

# LIFTING INTEGRABLE MODELS

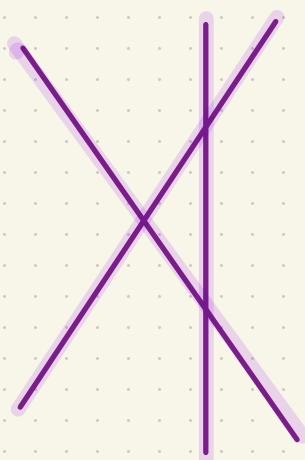
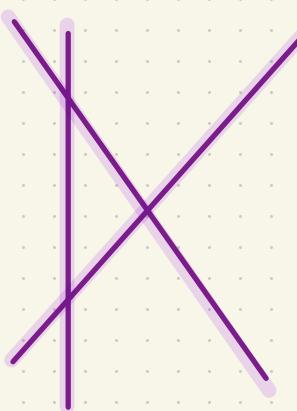
## AND LONG-RANGE SPIN CHAINS

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IGST 2022



# INTEGRABILITY APPLICATIONS

CONDENSED  
MATTER

open quantum  
systems

quantum  
circuits

...

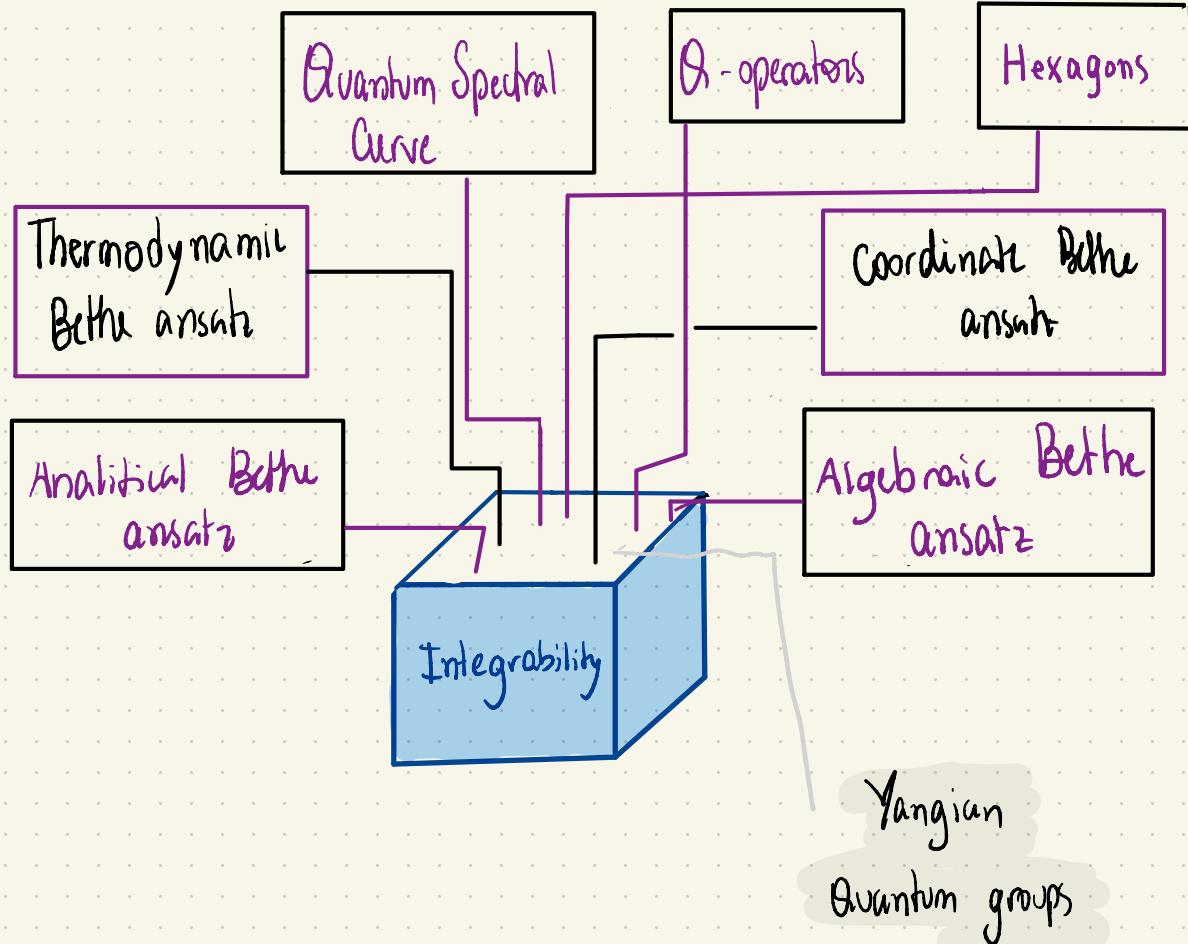
$AdS_5$

$AdS_3$

$N=4$  SYM

$AdS_2$

STRING  
THEORY



Therefore, questions like

- Is this model integrable ?
- Can I classify integrable models ?
- How to systematically construct new integrable models ?

are very relevant

The algebraic method: (Drinfeld, Faddeev, Kulish,

One assumes a symmetry Reshetikhin, Korepin ...)

for the R-matrix (like Yangian)

Solve linear equations

used in AdS context

(Beisert, Borsato, OhlssonSax,  
Sfondrini, Stefanski, Hoare,  
Pittelli, Torrielli, de Leeuw,  
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Direct method: R. Vieira, Lima-Santos ...

Expand R: Iizumi, ...

Method 1: Boost automorphism method

arXiv: 2003.04332, 2109.14280 (M. de Leeuw, C. Paletta, A. Prybytky,  
ALR, P. Ryan)

Method 2: Lifting constant Hamiltonians

three-loops in  $\text{su}(2)$  sector of  $N=4$  SYM

Examples:

range 3 6-vertex deformations

arXiv: 2206.08390 (M. de Leeuw, ALR)

## Method 1: Boost automorphism method

Goal: Classify integrable models that satisfy

$$R(u,u) \propto P \quad (\text{regularity})$$

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$$H(u) \propto P \left. \frac{dR(u,v)}{du} \right|_{v=u} \quad (\text{Hamiltonian density})$$

periodic chain, nearest-neighbor interaction

$$H(u) \propto P \frac{dR(u,v)}{du} \Big|_{v=u}$$

$$R(u,v) = R(u-v)$$

DIFFERENCE FORM R-MATRIX



Constant H

relativistic models are  
in this class

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DIFFERENCE FORM R-MATRIX

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$$R(u,v) \neq R(u-v)$$

H(u)

NON-DIFFERENCE FORM R-MATRIX

few known models  
(but very interesting)

If you start with an ansatz Hamiltonian,

$$H(u) = \begin{pmatrix} h_1(u) & 0 & 0 & 0 \\ 0 & h_2(u) & h_3(u) & 0 \\ 0 & h_4(u) & h_5(u) & 0 \\ 0 & 0 & 0 & h_6(u) \end{pmatrix}$$

How to say if it is Yang-Baxter integrable?

And for which values of  $h_i(u)$ ?

We could start by

$$R_{12}(u, v) R_{33}(u, w) R_{23}(v, w) = R_{23}(v, w) R_{13}(u, w) R_{12}(u, v)$$

$$\frac{d}{du} \left( YBE \right) \Big|_{v=u}, \quad R(u, u) \propto P, \quad \text{if } \alpha P \dot{R}(u, v) \Big|_{v \neq u}$$

We could start by

$$R_{J2}(u, v) R_{J3}(u, w) R_{23}(v, w) = R_{23}(v, w) R_{13}(u, w) R_{12}(u, v)$$

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$$[R_{J3} R_{23}, H_{12}(u)] = \dot{R}_{J3} R_{23} - R_{J3} \dot{R}_{23}$$

SUTHERLAND  
EQUATION

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SUTHERLAND  
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Unknown  
and depends on  
 $R_i(u, v)$

also unknown

Very hard task in general (especially for  
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What if we had an independent way to say if

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⇒ Boost automorphism  
mechanism

(Tetelman, 1982, Loebbert 2016, Grabowski and Mathieu, 1999)



Use the Boost operator to generate  $Q_3$ :

$$B[Q_2] = \partial_u + \sum_{n=-\infty}^{\infty} n K_{n,n+1}(u)$$

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Use the Boost operator to generate  $Q_3$ :

$$B[Q_2] = \partial_u + \sum_{n=-\infty}^{\infty} n \mathcal{H}_{n,n+1}(u)$$

$$Q_{r+1} \sim [B[Q_2], Q_r] \quad r > 1$$

$$\begin{aligned} Q_3 &= [B[Q_2], Q_r] \\ &= \sum_{i=1}^L [\mathcal{H}_{i-1,i}, \mathcal{H}_{i,i+1}] + \frac{dH}{du} \end{aligned}$$

$$[Q_2, Q_3] = 0$$



SYSTEM  
OF CUBIC  
POLYNOMIAL  
EQUATIONS



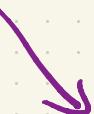
constant

H

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SYSTEM  
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$$Q_3 = \sum_{i=1}^L [H_{i-1,i}, H_{i,i+1}] + \frac{dH}{du}$$

SYSTEM  
OF ODE



constant

H

u-dependent

H

$[\theta_2, \theta_3] = 0 \Rightarrow$  each solution is a  
Potentially integrable H

What about  $[Q_2, Q_4] = 0$  ?

$[Q_3, Q_4] = 0$  ?  
⋮

Now we need to solve YBE

$$[R_{J3} R_{23}, H_{12}(u)] = \dot{R}_{J3} R_{23} - R_{J3} \dot{R}_{23}$$

SUTHERLAND  
EQUATION



but now we  
"know"  $H$

Much easier

Especially for non-difference form models

Summary:

ansatz for  
 $H = \theta_2$



Compute  $\Theta_3$

using the boost op.



require

$$[\theta_2, \theta_3] = 0$$

and solve the  
system of eqs.



Check  
YBE



Solve YBE by  
solving Sutherland eq.



Potentially  
integrable  
 $H$ 's

M. de Leeuw, A. Prybitok, P. Ryan, 2019

M. de Leeuw, C. Paquette, A. Prybitok,  
ALR, P. Ryan, 2020

## Applications

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- All  $4 \times 4$  difference form models
- $\text{su}(2) \oplus \text{su}(2)$  classification (Both diff. form and non-diff form models)
- New integrable deformations of (including an elliptic  $\text{AdS}_2$  and  $\text{AdS}_3$  deformation)
- M. de Leeuw, C. Paletta and B. Pozsgay applied the method to Lindblad system  
(First systematic construction of YB6 integrable Lindblad systems)

We found many new integrable models  
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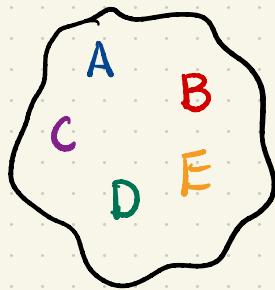
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## Disadvantages:

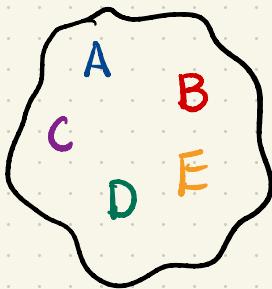
Expectation:



some interesting  
solutions

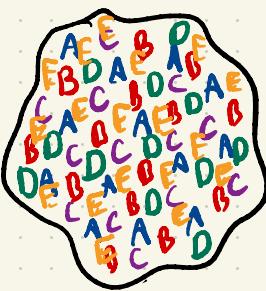
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Reality:

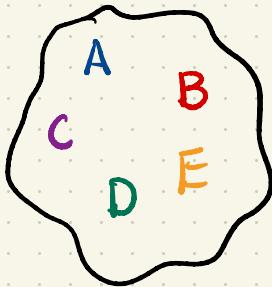


lots of dependent  
solutions

Normalization  
Reparametrization  
Basis transformation  
Discrete transformation  
Twist

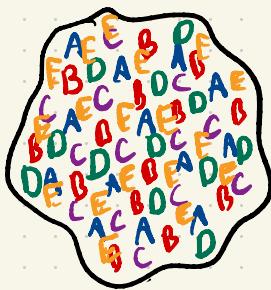
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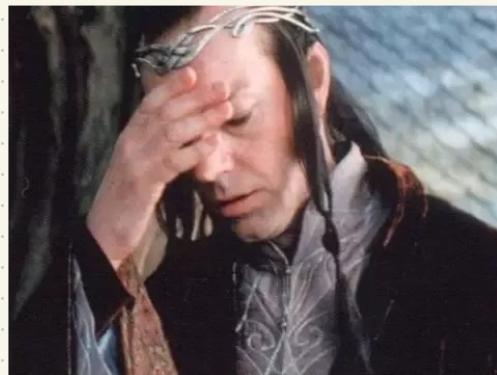
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Reality:



lots of dependent  
solutions

reasonable  
amount of  
work  
→



Normalization  
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 $\Rightarrow$  NOT INTEGRABLE

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- - - - -
- Hard to generalize to long-range spin chains

## Method 2: Lifting constant Hamiltonians

It is well known that



$$t(u) = \text{tra } \mathcal{L}_{aL}^{(u)} \cdots \mathcal{L}_{a1}^{(u)}$$

$$\Omega_{n+1} = \left. \frac{d^n}{du^n} \log t(u) \right|_{u=0}$$

$\mathcal{L}$  satisfies

$$R_{ab}(u, v) \mathcal{L}_{aj}(u) \mathcal{L}_{bj}(v) = \mathcal{L}_{bj}(v) R_{aj}(u) R_{ab}(u, v)$$

## Method 2: Lifting constant Hamiltonians

Start with a constant Hamiltonian

$$\begin{aligned} H = & h_1 \mathbb{I} + h_2 (\sigma^z \otimes \mathbb{I} - \mathbb{I} \otimes \sigma^z) + h_3 \sigma^+ \otimes \sigma^- + h_4 \sigma^- \otimes \sigma^+ \\ & + h_5 (\sigma^z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma^z) \end{aligned}$$

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- Still nearest-neighbour  $H$
- and periodic

$$\mathcal{L}'(u) = P H$$

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- and periodic

- but always constant now

$$\mathcal{L}(u) = P \left( 1 + u \mathcal{H} + \sum_{i>1} \frac{\mathcal{L}^{(i)} u^i}{i!} \right)$$

$$\mathcal{L}^{(i)} = \begin{pmatrix} \tilde{l}_1^{(i)} & 0 & 0 & 0 \\ 0 & \tilde{l}_2^{(i)} & \tilde{l}_3^{(i)} & 0 \\ 0 & \tilde{l}_4^{(i)} & \tilde{l}_5^{(i)} & 0 \\ 0 & 0 & 0 & \tilde{l}_6^{(i)} \end{pmatrix}$$

all  $u$ -dependence is now explicit

$l_j^{(i)}$  and  $\tilde{l}_j^{(i)}$  are constants

$$Q_2 = H = \frac{d}{du} \log t(u)$$

$$L=4$$

$$Q_3 = \frac{d^2}{du^2} \log t(u) = t^{-1}(0) t''(0) - Q_2^2$$



$$[Q_2, Q_3] = 0$$

NOT RELEVANT FOR  
COMUTATOR PURPOSES

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$$\tilde{\ell}_6^{(2)} = -8h_2^2 - 2h_3h_4 + 8h_5^2 - \tilde{\ell}_1^{(2)} + \tilde{\ell}_3^{(2)} + \tilde{\ell}_4^{(2)}$$

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$$Q_4 \propto t^{-1}(0) t'''(0)$$

$$[\theta_2, \theta_4] = 0$$

↓

$$\tilde{l}_6^{(3)} = \dots$$

again only one condition  
and all other  $\tilde{l}_i^{(3)}$   
remain free

$$[\theta_2, \theta_4] = 0$$

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$$Q_5 = t^{-1}(0) \ t^{(n)}(0)$$

⋮

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$$[\theta_2, \theta_5] = 0 \text{ fixes } \tilde{l}_6^{(4)}$$

⋮

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↓

$$\tilde{l}_6^{(3)} = \dots$$

$$Q_5 = t^{-1}(0) t^{(n)}(0)$$

⋮

$$Q_n = t^{-1}(0) t^{(n-1)}(0)$$

⋮

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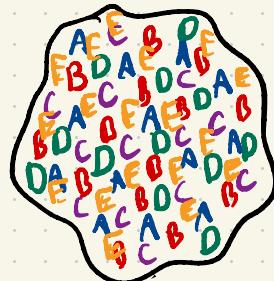
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⋮

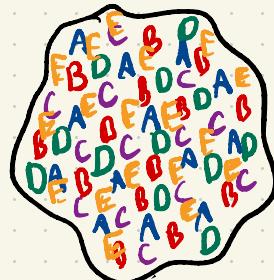
$$[Q_2, Q_n] = 0 \text{ fixes } \tilde{l}_6^{(n-1)}$$

⋮

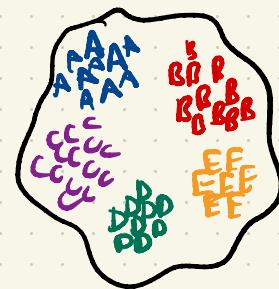
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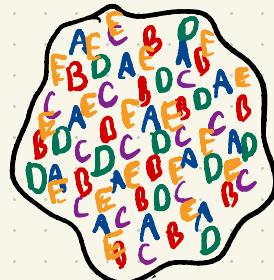
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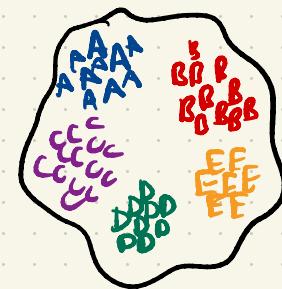


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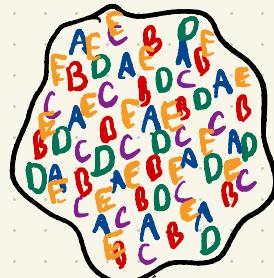


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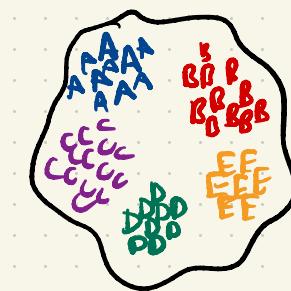
The freedom is on the  $\tilde{l}_i^{(j)}$



Instead of a "crowd" of  
solutions like



we find directly one like



The freedom is on the  $\tilde{b}_i^{(j)}$

We can chose the extra  $\tilde{b}_i^{(j)}$ 's to have the nicer  
possible Lax operator

There is an easier way

$$\mathcal{L}(u) = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & f_1(u) & * & 0 \\ 0 & * & f_2(u) & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

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$$L(u) = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & f_1(u) & * & 0 \\ 0 & * & f_2(u) & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

In order to fix  $f_1(u)$  and  $f_2(u)$  we can directly

$$[t(u), H] = 0$$

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Once we have  $\mathcal{L}$  we can use RLF relations

to find compute  $R$

↓  
linear in  $R$

(27)

## Advantages :

- Non-DIFFERENCE FORM models  
from a constant  $H$
- Not many dependent solutions;
- Easy to apply to long range spin chains

# Long-Range spin chains (perturbative)

- In the context of AdS/CFT for planar  $N=4$  SYM at one loop

DILATATION OPERATOR = INTEGRABLE NEAREST  
NEIGHBOUR SPIN CHAIN

(Minahan, Zarembo, 2002, Beisert 2004)

# Long-Range spin chains (perturbative)

- In the context of AdS/CFT for planar  $N=4$  SYM at one loop

DILATATION OPERATOR = INTEGRABLE NEAREST  
NEIGHBOUR SPIN CHAIN

(Minahan, Zarembo, 2002, Beisert 2004)

- At each order in perturbation theory the interaction range of  $H$  increases as well

(Beisert, Kristjánsson & Staudacher, 2003  
Beisert, Dippel & Staudacher, 2004)

$$\mathcal{H} = \mathcal{H}_{NN} + g^2 \mathcal{H}_{NNN} + \dots$$

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Full  $\mathcal{H}$  has infinite range

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A general framework for this type of spin chain was put forward by Bargheer, Beisert & Loebbert in 2007 & 2009 (for the charges)

$$\mathcal{H} = \mathcal{H}_{NN} + g^2 \mathcal{H}_{NNN} + \dots$$

Full  $\mathcal{H}$  has infinite range

A general framework for this type of spin chain was put forward by Bargheer, Beisert & Loebbert in 2007 & 2009 (for the charges)

Later used to classify all the integrable long range deformations of 6vertex model.  
(Beisert, Fiévet, de Leeuw, Loebbert, 2013)

we would like to apply the idea to

the  $\mathfrak{su}(2)$  sector in  $N=4$  SYM

we would like to apply the idea to

$\mathcal{H}$  of  $su(2)$  sector in  $N=4$  SYM

Range 3 deformation of Gavros  
model

One more ingredient : increase the dimension  
of the auxiliary space

For range 3, for example, if the physical space  
 $V$  has dimension  $d(V)$ , the dimension of the  
auxiliary space is  $d(V)^2$   
(Gombor, Pozsgay, 2021)

For a range  $r$ ,  $\mathcal{H}$

- dimension of aux. space =  $(\dim Y)^{r-1}$

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- minimum  $L = 2r$  sites. (to avoid wrapping effect)



T. Gombor 2022

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T. Gombor 2022

$$H \sim J - P_{12} + g^2 (1 - P_{13}) + \dots$$

$$\sim J - P_{12} + g^2 P_{13} + \dots$$

$$\dim(\text{aux}) = 4 = 2^2 \quad , \quad L = 6,$$

$$L_{12,3}(u) = P_{23} P_{12} \left( P_{12} L_{12}(u) - \frac{2g^2 u}{(u-2)(u^2-1)} P_{13} \right)$$

after solving for BC

$$L'(0) = P_{23} P_{12} H_{123}$$

$$L(0) = P_{23} P_{12} .$$

$$\mathcal{H}_{1234} = \mathcal{H}_{123} + g^4 \sum_i p_i$$

with

$$p_1 = A_1 \sum_{i=1}^3 \sigma_i \otimes \sigma_i \otimes \sigma_i,$$

$$p_2 = A_2 \sum_{i,j=1}^3 \sigma_i \otimes \sigma_j \otimes \sigma_j \otimes \sigma_i$$

$$p_3 = A_3 \sum_{i,j=1}^3 \sigma_i \otimes \sigma_j \otimes \sigma_i \otimes \sigma_j,$$

$$p_4 = A_4 \sum_{i,j=1}^3 \sigma_i \otimes \sigma_i \otimes \sigma_j \otimes \sigma_j$$

$$p_5 = A_5 \sum_{i,j,k=1}^3 \epsilon^{ijk} \sigma_i \otimes \sigma_j \otimes \sigma_k, \quad p_6 = A_6 \sum_{i,j,k=1}^3 \epsilon^{ijk} \sigma_i \otimes \sigma_j \otimes \sigma_k$$

is  $SU(2)$  invariant.

we find

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$\Rightarrow$  we have two deformations;

However one of them is actually  $\Omega_4$  for the original chain

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- we cannot put all  $A_i$  to zero at the same time
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- very interesting! It matches with the result by McLoughlin & Spiering 2022 which show chaos if one considers two-loops.

Explicitly,

$$\mathcal{L}^{(g^4)} = \lambda_1 1 +$$

$$\begin{aligned} & \lambda_2 \sum \sigma^i \otimes \sigma^i \otimes 1 \otimes 1 + \lambda_3 \sum 1 \otimes \sigma^i \otimes \sigma^i \otimes 1 + \lambda_4 \sum 1 \otimes 1 \otimes \sigma^i \otimes \sigma^i + \\ & \lambda_5 \sum \sigma^i \otimes 1 \otimes \sigma^i \otimes 1 + \lambda_6 \sum 1 \otimes \sigma^i \otimes 1 \otimes \sigma^i + \\ & \lambda_7 \epsilon_{ijk} \sigma^i \otimes \sigma^j \otimes \sigma^k \otimes 1 + \lambda_8 \epsilon_{ijk} 1 \otimes \sigma^i \otimes \sigma^j \otimes \sigma^k + \\ & \lambda_9 \sum \sigma^i \otimes 1 \otimes 1 \otimes \sigma^i + \lambda_{10} \sum \sigma^i \otimes \sigma^j \otimes \sigma^j \otimes \sigma^i + \lambda_{11} \sum \sigma^i \otimes \sigma^j \otimes \sigma^i \otimes \sigma^j + \\ & \lambda_{12} \sum \sigma^i \otimes \sigma^i \otimes \sigma^j \otimes \sigma^j + \lambda_{13} \epsilon_{ijk} \sigma^i \otimes \sigma^j \otimes 1 \otimes \sigma^k + \lambda_{14} \epsilon_{ijk} \sigma^i \otimes 1 \otimes \sigma^j \otimes \sigma^k \end{aligned}$$

$$\lambda_4 = \frac{(u-1)^3(3u-2)}{2(1-2u)^2u^2}(2A_1 - 1) + \left(\frac{2}{u} - 1\right)\lambda_{10}$$

$$\lambda_6 = \frac{(u-1)^2}{2(2u-1)^2} \left( \frac{4(2u-1)}{u^2} A_1 - 2A_1 + 1 \right) + \left(\frac{2}{u} - 1\right)\lambda_9$$

$$\lambda_8 = \frac{i(u-1)^3}{(1-2u)^2(u-2)} \left( A_1 - \frac{(2A_1-1)(u+2)}{4u} \right) + \left(\frac{2}{u} - 1\right)\lambda_{14}$$

$$\begin{aligned}\lambda_{11} &= \frac{A_1(u(u(2u-5)+9)-4)(u-1)^2}{2(1-2u)^2u^2} + \frac{A_2(u-1)^2}{u(2u-1)} - i\lambda_{14} + \left(\frac{2}{u} - 2\right)\lambda_9 \\ &\quad - \frac{(u(u(u(4u^2-22u+31)-17)-4)+4)(u-1)^2}{4(u-2)^2u(2u-1)^3} - \frac{u}{2}\lambda_3 + \left(1 - \frac{u}{2}\right)\lambda_5\end{aligned}$$

$$\lambda_{12} = \frac{(1-2A_1)(u-1)^3}{4(1-2u)^2u} + \lambda_{10}$$

$$\lambda_{13} = \frac{iA_1(3u-2)(u-1)^2}{2(1-2u)^2u} - \frac{3iu(u-1)^2}{4(1-2u)^2(u-2)} + i\lambda_9$$

Because of Boundary conditions :

$$\lambda_i \sim \frac{a_i}{u} + b_i + O(u)$$

$$a_g = A_1, \quad a_{10} = \frac{1}{2} - A_1, \quad b_g = \frac{A_1}{2} \quad b_{1+} = i \frac{(2A_1 + 1)}{8}$$

the remaining  $a_i, b_i$  vanish.

Finding  $R$  is again straightforward once  $L$  is known.

6 vertex model

$$[H, \sum_i \alpha_i^3] = 0$$

$$H_{12} = h_1 \mathbb{I} + h_2 (\sigma^z \otimes \mathbb{I} - \mathbb{I} \otimes \sigma^z)$$

$$+ h_3 \sigma^+ \otimes \sigma^- + h_4 \sigma^- \otimes \sigma^+$$

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$$\tilde{H}_{123} = H_{12} + g^2 \tilde{H}_{123}$$

$$\sum_{a,b,c}^{11} h_{a,b,c} \sigma^a \otimes \sigma^b \otimes \sigma^c$$

$$a, b, c = 0, \pm, z$$

(41)

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Conclusion: All range 3 deformations of 6 vertex model  
computed using the deformation equation  
by Beisut, L. Fiévet, M. de Leeuw & F. Loebbert  
in 2013 are Yang-Baxter integrable

(are all regular integrable models YB integrable? hard question)

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- Construct  $L$  and  $R$  at all orders;
- Apply the method to other physically interesting models;
- Use it to study chaos in other perturbatively integrable model;

THANK YOU!