Crosscap States, Integrability, and Holography

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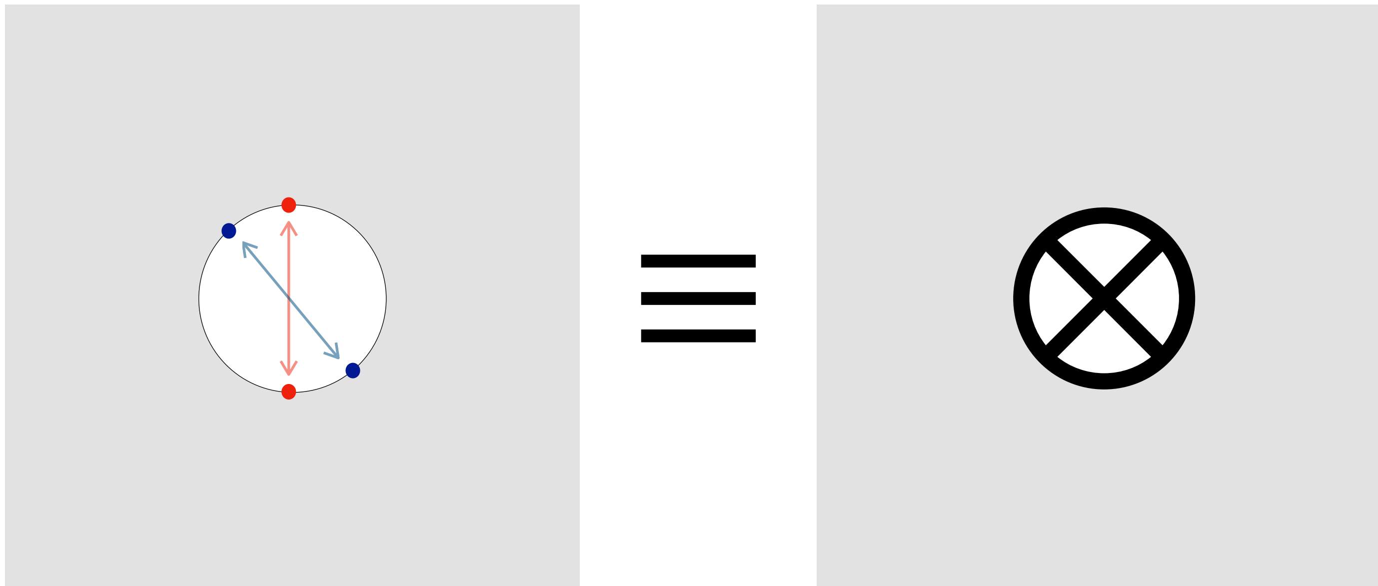
CERN

Based on 2111.09901 with **Joao Caetano**
+ work in progress with **Joao Caetano**, **Leonardo Rastelli**, **Paolo Soresina**

cf. 2207.10598 by **Tamas Gombor**

What is crosscap?



$$z \sim -1/\bar{z}$$

- Cut out a disk and perform antipodal identification.
- Surface becomes non-orientable.

Punchline

- Integrability survives in the presence of crosscaps.

- Overlap between the crosscap state and an excited state can be computed analytically both in field theory and spin chain.

$$\langle \mathscr{C} | \Psi \rangle$$

- They compute 1-pt functions in N=4 SYM on RP_4 with charge conjugation.

By-products

- Generalize Zamolodchikov's staircase model to D-series minimal model.
- "Fermionization" of integrable field theories. [Petkova], [Hsieh, Nakayama, Tachikawa]…

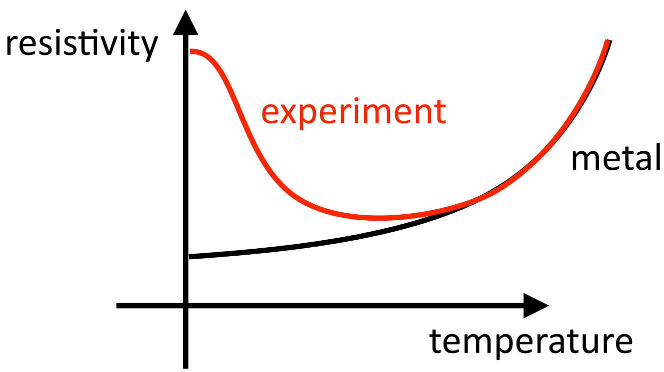
Plan of the talk

1. Why crosscaps?
2. Crosscap overlap in integrable field theories
3. Crosscap states in integrable spin chains
4. Crosscap states and N=4 SYM
5. Conclusion

Why ~~crosscaps~~? boundaries

- Boundaries in 1+1-d systems have attracted much attention.

- Successful example: Kondo problem.



Kondo model:
Free fermion + magnetic impurity

s-wave dimensional reduction

1+1 d fermion with boundary.
Solvable by Integrability, CFT

Andrei, Wiegmann, Affleck, Ludwig….

cf. Callan-Rubakov effect

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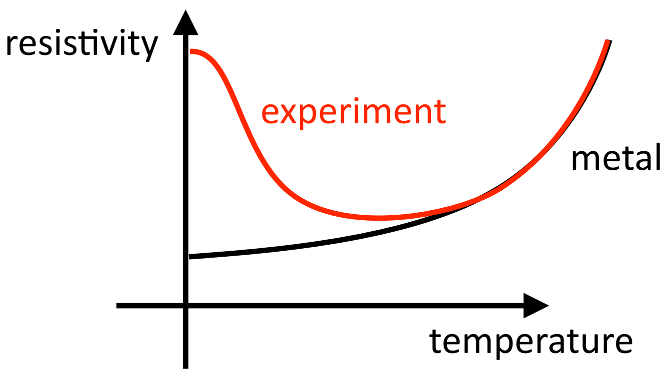

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- 2d CFT with boundary: D-branes in string theory

- Boundary state in CFT/spin chains: quantum quench.

Calabrese, Cardy, Caux, Essler, Brockman, De Nardis, Wouters, Pozsgay

- Application to N=4 SYM: defect 1pt, determinant 3pt,….

Bohl-Mortensen, de Leeuw, Kristjansen, Zarembo, Wilhelm, Jiang, Komatsu
Vescovi, Wang, Bajnok Gombor,……

Why crosscaps?

- In 2d CFTs, there exists another important class of states: Crosscap

- 2d system on non-orientable surfaces, 2+1 d Symmetry Protected Topological phase with time reversal symmetry.

- Orientifold in string theory: Important in string compactifications, in particular for the construction of de Sitter vacua.

cf. Maldacena, Nunez

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cf. Maldacena, Nunez

- Almost no literature for integrable field theories and spin chains.

- As we will see, the crosscap states in spin chain provide new “integrable” initial states for quantum quench: Long-range entangled, volume law…

- The analysis in N=4 SYM suggests a new class of “defects” in higher-d CFTs: “orientifold defects”

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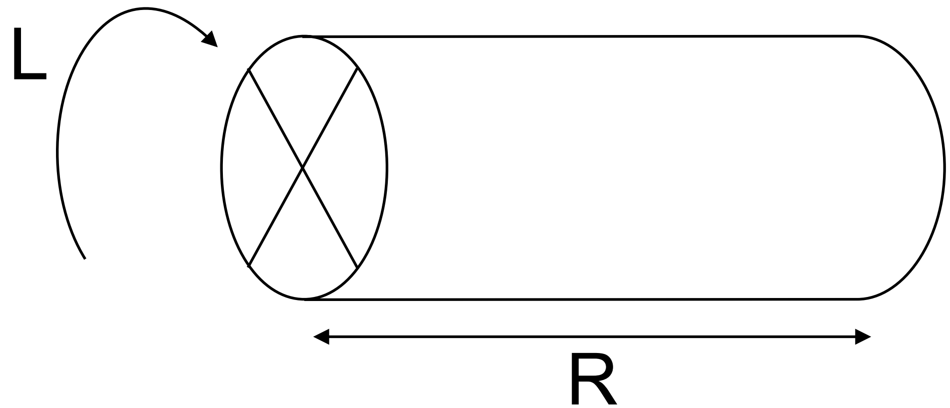
Crosscap overlap in integrable field theories

- Set up:



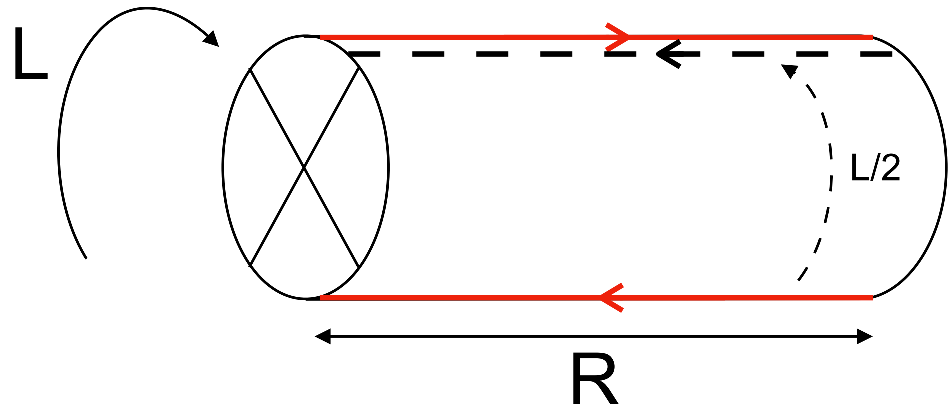
- Tree-channel expansion

$$Z_{\mathbb{K}} = \sum_{\psi_L} |\langle \mathscr{C} | \psi_L \rangle|^2 e^{-E_{\psi_L} R} \quad \overset{R\to\infty}{\to} \quad |\langle \mathscr{C} | \Omega_L \rangle|^2 e^{-E_{\Omega_L} R}$$

Compute the partition function and take $R \to \infty$ limit: Crosscap overlap

Crosscap overlap in integrable field theories

- Loop-channel expansion



$$Z_{\mathbb{K}} = \mathrm{tr}_{2R}\left[\Pi\, e^{-L\hat{H}/2}\right] = \sum_{\psi_{2R}} \epsilon_{\psi_{2R}} e^{-LE_{\psi_{2R}}/2}$$

Π : Parity operator

In integrable theories, states are labelled by rapidities of particles.

$$|\{p_j\}\rangle \qquad \Pi|\{p_j\}\rangle \propto |\{-p_j\}\rangle$$

- $\langle\{p_j\}|\Pi|\{p_j\}\rangle = 0$ unless $\{p_j\} = \{-p_j\}$.
- If $\{p_j\} = \{-p_j\}$, $\epsilon_\psi = +1$.

(Nontrivial but can be proven by using the structure of Bethe-ansatz wave function)

Crosscap overlap in integrable field theories

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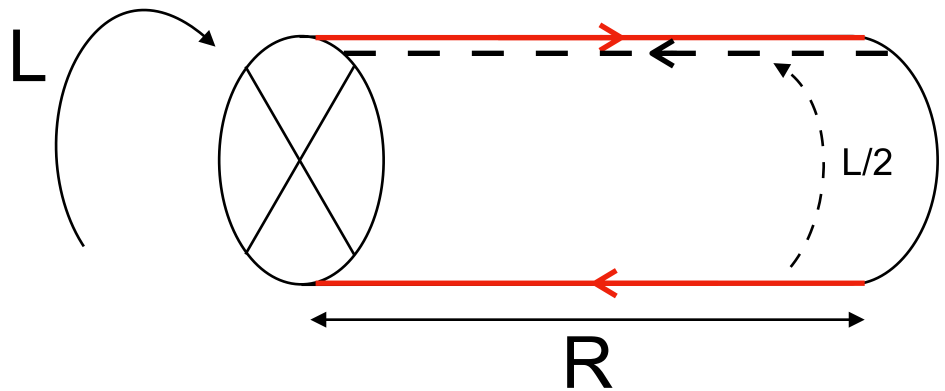

$$Z_{\mathbb{K}} = \mathrm{tr}_{2R}\left[\Pi\, e^{-L\hat{H}/2}\right] = \sum_{\underline{\{p_j\}=\{-p_j\}}} e^{-L\sum_j E(p_j)/2}$$

Parity-invariant solutions to Bethe eq.

Crosscap overlap in integrable field theories

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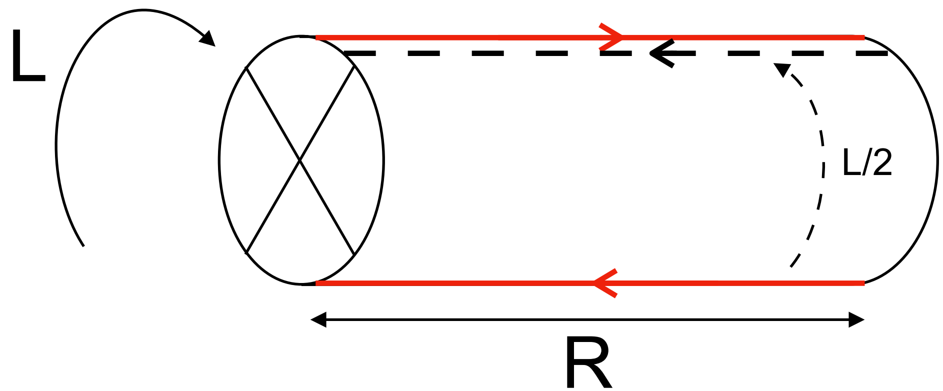

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Parity-invariant solutions to Bethe eq.

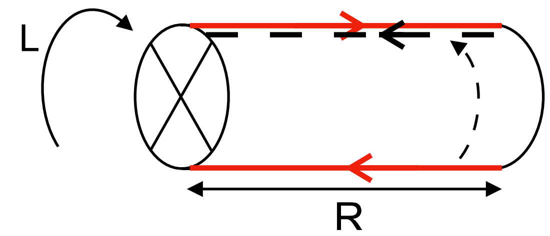
- Loop = Tree

$$\underline{\lim_{R\to\infty} \sum_{\{p_j\}=\{-p_j\}} e^{-L\sum_j E(p_j)/2}} = |\langle \mathscr{C} | \Omega_L \rangle|^2 e^{-E_{\Omega_L} R}$$

Could be dealt with by thermodynamic Bethe ansatz if there was no parity constraint

How to deal with parity constraint

$$\lim_{R\to\infty}\sum_{\{p_j\}=\{-p_j\}} e^{-L\sum_j E(p_j)/2} = |\langle \mathscr{C} | \Omega_L \rangle|^2 e^{-E_{\Omega_L} R}$$



- (Asymptotic) Bethe equation for parity invariant states:

$$\mathbf{S}: \qquad 1 = e^{2ip_jR} S(p_j,-p_j) \prod_{k\neq j} S(p_j,p_k) S(p_j,-p_k)$$

$$\mathbf{T}: \qquad 1 = e^{2ip_jR} S(p_j,-p_j) S(p_j,0) \prod_{k\neq j} S(p_j,p_k) S(p_j,-p_k)$$

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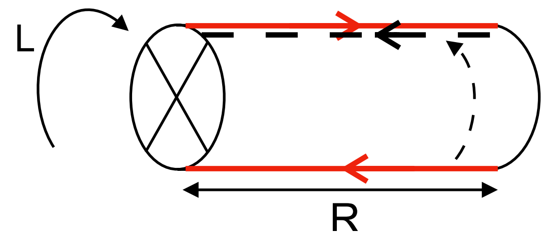

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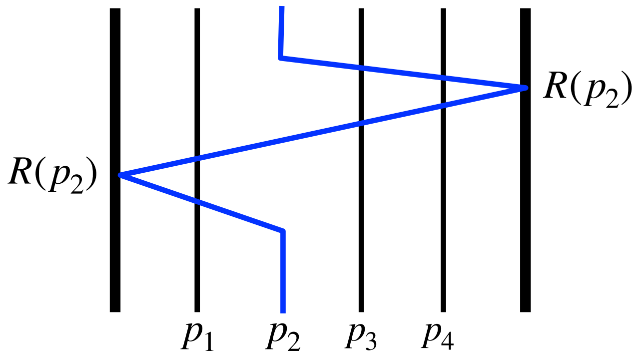
- Crucial observation: formally identical to boundary problem

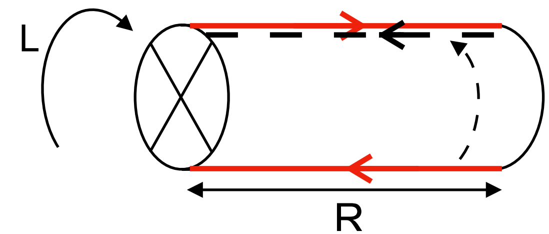


$$1 = e^{2iRp_j} (R(p_j))^2 \prod_{k\neq j} S(p_j,p_k) S(p_j,-p_k)$$

$$(R(p))^2 \leftrightarrow \begin{cases} S(p,-p) & \mathbf{S} \\ S(p,-p)S(p,0) & \mathbf{T} \end{cases}$$

How to deal with parity constraint

$$\lim_{R\to\infty} \sum_{\{p_j\}=\{-p_j\}} e^{-L\sum_j E(p_j)/2} = |\langle \mathscr{C} | \Omega_L \rangle|^2 e^{-E_{\Omega_L} R}$$



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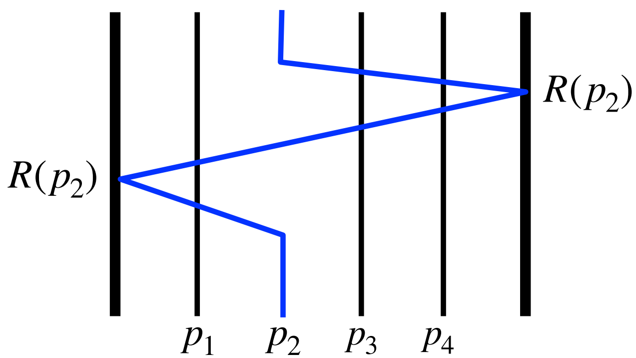

$$1 = e^{2iRp_j} (R(p_j))^2 \prod_{k\neq j} S(p_j, p_k) S(p_j, -p_k)$$

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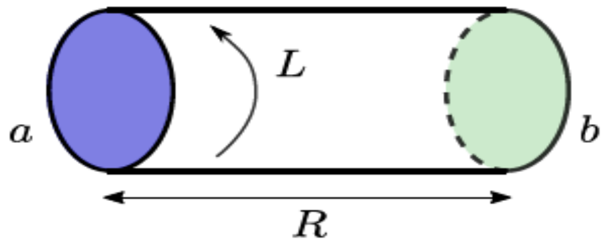
We can recycle the result for boundary problem

Boundary Entropy: Review



$$\overset{\text{closed string}}{\underset{R \gg 1}{=}} \sum_{\psi_n} |\langle B|\psi_n\rangle|^2 e^{-RE_n}$$

- Closed-form expression (diagonal S-matrix without bound states)

$$|\langle B|\Omega_L\rangle|^2 = \exp\left[2\int_0^\infty \frac{du}{2\pi}\Theta(u)\log(1+Y(u))\right]\frac{\det\left[1-\hat{G}_-\right]}{\det\left[1-\hat{G}_+\right]}$$

- TBA equation: $0 = LE(u) + \log Y(u) - \log(1+Y) \star \mathcal{K}_+$

$$A \star B = \int_0^\infty \frac{dv}{2\pi} A(u,v)B(v)$$

- Kernel: $\mathcal{K}_\pm(u,v) \equiv \frac{1}{i}\left(\log S(u,v) \pm S(u,-v)\right)$
- Fredholm det: $\hat{G}_\pm \cdot f(u) = \int_0^\infty \frac{dv}{2\pi}\frac{\mathcal{K}_\pm(u,v)}{1+1/Y(v)} f(v)$
- Prefactor: $\boxed{\Theta(u) = \frac{1}{i}\partial_u \log R(u) - \pi\delta(u) - \frac{1}{2i}\partial_u \log S(u,-u)}$

$$(R(p))^2 \leftrightarrow \begin{cases} S(p,-p) & \mathbf{S} \\ S(p,-p)S(p,0) & \mathbf{T} \end{cases}$$

Final result

$$|\langle \mathscr{C} | \Omega_L \rangle| = \sqrt{\left(1 + \sqrt{\frac{Y(0)}{1 + Y(0)}}\right) \frac{\det\left[1 - \hat{G}_-\right]}{\det\left[1 - \hat{G}_+\right]}}$$

Final result

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- Excited state:

  cf. [Dorey, Tateo]

$$|\langle \mathscr{C} | \Psi \rangle| = \sqrt{\left(1 + \sqrt{\frac{Y(0)}{1+Y(0)}}\right) \frac{\det\left[1 - \hat{G}_-^{\bullet}\right]}{\det\left[1 - \hat{G}_+^{\bullet}\right]}}$$

$$\hat{G}_{\pm}^{\bullet} \cdot f(u) = \sum_k \frac{i\mathscr{K}_{\pm}(u, u_k)}{\partial_u \log Y(u_k)} f(u_k) + \int_0^{\infty} \frac{du}{2\pi} \frac{\mathscr{K}_{\pm}(u, v)}{1 + 1/Y(v)} f(v)$$

- Asymptotic limit ($L \to \infty$): $\quad |\langle \mathscr{C} | \Psi \rangle| = \sqrt{\frac{\det\left[G_+\right]}{\det\left[G_-\right]}}$

$$\left(G_{\pm}\right)_{1 \le i,j \le \frac{M}{2}} = \left[L \partial_u p(u_i) + \sum_{k=1}^{\frac{M}{2}} \mathscr{K}_+(u_i, u_k)\right] - \mathscr{K}_{\pm}(u_i, u_j)$$

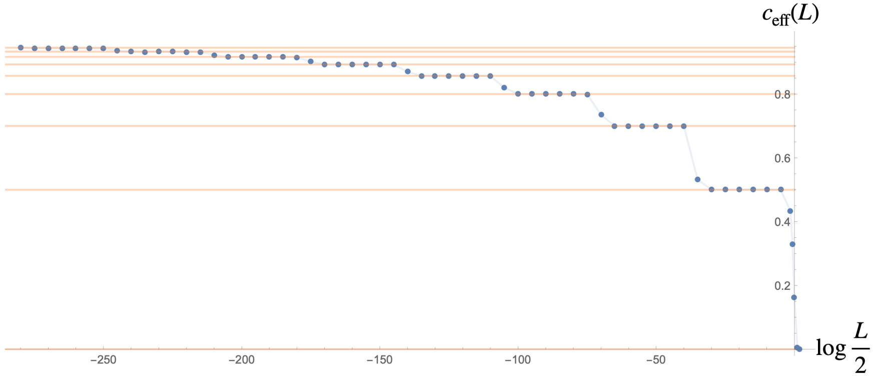
“Simplest possible” g-function

Check (Staircase model)

- Definition of the model:

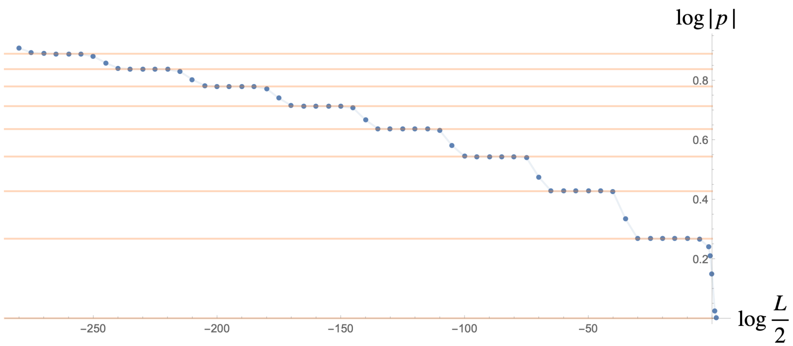
$$S(\theta) = \tanh\left(\frac{\theta - \theta_0}{2} - \frac{i\pi}{4}\right)\tanh\left(\frac{\theta + \theta_0}{2} - \frac{i\pi}{4}\right)$$

- Effective central charge (ground state energy):



RG flows interpolate between unitary minimal models.

- Crosscap overlap for the ground state.



Values at plateaux agree with A-series minimal models:

$$\left(\frac{2}{m(m+1)}\right)^{\frac{1}{4}}\sqrt{\cot\frac{\pi}{2m}\cot\frac{\pi}{2(m+1)}}$$

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1. Why crosscaps?
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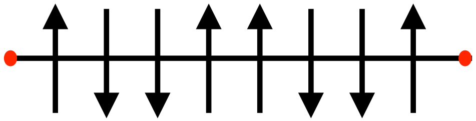
Boundaries in spin chain

- Open spin chain (= boundaries in space)



Well-studied in the (integrability) literature since 80's

$$H = H_{\text{bulk}} + H_{\text{boundary}}$$

- "Boundary states" in closed spin chain (= boundaries in time)

$$|B\rangle = \prod_j (|K\rangle\rangle_j)^{\otimes} \qquad |K\rangle\rangle_j = c_0 |\uparrow\uparrow\rangle_{j,j+1} + c_1 |\uparrow\downarrow\rangle_{j,j+1} + c_2 |\downarrow\uparrow\rangle_{j,j+1}$$

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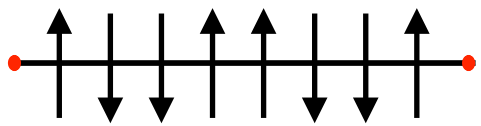

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  - Almost a product state. Defined locally, much like a boundary state in field theory.
  - Short-range entangled, ideal initial state for studying time evolution of entanglement.
  - Simple formula in integrable spin chains.

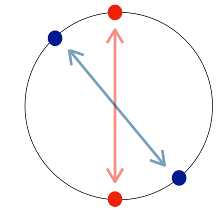
Caux, Essler, Brockman, De Nardis, Wouters, Pozsgay

$$|\langle \mathscr{B} | \Psi \rangle| = (\text{some prefactors})\sqrt{\frac{\det [G_+]}{\det [G_-]}} \qquad (G_\pm)_{1\le i,j\le \frac{M}{2}} = \left[L\partial_u p(u_i) + \sum_{k=1}^{\frac{M}{2}} \mathscr{K}_+(u_i,u_k) \right] - \mathscr{K}_\pm(u_i,u_j)$$

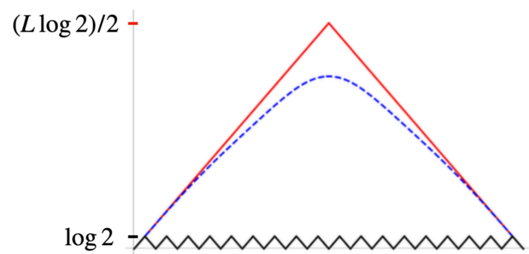
Can we define a crosscap state in spin chain?

- Naive guess (for XXX spin 1/2 chain):

$$|c\rangle\!\rangle_j \equiv |\uparrow\rangle_j \otimes |\uparrow\rangle_{j+\frac{L}{2}} + |\downarrow\rangle_j \otimes |\downarrow\rangle_{j+\frac{L}{2}} \qquad |\mathcal{C}\rangle \equiv \prod_{j=1}^{\frac{L}{2}} (|c\rangle\!\rangle_j)^{\otimes}$$



Long-range entangled.



- The state preserves infinitely many conserved charges:

$$\langle C|\left[\boxed{T(u)} - T(-u)\right] = 0$$

Transfer matrix in the fundamental rep.

Overlap formula

- We also computed the overlap between the crosscap and Bethe eigenstates.

- We found (numerically, without a proof) that they admit a very simple (Gaudin-like) determinant formula.

Proven recently by Gombor!
arXiv:2207.10598

$$|\langle C|\Psi\rangle| = \sqrt{\frac{\det\left[G_+\right]}{\det\left[G_-\right]}} \qquad \left(G_\pm\right)_{1\leq i,j\leq \frac{M}{2}} = \left[L\partial_u p(u_i) + \sum_{k=1}^{\frac{M}{2}} \mathcal{K}_+(u_i,u_k)\right] - \mathcal{K}_\pm(u_i,u_j)$$

1. Coincides with the asymptotic limit of the overlap formula for integrable QFTs.
2. Similar to but simpler than an overlap formula for the spin-chain boundary states.

Caux, Essler, Brockman, De Nardis, Wouters, Pozsgay

$$|\langle B|\Psi\rangle| = (\text{non-universal prefactor})\sqrt{\frac{\det\left[G_+\right]}{\det\left[G_-\right]}}$$

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A new tractable initial condition for quantum quench…?

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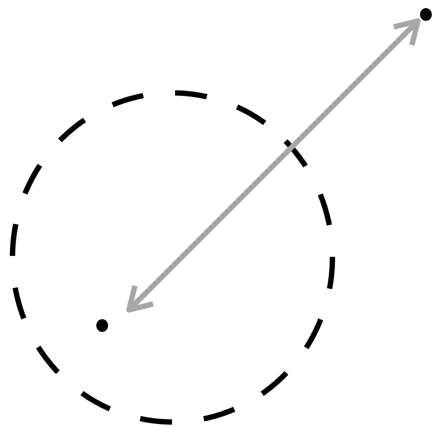
Crosscap and N=4 SYM

- Crosscap overlaps on the world sheet = 1pt functions in N=4 SYM on RP_4 with charge conjugation.

- 4d CFT on RP_4.



$$x^\mu \quad \leftrightarrow \quad -\frac{x^\mu}{|x|^2}$$

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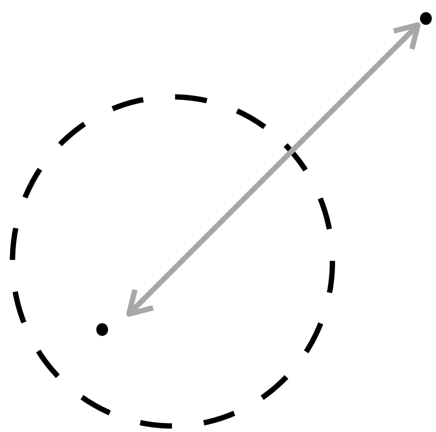
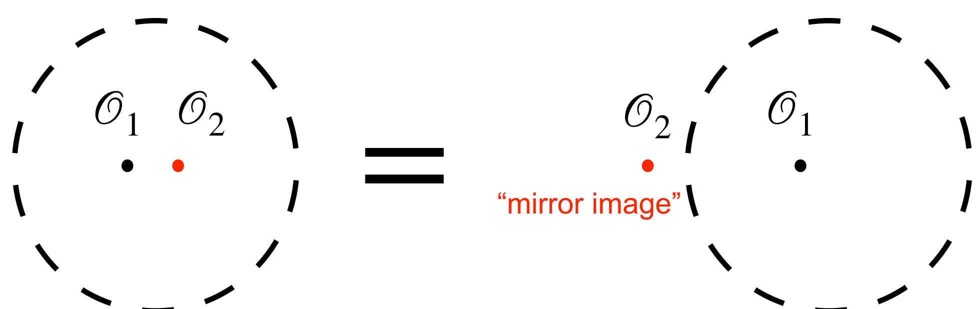


$$x^\mu \quad \leftrightarrow \quad -\frac{x^\mu}{|x|^2}$$

- New conformal data: $\langle \mathcal{O} \rangle_{\mathrm{RP}_4}$
- Crossing equation:



N=4 SYM on RP_4

- In N=4 SYM, we can define a "half-BPS version" of RP_4.

$$x^\mu \quad \leftrightarrow \quad -\frac{x^\mu}{|x|^2} \qquad \boldsymbol{+} \qquad \Phi_{1,2,3} \leftrightarrow \Phi_{1,2,3} \quad \Phi_{4,5,6} \leftrightarrow -\Phi_{4,5,6}$$

- This turns out to be **non-integrable.** [Caetano, Rastelli, 2022]

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- Integrability gets restored if we combine it with "charge conjugation"

$$\Phi_{1,2,3} \leftrightarrow -\Phi^T_{1,2,3} \qquad \Phi_{4,5,6} \leftrightarrow \Phi^T_{4,5,6}$$

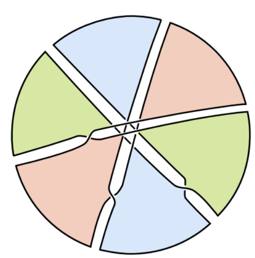
Charge conjugation of SU(N)
$T_A \to -(T_A)^T$

- Propagator on RP_4 :

$$\frac{}{|x-y|^2} \pm \frac{}{|1+x\cdot y|^2}$$

self contraction   relevant for 1pt function

- One-point function at tree level:



Leading large N answer
= antipodal contraction of single-trace op.
= crosscap overlap

Conjecture for the asymptotic 1pt functions

$$\langle \mathcal{O} \rangle_{\mathrm{RP}_4} \overset{L \gg 1}{=} \boxed{\mathrm{Sdet}} \boxed{G_{\mathrm{Gaudin}}}$$

- Derivatives of logarithms of Bethe equation.
- Same Z_2 grading structure as the defect 1pt function (but without non-universal prefactors) cf. talk by Kostya [Komatsu, Wang], [Bajnok, Gombor]

New class of defects in higher-d CFT?

- In embedding coordinates, RP_4 corresponds to

$$X_{-1} \sim X_{-1} \qquad X_{1,2,3,4,} \sim -X_{1,2,3,4}$$

- We can define "higher-codimension versions" of RP_4 : "orientifold defects"

e.g. $$X_{-1,1} \sim X_{-1,1} \qquad X_{2,3,4,} \sim -X_{2,3,4}$$

- Can we define them consistently in any CFT? What are crossing equations?
- In N=4 SYM, there should be half-BPS versions of these orientifold defects.
- Combined with charge conjugation, they probably lead to crosscaps on the world sheet, which preserve integrability.

Plan of the talk

1. Why crosscaps?

2. Crosscap overlap in integrable field theories

3. Crosscap states in integrable spin chains

4. Spacetime crosscap and world sheet crosscap

5. Conclusion

Conclusion

- Crosscap is integrable!

- Overlap with excited states can be computed analytically.
  "Simplest possible g-function"

- Relevant for N=4 SYM on RP_4.

  Conjecture for the asymptotic formula at finite λ
  1-loop check + susy localization in progress with
  Caetano, Rastelli, Soresina

- Study quench dynamics, measurement-induced phase transition.
  In progress with Caetano, Claeys, Miao

- Generalization to non-diagonal S-matrix, theories with bound states….

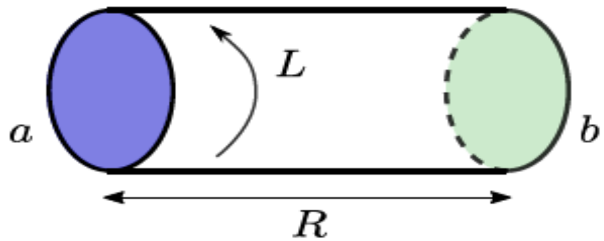
- Best arena for developing QSC approaches to correlation functions.
  cf.[Caetano, Komatsu] [Cavaglia, Gromov, Levkovich-Maslyuk]

- More orientifold defects in N=4 SYM and general CFTs.
  cf. [Gaiotto, Witten]

- Crosscap for fishnet, Yangian invariance of crosscap type Feynman diagrams?
  cf.talk by Edoardo, talk by Florian

Back ups

Boundary Entropy: Review

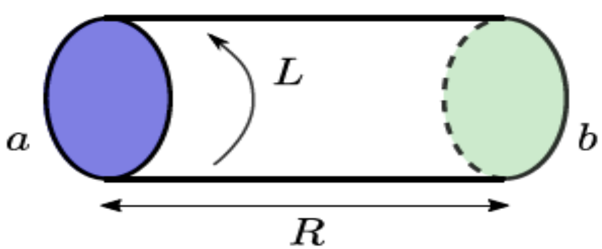


closed string $\underset{R \gg 1}{=} \sum_{\psi_n} |\langle B|\psi_n\rangle|^2 e^{-RE_n}$

- Closed-form expression (diagonal S-matrix without bound states)

$$|\langle B|\Omega_L\rangle|^2 = \exp\left[2\int_0^\infty \frac{du}{2\pi}\Theta(u)\log(1+Y(u))\right]\frac{\det\left[1-\hat{G}_-\right]}{\det\left[1-\hat{G}_+\right]}$$

- Derivation



$$= \sum_{\psi_{\rm open}} e^{-LE_{\psi_{\rm open}}} \overset{R\gg 1}{\rightarrow} \int D\rho\, e^{-L\,R\,s_{\rm eff}[\rho]}$$

Density of particles

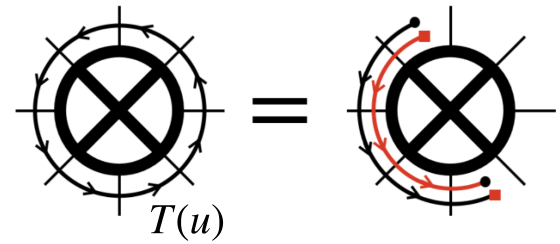
different saddles = $\sum_{\psi_n}$ | Saddle pt eq = TBA eq | 1-loop det = $|\langle B|\psi_n\rangle|^2$

Proof of $\langle C|\left[T(u)-T(-u)\right]=0$

- **Step 1**

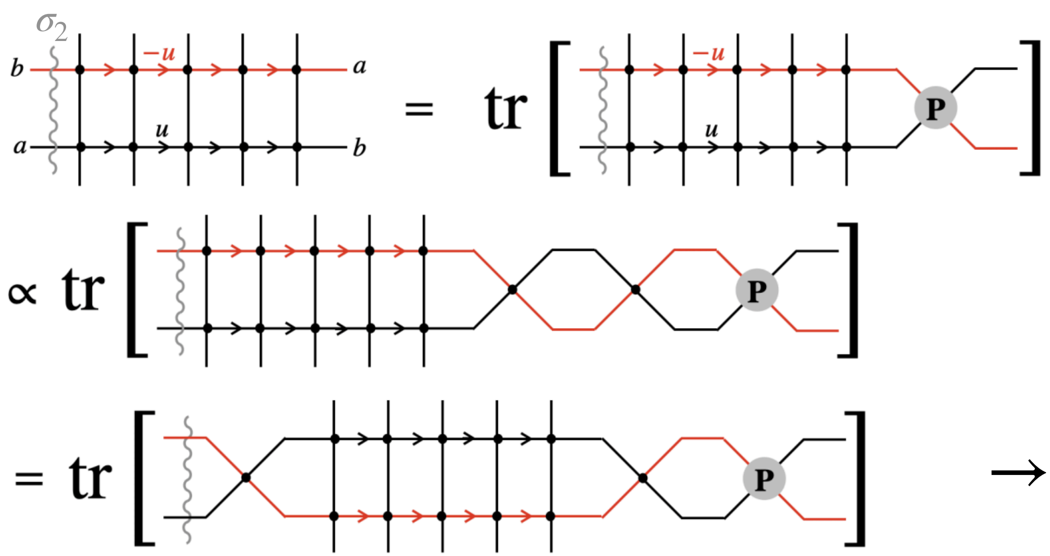
$$T(u)=\mathrm{tr}\left[L_1(u)L_2(u)\cdots\right]$$

$${}_j\langle\!\langle c|L_{j+\frac{L}{2}}(u)=-{}_j\langle\!\langle c|\left(\sigma_2 L_j(-u)\sigma_2\right)$$



$$\begin{aligned}\langle\mathcal{C}|T(u)&=\langle\mathcal{C}|\left(L_1(u)\cdots L_{\frac{L}{2}}(u)\right)_{ab}\left(L_{\frac{L}{2}+1}(u)\cdots L_L(u)\right)_{ba}\\&=(-1)^{\frac{L}{2}}\langle\mathcal{C}|\left(\sigma_2L_1(-u)\cdots L_{\frac{L}{2}}(-u)\sigma_2\right)_{ba}\left(L_1(u)\cdots L_{\frac{L}{2}}(u)\right)_{ab}\\&=(-1)^{\frac{L}{2}}\langle\mathcal{C}|\left(\sigma_2L_1(-u)\cdots L_{\frac{L}{2}}(-u)\right)_{ba}\left(\sigma_2L_1(u)\cdots L_{\frac{L}{2}}(u)\right)_{ab}\end{aligned}$$

- **Step 2**



$\rightarrow \langle C|T(-u)$

Exact p-function

$$|\langle \mathscr{C} | \Omega_L \rangle|^2 \supset \exp\left[2\int_0^\infty \frac{du}{2\pi}\tilde{\Theta}(u)\log(1+Y(u))\right]\frac{\det\left[1-\hat{G}_-\right]}{\det\left[1-\hat{G}_+\right]}$$

- **S**-sector

$$\mathbf{S}: \qquad 1 = e^{2ip_jR}S(p_j,-p_j)\prod_{k\neq j}S(p_j,p_k)S(p_j,-p_k)$$

$$S(0,0) = -1$$

$$\tilde{\Theta}_{\mathbf{S}}(u) = \frac{1}{i}\partial_u \underbrace{\log R(u)}_{\frac{1}{2}\log S(u,-u)} - \cancel{\pi\delta(u)} - \frac{1}{2i}\partial_u \log S(u,-u) = 0$$

$$\boxed{|\langle \mathscr{C} | \Omega_L \rangle|^2 \overset{\mathbf{S}}{\supset} \frac{\det\left[1-\hat{G}_-\right]}{\det\left[1-\hat{G}_+\right]}}$$

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$$e^{-mL/2}\sum_{\mathbf{T}} e^{-L\sum_{p_j>0}E(p_j)}$$

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Orbifolded staircase model

- For $m \geq 5$, there are two (or more) unitary minimal models.

  $m = 3$: Ising model, $m = 4$: Tricritical Ising model,
  $m = 5$: Tetracritical Ising (A-series), three-state Potts model (D-series)

- The two series are related by Z_2-gauging: A-series $\overset{\text{gauging}}{\leftrightarrow}$ D-series

- Gauging in 2d QFT:

  1. Introduce a twisted sector : $\phi(\sigma + 2\pi) = -\phi(\sigma)$.
  2. Restrict to Z_2-invariant states.

- Modification to (parity-invariant) Bethe eq.

$$\mathbf{S}: \qquad 1 = e^{2ip_jR} S(p_j, -p_j) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k)$$

$$\cancel{\mathbf{T}: \qquad 1 = e^{2ip_jR} S(p_j, -p_j) S(p_j, 0) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k)}$$

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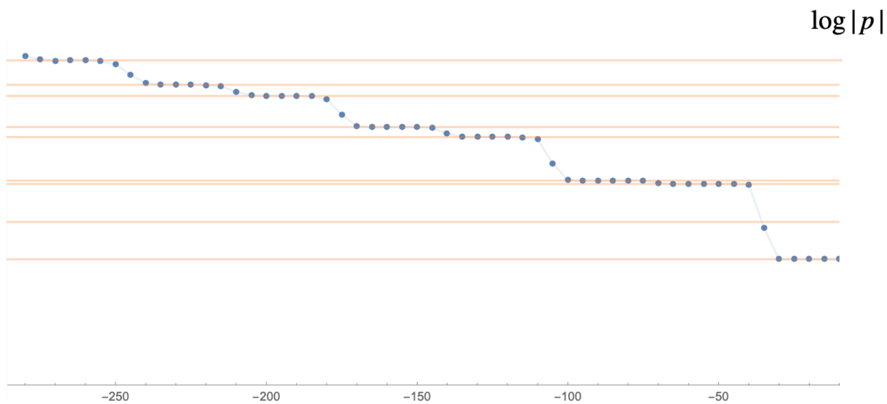
$$\mathbf{U}: \qquad -1 = e^{2ip_jR} S(p_j, -p_j) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k)$$

Answer for orbifolded theory

$$|\langle \mathscr{C} | \Omega_L \rangle| = \sqrt{\left(1 + \boxed{\sqrt{\frac{1}{1+Y(0)}}}\right) \frac{\det\left[1 - \hat{G}_-\right]}{\det\left[1 - \hat{G}_+\right]}}$$

U-sector, $-\pi\delta(u)$

- p-function



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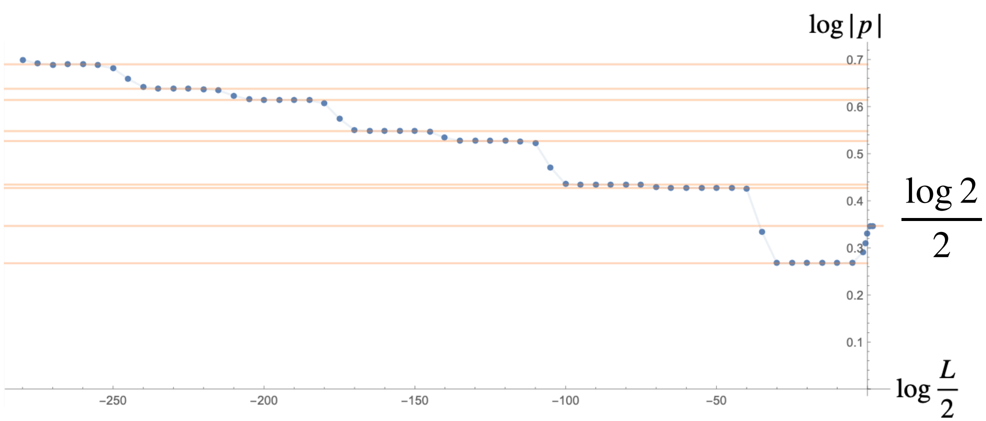

1. Reproduces the answer for D-series.
2. Starts to increase in the deep infrared.
3. IR limit is the Z_2-symmetry broken phase.