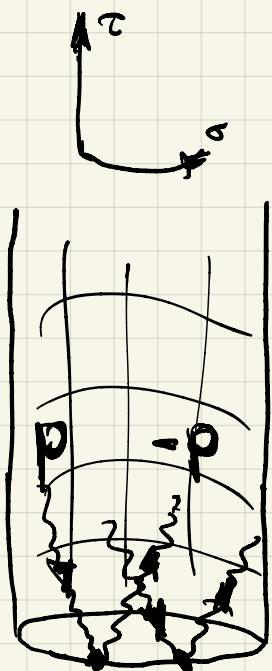
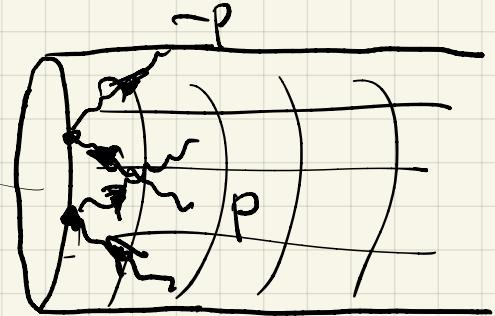
Boundary state is a Valence Bond State:

$$VBS_{\alpha_1 \dots \alpha_L}^{d_1 \dots d_L} = \delta_{\alpha_1}^{d_1} \dots \delta_{\alpha_L}^{d_L}$$

$$\alpha_i, \beta_i = 1, 2 \quad \text{otherwise } 0.$$



Both MPS and VBS are integrable



Integrable boundary state:

$$|B\rangle = \sum C_{\{p_i, -p_i\}}^3 | \{p_i, -p_i\} \rangle$$

Ghoshal, Zamolodchikov '93

Pivoli, Razguly, Vernier '17

- Magnons come in parity-symmetric pairs

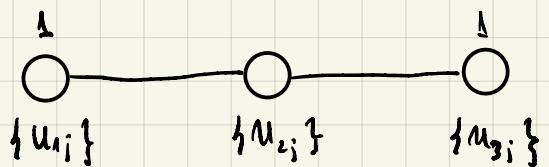
$|B\rangle$

One-point functions and integrability

$$\langle J \rangle \propto \frac{\langle B | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle^{\frac{1}{2}}}$$

$J \hookrightarrow$ Solution of Bethe ansatz eqs. $\{u_{\alpha j}\}$

$$\alpha = 1, 2, 3$$



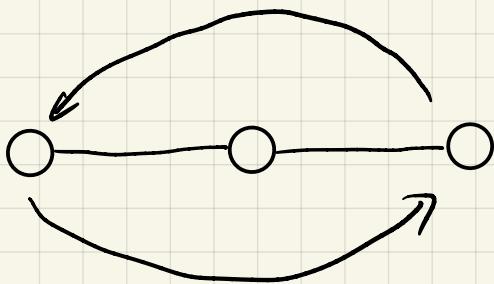
$$\langle VBS | \{u_{\alpha j}\} \rangle = ?$$

Selection rules

- Integrability $\iff \mathbb{Z}_2$ symmetry:

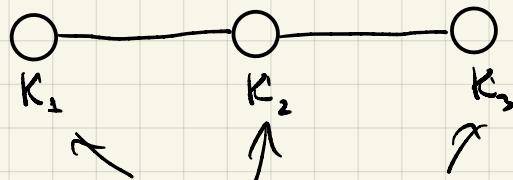
$$\Omega_h : u_{\alpha j} \mapsto -u_{\sigma(\alpha)j}$$

$$\sigma = (1\ 3)$$



VBS is $SU(2) \times SU(2)$ inv. and overlaps with highest weights for $SU(4)$

singlets only



$$K_1 = K_2 = K_3 = L$$

of Bethe roots

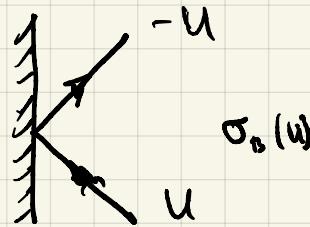
Overlap formulae

Overlap of Bethe states with an integrable boundary state:

$$\frac{\langle B | \psi_{u_i} \rangle}{\langle \psi_{u_i} | \psi_{u_i} \rangle^{1/2}} = \sqrt{\prod_j \sigma_b(u_j) \det_{ij} G}$$

Brockmann, De Nardis, Wouters, Caux '14,
de Leeuw, Kristjansen, Mori '15
Gombor'21
...

$\sigma_b(u)$ - boundary reflection phase



G - Gaudin matrix:

$$G_{jk} = \frac{\partial}{\partial u_k} \ln \text{BAE}_j$$

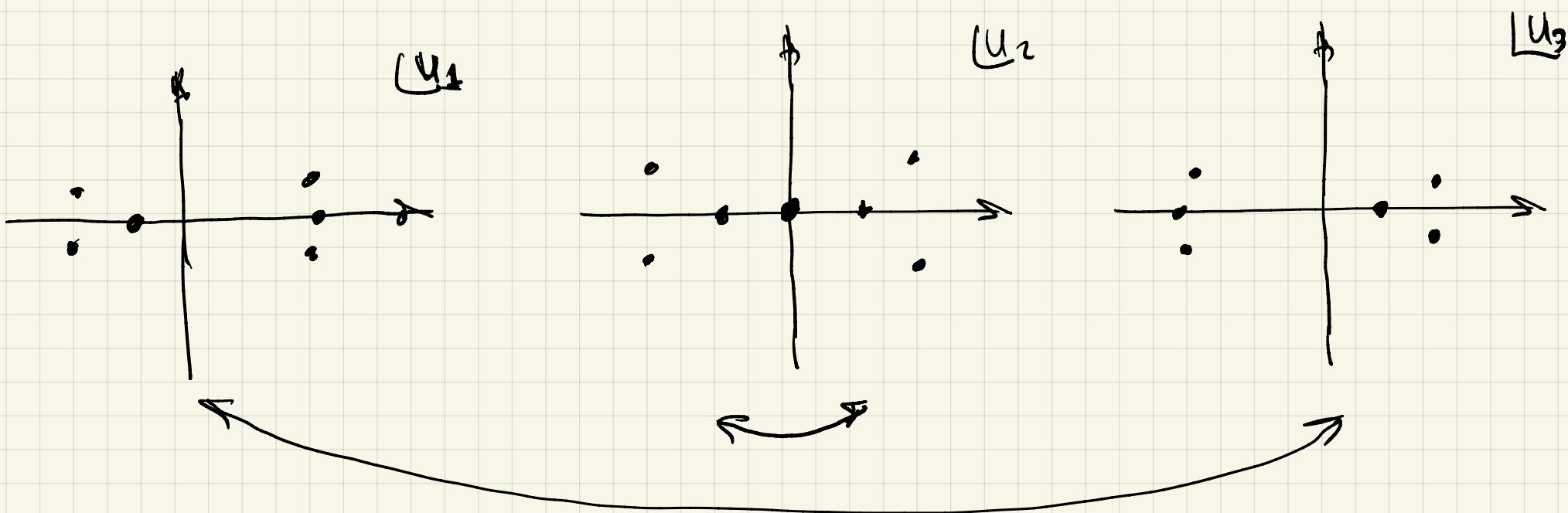
Nested BAE:

$$e^{iX_{\alpha j}} = \left(\frac{u_{\alpha j} - \frac{i\eta_\alpha}{2}}{u_{\alpha j} + \frac{i\eta_\alpha}{2}} \right)^L \prod_{b \neq j} \frac{u_{\alpha j} - u_{\alpha b} + \frac{iM_{\alpha}}{2}}{u_{\alpha j} - u_{\alpha b} - \frac{iM_{\alpha}}{2}} = -1$$

Gaudin matrix:

$$G_{\alpha j, \beta k} = \frac{\partial x_{\alpha j}}{\partial u_{\beta k}}$$

Selection rules:



Gaudin

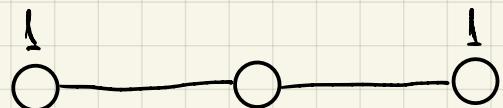
superdeterminant

$$S \det G = \frac{\det G^+}{\det G^-}$$

$$G_{aj, bk}^\pm = \left(\frac{L q_a}{M_{aj}^2 + \frac{q_a^2}{4}} - \sum_{cl} K_{aj, cl}^+ \right) \delta_{ab} \delta_{jk} + K_{aj, bk}^\pm$$

$$K_{aj, bk}^\pm = \frac{M_{ab}}{(M_{aj} - M_{bk})^2 + \frac{M_{ab}^2}{2}} \pm \frac{M_{ab}}{(M_{aj} + M_{bk})^2 + \frac{M_{ab}^2}{4}}$$

$$M = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad q = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$



Determinant representation of Ipt functions

$$\frac{\langle VBS | u \rangle}{\langle u | u \rangle^{\frac{1}{2}}} = 2^{-L} Q_2(i) \sqrt{\frac{S \det G}{Q_2(0) Q_2(\gamma_2)}}$$

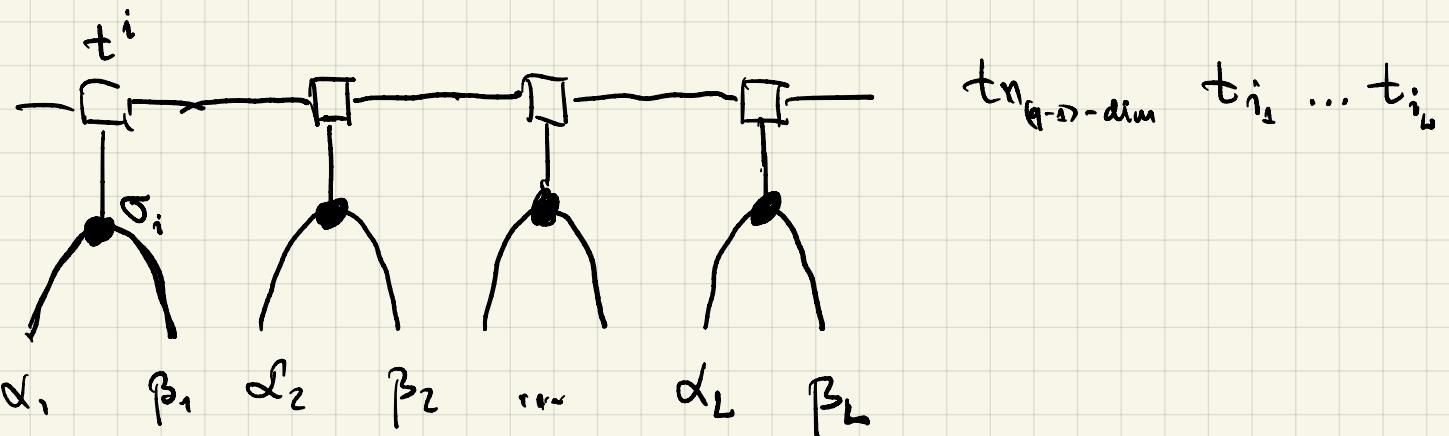
Gombor'21

$$Q_a(u) = \prod_j (u - u_{a,j}) \quad \text{- Baxter polynomial}$$

Ipt function of generic scalar operator, for $\boxed{\tau=2}$:

$$\boxed{\langle J(x) \rangle = \frac{L}{2\pi^L} \cdot \frac{1}{(2\lambda)^L L^{\frac{1}{2}}} \cdot \sqrt{\prod_j \frac{(u_{2j}^2 + 4)^2}{u_{2j}^2 (u_{2j}^2 + \frac{1}{4})}} \frac{\det G^+}{\det G^-}}$$

MPS:



bond dimension = $q-1 \rightsquigarrow$ character of $(q-1)$ -dim rep.

$$\text{overlap} = \sum_{l=1}^{q-1} (\dots)$$

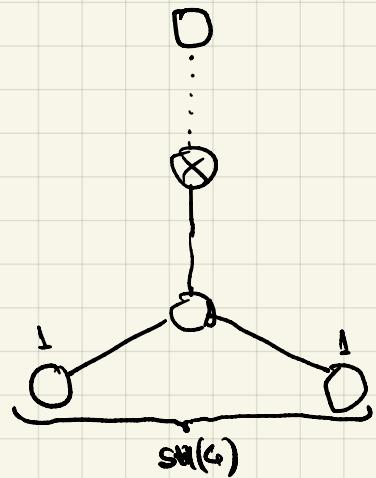
$$\langle \mathcal{O}(x) \rangle = \frac{1}{L^{d/2}} \sum_{l=1}^{q-1} \left(\frac{1}{2^{d/2}} \right)^L \prod_j \frac{\left[u_{2j}^2 + \frac{1}{4} \right] \left[u_{2j}^2 + \left(q - \frac{l}{2} \right)^2 \right]}{\left[u_{2j}^2 + \left(l - \frac{1}{2} \right)^2 \right] \left[u_{2j}^2 + \left(l + \frac{1}{2} \right)^2 \right]}$$

$$\sqrt{\prod_j \frac{\left(u_{2j}^2 + k^2 \right)^2}{u_{2j}^2 \left(u_{2j}^2 + \frac{1}{4} \right)} \cdot \frac{\det G^+}{\det G^-}}$$

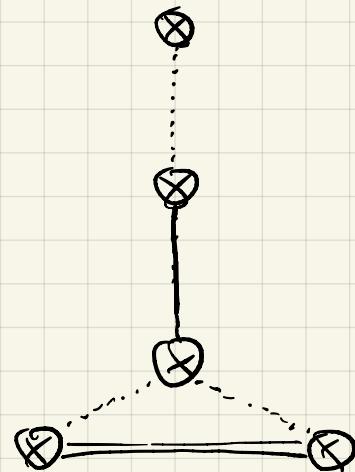
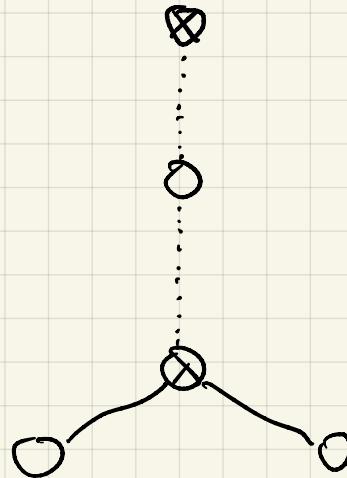
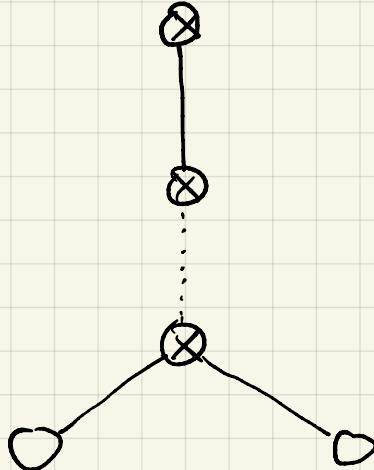
Gombor, Kristjánus'22

Arbitrarily Operations

$OSp(6|4)$ Dynkin diagram:



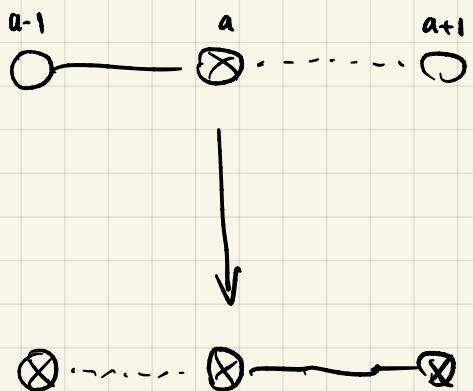
Different gradings:



• related by fermionic duality

Fermionic duality

Tsuboi '98



$$(Q_{a-1}^-, Q_{a+1}^+ - Q_{a-1}^+ Q_{a+1}^-) = i(K_{a+1} - K_{a-1}) Q_a \tilde{Q}_a$$

$\deg K_{a+1} + \deg K_{a-1} - \deg K_a - 1$
 \uparrow
 \downarrow
 $\deg K_a$
 $f^\pm(u) \equiv f(u \pm \frac{i}{2})$

Gaudin superdeterminant transforming covariantly:

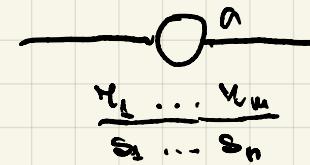
$$Q_{a-1}\left(\frac{i}{2}\right) Q_{a+1}\left(\frac{i}{2}\right) S\det \tilde{G} = Q_a(0) \tilde{Q}_a(0) S\det G$$

Kristjansen, Müller, Z. '21

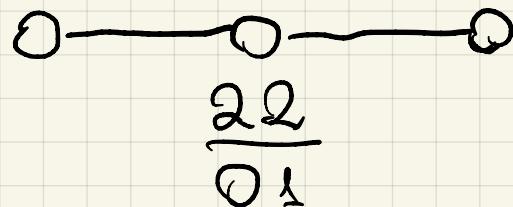
Transformation rules for overlaps

Graphic notations:

$$\frac{\langle B | u \rangle}{\langle u | u \rangle^{\frac{1}{2}}} = \sqrt{\prod_a \frac{\pi Q_a(\frac{i\eta_a}{2})}{\pi Q_a(\frac{i\sigma_a}{2})}} \quad S \det G$$



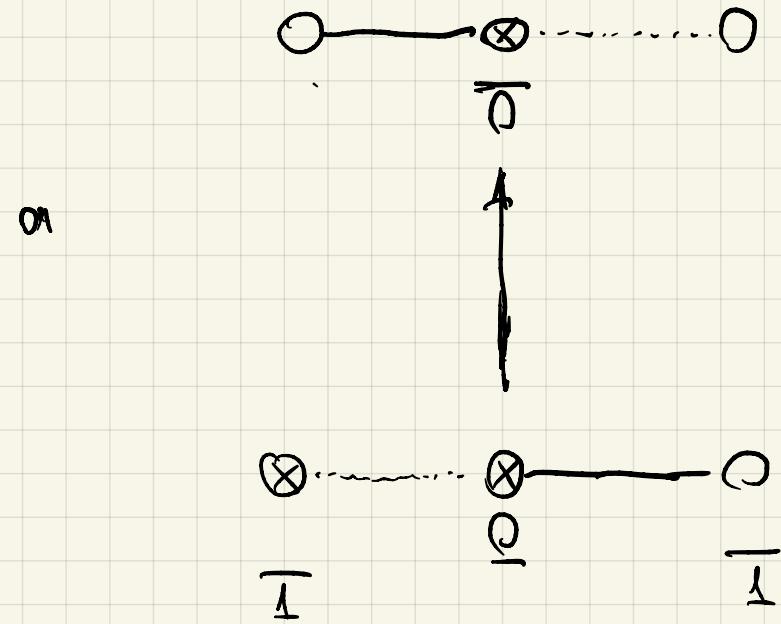
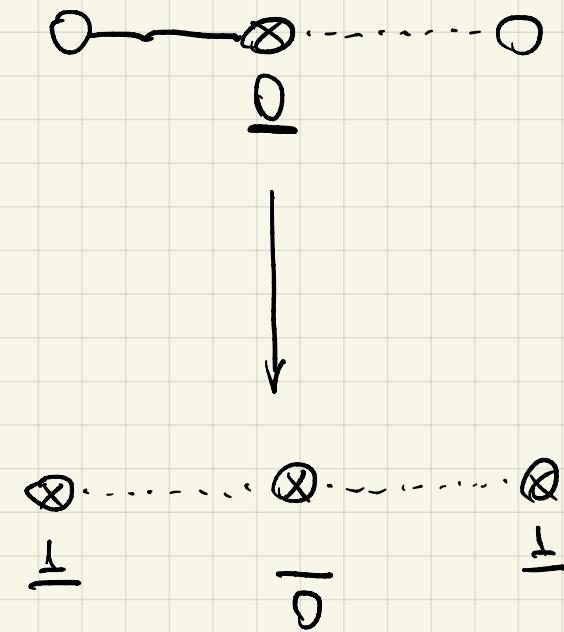
Ex: su(4) overlap



$$\frac{\langle VBS | u \rangle}{\langle u | u \rangle^{\frac{1}{2}}} = 2^{-L} \frac{S \det G}{Q_2(0) Q_2(\frac{1}{2})}$$

Overlap transforms nicely under fermionic duality

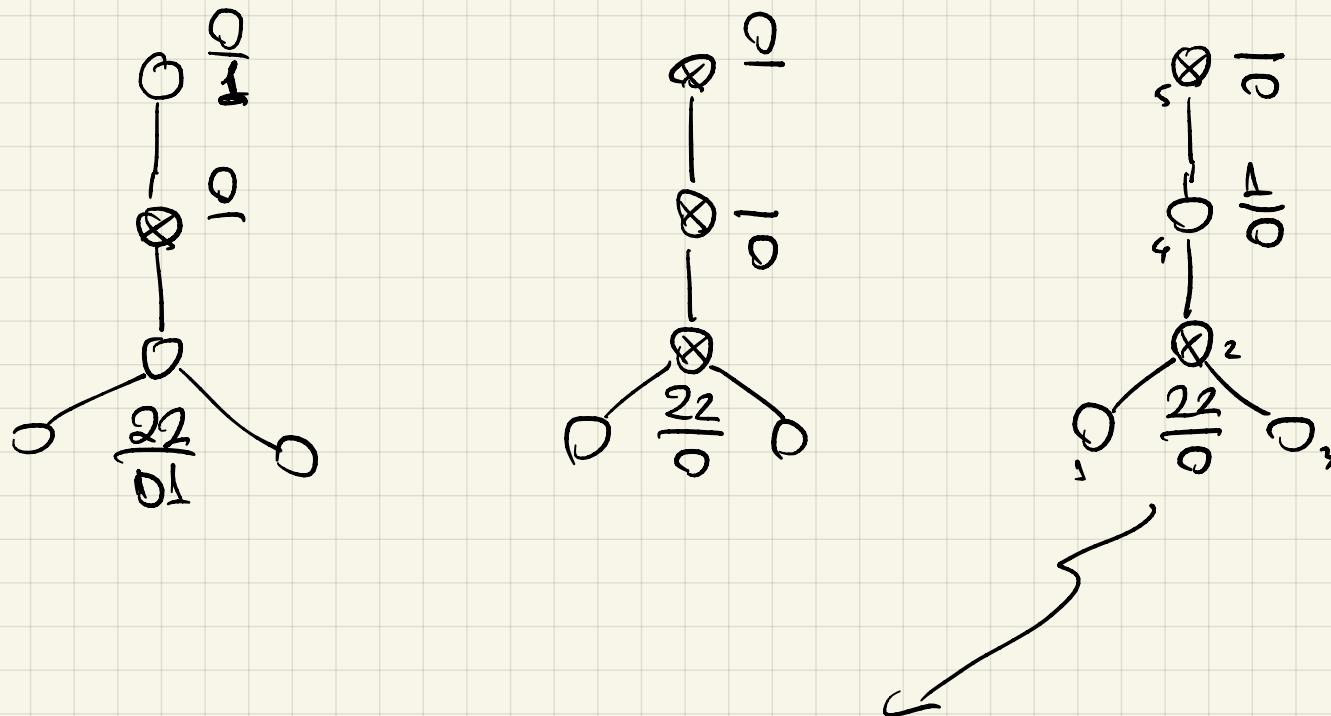
provided the pre-factor contains $Q_a(\theta)$ or $\frac{1}{Q_a(\theta)}$:



This by itself imposes strong constraints
on the possible form of the overlap formula.

Complete overlap formulae for $Osp(6|4)$

Extension of overlap formula from $su(k)$ to $Osp(6|4)$, consistent with fermionic dualities,
 is unique:



$$\frac{\langle VBS | u \rangle}{\langle u | u \rangle^{\frac{1}{2}}} = Q_2(i) \sqrt{\frac{Q_4(\frac{i}{2})}{Q_2(0) Q_4(0) Q_6(0)}} S \det G$$

Conclusions

- Integrable D-branes in $AdS_{d+1} \times X_{d-1}$ ↪ Integrable boundary states in the spin chain
- Exact universal formula for 1pt functions based on integrable overlaps
- Ex.: BPS domain walls in ABJM

- Extension to higher loops?

Buhk-Mortensen, de Buyl, Ipsen, Kristjansen, Wilhelms

Komatsu, Wang'20

Balogh, Gombor'20

- TBA?

Cartano, Komatsu'20

- Other setups: heavy-heavy-light 3pt functions, Coulomb branch, Wilson loops, surface operators, ...?