

Integrability in Gauge and String Theory 2022

Wrapping corrections for long range spin chains



Tamas Gombor



Based on: 2108.02053 TG, B. Pozsgay 2206.08679 TG

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N=4 SYM at the planar limit

Two-point function in the planar limit at weak coupling

$$\langle \mathcal{O}_I(x)\mathcal{O}_{I'}(y)\rangle = \frac{\delta_{I,I'}}{|x-y|^{2\Delta}} = \frac{\delta_{I,I'}}{|x-y|^{2\Delta_0}}(1-2\gamma\log|x-y|+\dots) \qquad \Delta(\lambda) = \Delta_0 + \gamma(\lambda)$$

$$\mathcal{O}_I = \mathcal{F}_I^{i_1 \dots i_J} \operatorname{Tr} \left[\phi_{i_1} \dots \phi_{i_J} \right] \longrightarrow \Delta_0 = J$$

$$\operatorname{span} \{\phi_i\} = V$$
 one site

 $\mathcal{F}_I \in V^{\otimes J}$ spin chain with length J

Single-trace operators



AdS/CFT duality

	Gauge theory side	String theory side
	Long range spin chain with length J	1+1 dimensional sigma model with length J
$J \to \infty$	$\begin{array}{l} \mbox{Dilatation operator:}\\ \mathcal{H}(\lambda) = H^{(0)} + \lambda H^{(1)} + \lambda^2 H^{(2)} + \dots \\ \mbox{Integrability:} \qquad [\mathcal{H}(\lambda), \mathcal{Q}_3(\lambda)] = 0\\ \mathcal{Q}_3(\lambda) = Q_3^{(0)} + \lambda Q_3^{(1)} + \lambda^2 Q_3^{(2)} + \dots \\ H^{(0)}: 2\mbox{-site int.} \qquad Q_3^{(0)}: 3\mbox{-site int.} \\ H^{(1)}: 3\mbox{-site int.} \qquad Q_3^{(1)}: 4\mbox{-site int.} \\ H^{(2)}: 4\mbox{-site int.} \qquad Q_3^{(2)}: 5\mbox{-site int.} \\ \mbox{Asymptotic Bethe Ansatz} \\ \mbox{N=4 SYM: inf. local dimension, "dynamical"} \\ \mbox{Toy models: gl(N) long range spin chains} \end{array}$	Anomalous dimension = string energies Dispersion relation: $e(p)$ Scattering matrix: $\mathbb{S}(p_1, p_2) \sim S(p_1, p_2)^{\otimes 2}$ $\operatorname{su}(2 2)_c$ scattering matrix $E = \sum e(p_j)$ Asymptotic Bethe Ansatz Toy models: simpler 1+1d field theories
Finite J	Wrapping corrections ??? + + + Toy models: ???	Lüscher corrections (finite size corrections) TBA, Y-, T-, Q-systems Toy models: 1+1d field theories in finite volume

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Integrability at higher loops

one-loop: nearest neighbor interaction (2-site) two-loop: next-to-nearest neighbor interaction (3-site) ...

r,s

k-loop: (k+1)-site interaction

 $H^{(0)} = \sum_{j} \prod_{j=1}^{j} \prod_{j=1}^{j}$

N. Beisert, C. Kristjansen, M. Staudacher '03

 $\mathcal{H}(\lambda) = H^{(0)} + \lambda H^{(1)} + \lambda^2 H^{(2)} + \ldots = H^{(\ell)}$ is defined only for $J \ge \ell + 2$ (asymptotic region) $Q_3(\lambda) = Q_3^{(0)} + \lambda Q_3^{(1)} + \lambda^2 Q_3^{(2)} + \dots$ Integrability: $[\mathcal{H}(\lambda), \mathcal{Q}_3(\lambda)] = 0$ $Q_3^{(0)}$: 3-site interaction Oth-order: $[H^{(0)}, Q_3^{(0)}] = 0$ $Q_3^{(1)}$: 4-site interaction 1th-order: $[H^{(1)}, Q_3^{(0)}] + [H^{(0)}, Q_3^{(1)}] = 0$ $Q_3^{(2)}$: 5-site interaction 2th-order: $[H^{(2)}, Q_3^{(0)}] + [H^{(1)}, Q_3^{(1)}] + [H^{(0)}, Q_3^{(2)}] = 0$ N. Beisert, T. Klose '06 We obtain $H^{(k)}, Q_3^{(k)}$ order by order Starting from an integrable 2-site spin chain which contains unfixed parameters $\frac{d}{d\lambda}\mathcal{Q}_r(\lambda) = [\mathcal{X}(\lambda), \mathcal{Q}_r(\lambda)] \longrightarrow \begin{bmatrix} \mathcal{Q}_r(\lambda), \mathcal{Q}_s(\lambda) \end{bmatrix} = 0$ $\mathcal{H}(\lambda) = \mathcal{Q}_2(\lambda)$ $\tilde{\mathcal{Q}}_r = e^{\mathcal{U}} \mathcal{Q}_r e^{-\mathcal{U}} \qquad \mathcal{U} = \sum \epsilon_k(\lambda) \mathcal{L}^k$ $\tilde{\mathcal{Q}}_r = \sum \gamma_{r,s}(\lambda) \mathcal{Q}_s$ T. Bargheer, N. Beisert, and F. Loebbert '08 Physical paramters: $\alpha_r(\lambda), \beta_{r,s}(\lambda)$ $\mathcal{X}(\lambda) = \sum \alpha_r(\lambda) \mathcal{B}[\mathcal{Q}_r(\lambda)] + \sum \beta_{r,s}(\lambda) [\mathcal{Q}_r(\lambda)] \mathcal{Q}_s(\lambda)]$ Unphysical paramters: $\gamma_{r,s}(\lambda), \epsilon_k(\lambda)$

Asymptotic Bethe Ansatz

Example: long range deformation of the XXX spin chain.

$$u(x) = x + \sum_{\ell=0}^{\infty} \frac{\alpha_{\ell}(\lambda)}{x^{\ell+1}} \qquad \qquad \Theta(u,v) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \beta_{r,s}(\lambda)(\bar{q}_r(u)\bar{q}_s(v) - \bar{q}_s(u)\bar{q}_r(v))$$

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B. Pozsgay '21 Medium range spin chains TG, B. Pozsgay '21 Interaction range $\ell + 2$ Length J $J \ge \ell + 2$ $H_J^{(\ell)} = \sum_{j=1}^J h_{j,j+1,\dots,j+\ell+1} = \sum_j h_j^{(\ell)}$ $\check{L}_{1}^{(\ell)}(u) = \check{L}_{1,2,\dots,\ell+2}^{(\ell)}(u) = 1 + uh_{1}^{(\ell)} + \mathcal{O}(u^{2}), \quad u = 1 + uh_{1}^{(\ell)} + uh_{$ $+\mathcal{O}(u^2)$ $\check{R}_{1}^{(\ell)}(u,v) = \check{R}_{1,2,\ldots,2\ell+2}^{(\ell)}(u,v), \qquad \check{R}_{2}^{(\ell)}(u,v)\check{L}_{1}^{(\ell)}(u)\check{L}_{\ell+2}^{(\ell)}(v) = \check{L}_{1}^{(\ell)}(v)\check{L}_{\ell+2}^{(\ell)}(u)\check{R}_{1}^{(\ell)}(u,v).$ $\left. \frac{d}{du} T_J^{(\ell)}(u) \right|_{u=0} = H_J^{(\ell)}$ $T_J^{(\ell)}(u) = \widehat{\mathrm{Tr}}_J \left[\check{L}_J^{(\ell)}(u) \dots \check{L}_1^{(\ell)}(u) \right] \qquad \qquad T_J^{(\ell)}(0) = \mathbf{1}$ $1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7$ $1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9$ $1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9$ v \mathcal{U} \mathcal{U} = \mathcal{U} uuU \mathcal{U} v $1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8$ $1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9$ 9 $5\ 6\ 7\ 1\ 2$ 3

Properties of transfer matrices



 $T_J^{(\ell)}(u) = U T_J^{(\ell)}(u) U^{-1}$

Properties of transfer matrices



u

Generalization to long range

$$\begin{split} \check{\mathcal{L}}_{1}^{(\ell)}(u,\lambda) &= \check{\mathcal{L}}_{1}^{(\ell)}(u) = \check{L}_{1}^{(0)}(u) + \sum_{j=1}^{\ell} \lambda^{j} \check{L}_{1}^{(j)}(u) & \check{L}_{1}^{(j)} \quad \text{range j+2} \\ \check{\mathcal{R}}_{1}^{(\ell)}(u,v,\lambda) &= \check{\mathcal{R}}_{1}^{(\ell,0)}(u,v) + \sum_{j=1}^{\ell} \lambda^{j} \check{\mathcal{R}}_{1}^{(\ell,j)}(u,v) & \check{\mathcal{R}}_{1}^{(j,k)} \quad \text{range 2j+2} \\ \check{\mathcal{R}}_{2}^{(\ell)} \check{\mathcal{L}}_{1}^{(\ell)}(u) \check{\mathcal{L}}_{\ell+2}^{(\ell)}(v) = \check{\mathcal{L}}_{1}^{(\ell)}(v) \check{\mathcal{L}}_{\ell+2}^{(\ell)}(u) \check{\mathcal{R}}_{1}^{(\ell)} + \mathcal{O}(\lambda^{\ell+1}) & \check{\mathcal{R}}_{1}^{(\ell)} \equiv \check{\mathcal{R}}_{1}^{(\ell)}(u,v) \end{split}$$

It is I+1 equation for $\lambda^0, \lambda^1, \dots, \lambda^\ell$ Let assume there exist $\check{\mathcal{L}}^\ell, \check{\mathcal{R}}^\ell$ and let us take the next order $\check{\mathcal{R}}_2^{(\ell+1)}\check{\mathcal{L}}_1^{(\ell+1)}(u)\check{\mathcal{L}}_{\ell+3}^{(\ell+1)}(v) = \check{\mathcal{L}}_1^{(\ell+1)}(v)\check{\mathcal{L}}_{\ell+3}^{(\ell+1)}(u)\check{\mathcal{R}}_1^{(\ell+1)} + \mathcal{O}(\lambda^{\ell+2})$ For the orders $\lambda^0, \lambda^1, \dots, \lambda^\ell$ this eqn contains only $\check{L}^{(0)}, \dots \check{L}^{(\ell)}$

$$\check{\mathcal{R}}_{2}^{(\ell+1)}\check{\mathcal{L}}_{1}^{(\ell)}(u)\check{\mathcal{L}}_{\ell+3}^{(\ell)}(v) = \check{\mathcal{L}}_{1}^{(\ell)}(v)\check{\mathcal{L}}_{\ell+3}^{(\ell)}(u)\check{\mathcal{R}}_{1}^{(\ell+1)} + \mathcal{O}(\lambda^{\ell+1})$$

Question: Is it guarantied that these equations have solutions $\check{R}^{(\ell+1,j)}$ for $j = 0, ..., \ell$?

$$\check{\mathcal{R}}_{1}^{(\ell+1)}(u,v) + \mathcal{O}(\lambda^{\ell+1}) = \check{\mathcal{L}}_{\ell+2}^{(\ell)}(u)\check{\mathcal{R}}_{1}^{(\ell)}(u,v)\check{\mathcal{J}}_{\ell+2}^{(\ell)}(v) \qquad \check{\mathcal{J}}_{1}^{(\ell)}(u)\check{\mathcal{L}}_{1}^{(\ell)}(u) = 1 + \mathcal{O}(\lambda^{\ell+1})$$

The matrices
$$\check{R}^{(\ell,j)}$$
 are fixed by $\check{R}^{(\ell,\ell)}$ \longrightarrow Unknowns: $\check{L}^{(\ell)}, \check{R}^{(\ell,\ell)}$
$$\sum_{j=0}^{\ell} \sum_{k=0}^{\ell-j} \check{R}_{2}^{(\ell,\ell-j-k)} \check{L}_{1}^{(j)}(u) \check{L}_{\ell+2}^{(k)}(v) = \sum_{j=0}^{\ell} \sum_{k=0}^{\ell-j} \check{L}_{1}^{(j)}(v) \check{L}_{\ell+2}^{(k)}(u) \check{R}_{1}^{(\ell,\ell-j-k)}$$

Lax operators and long range Hamiltonians

The transfer matrix $\mathcal{T}_{J}^{(\ell)}(u) = \widehat{\operatorname{Tr}}_{J} \left[\check{\mathcal{L}}_{J}^{(\ell)}(u) \dots \check{\mathcal{L}}_{1}^{(\ell)}(u) \right] + \mathcal{O}(\lambda^{\ell+1})$ Truncated RLL $\longrightarrow [\mathcal{T}_{J}^{(\ell)}(u), \mathcal{T}_{J}^{(\ell)}(v)] = \mathcal{O}(\lambda^{\ell+1})$

The regular solutions of the truncated RLL gives integrable long range Hamiltonains by definition!

Question: Is the reverse direction also true?

Are there Lax operators for every integrable long range Hamiltonains?

For general long range deformed GL(N) spin chains, there exist $\check{L}_1^0, \check{L}_1^1, \check{L}_1^2$ (for every physical parameters $\alpha_r(\lambda), \beta_{r,s}(\lambda)$)

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The transfer matrix $\mathcal{T}_{J}^{(\ell)}(u) = \widehat{\mathrm{Tr}}_{J} \left[\check{\mathcal{L}}_{J}^{(\ell)}(u) \dots \check{\mathcal{L}}_{1}^{(\ell)}(u) \right] + \mathcal{O}(\lambda^{\ell+1})$

is well defined even for $J < \ell + 2$



Inozemtsev spin chain

Let us take a finite volume long range spin chain \mathcal{H}_J for every length JIt defines the asymptotic Hamiltonian $\mathcal{H} = \lim_{J \to \infty} \mathcal{H}_J$

"Physical" consistency: Does out method give the initial finite volume Hamiltonians \mathcal{H}_J ?

The Inozemtsev's spin chain is
$$\mathcal{H}_J = \sum_{1 \le j,k \le J} \left(\wp(k-j) + \frac{2}{\omega} \zeta(\frac{\omega}{2}) \right) P_{j,k} \qquad \omega = i \frac{\pi}{\kappa}$$

. 2

The asymptotic limit is
$$\mathcal{H} = \sum_{-\infty < j < k < \infty} V(k-j)P_{j,k} \qquad V(j) = \left(\frac{\kappa}{\sinh(j\kappa)}\right)^2$$
The transformation
$$\tilde{h}^{(J,\ell)} = \widehat{\operatorname{Tr}}_J h^{(\ell)} \qquad 1 2 3 4 5 6 7 8$$
acts on the permutation as
$$P_{1,k} \rightarrow \begin{cases} N, & \operatorname{Mod}(k,J) = 1, \\ P_{1,\operatorname{Mod}(k,J),} & \operatorname{Mod}(k,J) = 2, \dots, J. \end{cases}$$

$$\mathcal{H}_J = const. + \sum_{1 \le j < k \le J} P_{j,k} \sum_{-\infty < l < \infty} V(k-j+lJ) \qquad \sum_{-\infty < l < \infty} V(k+lJ) = \left(\wp(k) + \frac{2}{\omega}\zeta(\frac{\omega}{2})\right)$$

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The anomalous dimensions
of single trace operatorsEnergy levels of a finite
volume 1+1d field theoryField theory description:Asymptotic data
 $e(p), S(p_1, p_2)$ The finite volume spectrumSpin chain description:Asymptotic
HamiltonianFinite size
Hamiltonians

 ${\cal H}_J$

Our method provides unique finite size Hamiltonians

 \mathcal{H}



Let us consider three asymptotic Hamiltonians $\mathcal{H}^{\mathcal{N}=4}, \mathcal{H}^{SU(N)}, \mathcal{H}^{SU(2)}$

for which
$$\mathcal{H}^{\mathcal{N}=4}\Big|_{SU(2)} = \mathcal{H}^{SU(N)}\Big|_{SU(2)} = \mathcal{H}^{SU(2)}$$

In the AdS/CFT, the wrapping corrections of the SU(2) sector includes contributions from the full spectrum

The finite volume Hamiltonian in a closed sector $\mathcal{H}_J^{\mathcal{N}=4}\Big|_{SU(2)} \neq \mathcal{H}_J^{SU(2)}$ depends on the full asymptotic Hamiltonian

Our definition for the wrapping region contains sums for the full spectrum



Our finite volume Hamiltonian in a closed sector depends also on the full asymptotic Hamiltonian

$$\left.\mathcal{H}_{J}^{SU(N)}\right|_{SU(2)} \neq \mathcal{H}_{J}^{SU(2)}$$

The four-loop asymptotic dilatation operator in the SU(2) sector contains: $\zeta(3)$

The four-loop length 4 dilatation operator in the SU(2) sector contains: $\zeta(3), \zeta(5)$ F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon '08

The finite volume Hamiltonian could contain extra transcendental numbers

Our definition for the wrapping region contains sums for the full spectrum



For infinite dimensional local Hilbert space, we have an infinite sums which may result in extra transcendental numbers.

Conclusions

The main message:



- I showed a perturbative construction for transfer matrices which give perturbative long range commuting charges.
- I checked that this construction gives all the SU(N) symmetric long range integrable spin chain up to λ^2
- This transfer matrix is well defined even when the interaction range is smaller than the spin chain length therefore it defines long range Hamiltonian with finite length.
- We saw that these finite volume Hamiltonians satisfy some properties of AdS/CFT wrapping corrections.

1)The finite volume spectrum is independent from the parameters $\epsilon_k(\lambda)$ 2)The wrapping corrections of the closed sectors depends on the full spectrum 3)Extra transcendental numbers can appear in the wrapping corrections