The AdS Virasoro Shapiro amplitude from dispersive sum rules and integrability

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Based on work with Tobias Hansen and Joao Silva.

What will this talk be about?

Conformal Field Theories techniques to study String scattering amplitudes on AdS

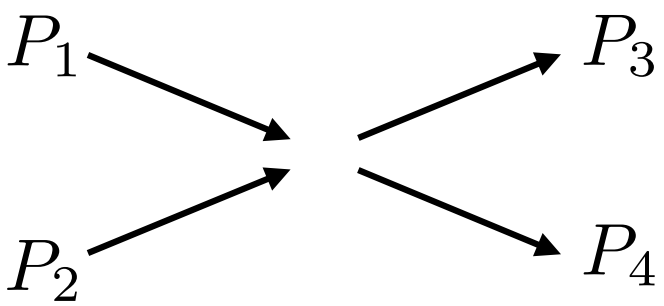
Why scattering amplitudes?

- They allow to test the predictions of our theory.
- They can teach us much about its structures/symmetries
  - On-shell and twistor Methods.
  - Color/kinematic duality and double copy.
  - Positive geometry.
  - ...

We will study graviton scattering amplitudes in String theory on AdS.

Scattering Amplitudes

Probability that two particles colliding (with momenta p_1, p_2) result into two other particles (with momenta p_3, p_4).



$$= \quad A(g, s, t, u)$$

- $A(g, s, t, u)$ depends on many things:
  - The parameters of your theory g.
  - The particles you are scattering (their masses, polarizations, etc)
  - The momenta of the particles being scattered:

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2$$

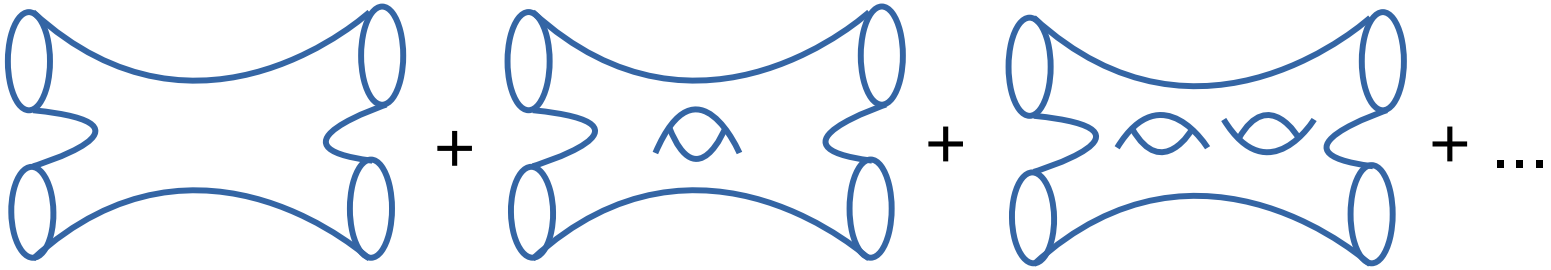
String theory scattering amplitudes - Flat space

4pt graviton amplitude in ST

- The parameters of the theory are g_s and α'.
- The amplitude depends on the momenta p_i and polarisations ϵ_i of the gravitons.
- SUSY fixes the dependence on the polarisation:

$$A(g_s, \alpha', p_i, \epsilon_i) = pref(\epsilon_i, p_i) \times A(g_s, \alpha', s, t, u)$$

- The computation organises in a genus expansion



$$A^{(genus\ 0)}(\alpha', s, t, u) + g_s^2 A^{(genus\ 1)}(\alpha', s, t, u) + g_s^4 A^{(genus\ 2)}(\alpha', s, t, u) + \cdots$$

String theory scattering amplitudes - Flat space

At tree level and in flat space: Virasoro-Shapiro amplitude

$$A^{(genus\ 0)}(\alpha', s, t, u) = \frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})}$$

- Crossing symmetric.
- Poles due to the exchange of particles (of mass $m = 2\sqrt{n/\alpha'}$ and spin ℓ)

$$A^{(genus\ 0)}(\alpha', s, t, u) \sim \frac{P_\ell(t,u)}{s - 4n/\alpha'}$$

- Regge behaviour

$$A^{(genus\ 0)}(\alpha', s, t, u) \sim s^{-2+\alpha' \frac{t}{2}}, \quad \text{for large } |s|$$

- α' expansion

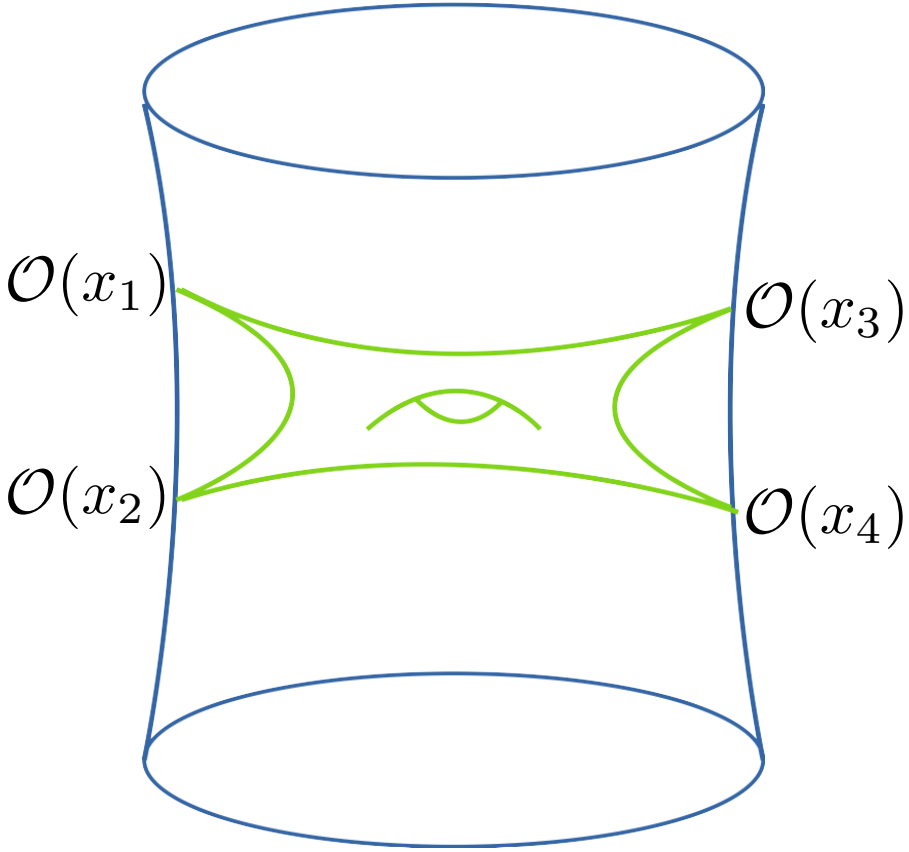
$$A^{(genus\ 0)}(\alpha', s, t, u) \sim \underbrace{\frac{1}{s\,t\,u}}_{\text{sugra}} + \underbrace{\alpha'^3 + \alpha'^5(s^2+t^2+u^2) + \cdots}_{\text{stringy corrections}}$$

VS in AdS

What can we say about the Virasoro-Shapiro amplitude on AdS?

AdS/CFT

String amplitudes on *AdS* $\leftrightarrow$ Correlators of local operators in the CFT at the boundary.



$$\mathcal{A}(g_s, \alpha', s, t, u) \leftrightarrow \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$$

AdS/CFT

$$\text{String theory on } AdS_5 \times S^5 \quad \leftrightarrow \quad \text{4d } \mathcal{N}=4 \text{ SYM}$$
$$(g_s, R) \qquad\qquad\qquad (g_{YM}, N)$$

$$g_s \approx \frac{1}{N}, \qquad \frac{R^2}{\alpha'} = \sqrt{g_{YM}^2 N} \equiv \sqrt{\lambda}$$

String amplitudes on $AdS_5 \times S^5$	Correlators in $\mathcal{N}=4$ SYM
Genus expansion	$1/N$ expansion
Stringy corrections to sugra	$1/\lambda$ corrections
Graviton on AdS	$\mathcal{O}_2$: protected scalar of dim. 2 in the stress-tensor multiplet
KK-modes on the S^5	$\mathcal{O}_k$: tower of scalar operators of dim. k

Compute $\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$ in a $1/N$ and $1/\lambda$ expansion.

4d $\mathcal{N} = 4$ SYM - Kinematics

The symmetry

$$PSU(2,2|4) \supset \underbrace{SO(2,4)}_{\text{conformal symmetry}} \oplus \underbrace{SO(6)}_{\text{R-symmetry}}$$

The operators

$$\underbrace{\mathcal{O}_{IJ}(x)}_{\text{sym. traceless of } SO(6)} = Tr\varphi^{(I}\varphi^{J)} \rightarrow \quad \mathcal{O}_2(x,y) = \mathcal{O}_{IJ}(x)y^I y^J, \quad y^2 = 0$$

The observable

$$\langle \mathcal{O}_2(x_1,y_1)\mathcal{O}_2(x_2,y_2)\mathcal{O}_2(x_3,y_3)\mathcal{O}_2(x_4,y_4)\rangle$$

4d $\mathcal{N}=4$ SYM - Kinematics

Bosonic symmetry

$$\langle \mathcal{O}_2(x_1,y_1)\cdots\mathcal{O}_2(x_4,y_4)\rangle = \left(\frac{y_{12}y_{34}}{x_{12}^2x_{34}^2}\right)^2 \underbrace{\mathcal{G}(U,V;\sigma,\tau)}_{\text{degree 2 pol. in } \sigma,\tau}$$

$$U=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\ V=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2},\ \sigma=\frac{y_1\cdot y_3 y_2\cdot y_4}{y_1\cdot y_2 y_3\cdot y_4},\ \tau=\frac{y_1\cdot y_4 y_2\cdot y_3}{y_1\cdot y_2 y_3\cdot y_4}$$

Super-conformal Ward identities

$$(z\partial_z-\alpha\partial_\alpha)\,\mathcal{G}(z,\bar z;\alpha,\bar\alpha)|_{\alpha=1/z}=0$$
$$(z\partial_z-\bar\alpha\partial_{\bar\alpha})\,\mathcal{G}(z,\bar z;\alpha,\bar\alpha)|_{\bar\alpha=1/z}=0$$

$$U=z\bar z,\ V=(1-z)(1-\bar z),\quad \sigma=\alpha\bar\alpha,\ \tau=(1-\alpha)(1-\bar\alpha)$$

$1/N$ expansion

- Genus-0 amplitudes $\rightarrow$ leading non-trivial term in a $1/N$ expansion.
- Dropping $1/\lambda$ corrections gives the supergravity approximation (computed by Arutyunov and Frolov 22 years ago)

$$\mathcal{G}(U,V;\sigma,\tau) = \underbrace{\mathcal{G}_{disc}(U,V;\sigma,\tau)}_{disconnected} + \frac{1}{N^2}\left(\boxed{\mathcal{G}^{(sugra)}(U,V;\sigma,\tau)} + \cdots\right) + \cdots$$

- We are interested in keeping all $1/\lambda$ corrections, to leading non-trivial order in $1/N$.

$$\mathcal{G}(U,V;\sigma,\tau) = \underbrace{\mathcal{G}_{disc}(U,V;\sigma,\tau)}_{disconnected} + \frac{1}{N^2}\underbrace{\left(\mathcal{G}^{(sugra)}(U,V;\sigma,\tau) + \cdots\right)}_{\text{Virasoro-Shapiro on AdS}} + \cdots$$

The right language: Mellin space

$\mathcal{G}(U,V;\sigma,\tau) \to \mathcal{M}(s,t,u;\sigma,\tau) \equiv \mathcal{M}(s,t;\sigma,\tau)$, with $s+t+u=4$.

$$\boxed{\mathcal{G}(U,V;\sigma,\tau) = \int_{-i\infty}^{i\infty} dsdtU^s V^t \underbrace{\Gamma_{\{k_i\}}(s,t,u)}_{\text{A prefactor}} \underbrace{\mathcal{M}(s,t;\sigma,\tau)}_{\text{string amplitude in } AdS_5\times S^5}}$$

$\mathcal{M}(s,t;\sigma,\tau)$ is a meromorphic function with very nice properties:

1. Crossing symmetry. ✓
2. Exchanged operators lead to simple poles:

$$\mathcal{M}_{exch}(s,t) = \sum_{m=0}^{\infty} C^2_{\Delta,\ell} \frac{Q_{\ell,m}(u,t)}{s-(\Delta-\ell)-2m} + \text{regular} \quad ✓$$

The right language: Mellin space

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3. Superconformal Ward identities:

$$\mathcal{M}(s+1,t;\sigma,\tau) + \sigma\partial_\sigma \mathcal{M}(s,t;\sigma,\tau) + \cdots = 0 \quad \checkmark$$

Solved by

$$\mathcal{M}(s,t;\sigma,\tau) = (\text{Shift operator}) \circ \mathcal{M}(s,t)$$

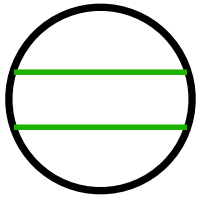
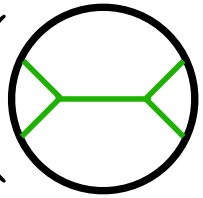
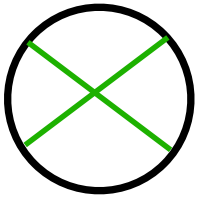
- Analogous to what happens in flat space!

$$A(p_i,\epsilon_i) = pref(\epsilon_i,p_i) \times A(s,t,u) \to \mathcal{M}(s,t;\sigma,\tau) = \mathcal{S}_{\sigma,\tau} \circ \mathcal{M}(s,t)$$

AdS VS amplitude: $\mathcal{M}(s,t)$ at leading non-trivial order in $1/N$.

AdS Virasoro-Shapiro amplitude

Diagrammatic perspective (perturbative in $1/\lambda$)

$\mathcal{M}(s,t) =$  $+ \frac{1}{N^2}\Big($  $+ 1/\lambda^{3/2}$  $+ \cdots\Big) + \cdots$

$\mathcal{M}_{tree}(s,t)$

- Sugra $\rightarrow$ exchange of sugra fields $\rightarrow$ symmetric meromorphic function.
- Stringy corrections $\rightarrow$ contact terms $\rightarrow$ symmetric polynomials.

$$\mathcal{M}_{tree}(s,t) = \underbrace{\frac{1}{(s-2)(t-2)(u-2)}}_{sugra} + \frac{\alpha_{0,0}}{\lambda^{3/2}} + \frac{\alpha_{1,0}(s^2+t^2+u^2)+\gamma_{0,0}}{\lambda^{5/2}} + \frac{\alpha_{0,1}\, s\, t\, u + \cdots}{\lambda^3} + \cdots$$

AdS Virasoro-Shapiro amplitude

General expansion

$$\mathcal{M}_{tree}(s,t) = \underbrace{\frac{1}{(s-2)(t-2)(u-2)}}_{sugra} + \underbrace{\sum_{a,b=0}^{\infty} \frac{\sigma_2^a \sigma_3^b}{\lambda^{\frac{3}{2}+a+\frac{3}{2}b}} \left(\alpha_{a,b} + \frac{\beta_{a,b}}{\lambda^{1/2}} + \cdots \right)}_{\text{stringy corrections}}$$

$$\sigma_2 = s^2 + t^2 + u^2, \quad \sigma_3 = s\,t\,u$$

What is known about the stringy corrections?

AdS Virasoro-Shapiro amplitude

Flat space limit

- Large s, t, u, λ, with $s/\sqrt{\lambda} \sim$ fixed.
- Only leading terms survive

$$\mathcal{M}_{tree}(s,t) \to \frac{1}{s\,t\,u} + \frac{\alpha_{0,0}}{\lambda^{3/2}} + \frac{\alpha_{1,0}(s^2+t^2+u^2)}{\lambda^{5/2}} + \frac{\alpha_{0,1}\,s\,t\,u}{\lambda^3} + \cdots$$

- In this limit we should recover the flat space answer

$$\mathcal{M}_{tree}(s,t) \to A^{(genus\ 0)}(\alpha', s, t, u)$$

$$\downarrow$$

$$\alpha_0 = \zeta_3, \quad \alpha_2 = \zeta_5, \quad \alpha_3 = \zeta_3^2, \cdots$$

Localisation constraints

- Two further constraints at each order.

To be able to say more we need to change our perspective!

AdS Virasoro-Shapiro amplitude

Non-perturbative perspective

1. Regge limit

$$\mathcal{M}_{tree}(s,t) \sim \frac{1}{s^2}, \quad \text{for large } s \text{ and } Re(t) < 2$$

2. Heavy/stringy operators with $\Delta \sim \lambda^{1/4}$ are exchanged.

These heavy operators correspond to 'short' strings, dual to *e.g.* the Konishi Operator $\mathcal{K} = Tr\varphi^I\varphi^I$

$$\Delta_{\mathcal{K}} = 2\lambda^{1/4} - 2 + 2\lambda^{1/4} + \cdots$$

The Mellin amplitude should have poles there. Can we see them?

Properties 1 and 2 are intimately connected!

AdS Virasoro-Shapiro amplitude

Toy Model: (partially-symetrized) exchange of a heavy operator:

$$M_{toy}(s,t) = \frac{1}{s-\Delta} + \frac{1}{u-\Delta} = \frac{1}{s-\Delta} + \frac{1}{(4-s-t)-\Delta}$$

- Correct Regge behaviour
- Its $1/\Delta$ expansion results in polynomials:

$$M_{toy}(s,t) = -\frac{-2}{\Delta} - \frac{s+u}{\Delta^2} - \frac{s^2+u^2}{\Delta^3} + \cdots$$

Real life: an infinite sum over such exchanges...[poles at $\tau = \Delta - \ell$]

$$\mathcal{M}_{tree} \sim \sum_{\Delta,\ell} C^2_{\Delta,\ell} \left(\frac{P_\ell(t,u)}{s-\tau} + \frac{P_\ell(s,t)}{u-\tau} \right)$$

- Note: full crossing symmetry is not automatic, and it should impose constraints on what can be exchanged!

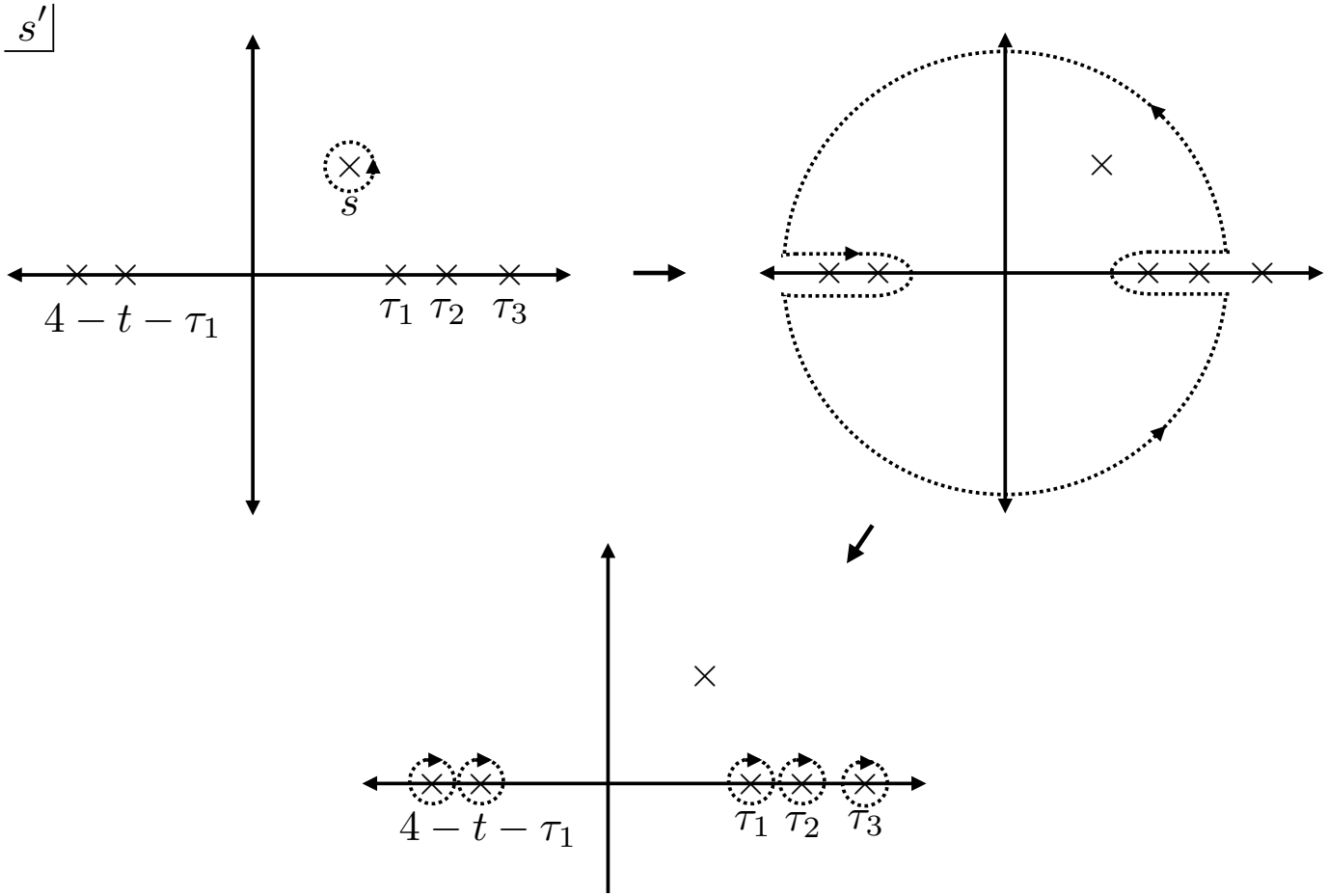
Can we make this precise?

Dispersive sums

1. Use Cauchy theorem

$$\mathcal{M}_{tree}(s,t) = \oint \frac{ds'}{2\pi i} \frac{\mathcal{M}_{tree}(s',t)}{s'-s}$$

2. Blow up the contour, pick contributions from the poles



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2. Blow up the contour, pick contributions from the poles

$$\mathcal{M}_{tree}(s,t) = \sum_{\tau,\ell} Res \frac{\mathcal{M}_{tree}(s',t)}{s'-s}\Bigg|_{s'=poles}$$

3. Expanding both sides in $1/\lambda$:

$$\underbrace{\mathcal{M}_{tree}(s,t)}_{\text{Polynomials in the } 1/\lambda \text{ expansion}} = \underbrace{\sum_{\tau,\ell} Res \frac{\mathcal{M}_{tree}(s',t)}{s'-s}\Bigg|_{s'=poles}}_{\text{CFT data of heavy operators}}$$

AdS Virasoro-Shapiro amplitude

Leading order coefficient

$$\boxed{\frac{\alpha_{0,0}}{\lambda^{3/2}}} + \frac{\alpha_{1,0}(s^2+t^2+u^2)+\gamma_{0,0}}{\lambda^{5/2}} + \frac{\alpha_{0,1}\, s\, t\, u + \cdots}{\lambda^3} + \cdots$$

$$\Downarrow$$

Tower of heavy operators labelled by $\delta = 1, 2, \cdots$ and $\ell = 0, 2, \cdots, 2\delta - 2$

$$\tau(\delta,\ell) = \tau_0(\delta)\lambda^{1/4} + \tau_1(\delta,\ell) + \tau_2(\delta,\ell)\lambda^{-1/4} + \cdots$$

$$C^2_{\delta,\ell} = f_0(\delta,\ell)\left(1 + f_1(\delta,\ell)\lambda^{-1/4} + f_2(\delta,\ell)\lambda^{-1/2} + \cdots\right)$$

Exactly as expected!

AdS Virasoro-Shapiro amplitude

<u>General structure</u>

$$\tau(\delta,\ell) = \tau_0(\delta)\lambda^{1/4} + \tau_1(\delta,\ell) + \tau_2(\delta,\ell)\lambda^{-1/4} + \cdots$$

Related to $\alpha_{a,b}$ Related to subleading coef. $\beta, \gamma, \cdots$

$$C^2_{\delta,\ell} = f_0(\delta,\ell)\left(1 + f_1(\delta,\ell)\lambda^{-1/4} + f_2(\delta,\ell)\lambda^{-1/2} + \cdots\right)$$

And we get precise equations relating the two!

Note: $\tau(\delta,\ell)$ and $C^2_{\delta,\ell}$ is CFT data of planar $\mathcal{N} = 4$ SYM, so it should be in principle computable from integrability!

AdS Virasoro-Shapiro amplitude

Simplest equation to leading order

$$\sum_{\delta=1,2,\cdots}\left(\sum_{\ell=0}^{2\delta-2}\frac{4^{2q+3}}{\tau_0(\delta)^{4q+6}}f_0(\delta,\ell)\right)=\alpha_{q,0}=\zeta(2q+3)$$

Large q expansion...$[\tau_0(1)<\tau_0(2)<\cdots\ ]$

$$\frac{4^{2q+3}}{\tau_0(1)^{4q+6}}f_0(1,0)+\frac{4^{2q+3}}{\tau_0(2)^{4q+6}}(f_0(2,0)+f_0(2,2))+\cdots=1+\frac{1}{2^{2q+3}}+\cdots$$

$$\downarrow$$

$$\tau_0(\delta)=2\sqrt{\delta},\qquad \sum_{\ell=0}^{2\delta-2}f_0(\delta,\ell)=1$$

- The twist agrees exactly with the GKP analysis!

AdS Virasoro-Shapiro amplitude

- All terms $\alpha_{a,b}, \beta_{a,b}, \cdots$ follow from data for heavy short operators + our equations!

$$\alpha_{a,b} = \sum_{\delta=1}^{\infty} \sum_{\ell=0}^{2\delta-2} \text{terms with } f_0$$

$$\beta_{a,0} = \sum_{\delta=1}^{\infty} \sum_{\ell=0}^{2\delta-2} \frac{f_2(\delta,\ell) - (2a+3) f_0(\delta,\ell)\tau_2(\delta,\ell)}{\delta^{4+2a}} + \text{terms with } f_0$$

- Crossing imposes non-trivial linear constraints on the CFT data!

$$\sum_{\delta=1}^{\infty} \sum_{\ell=0}^{2\delta-2} \frac{1}{\delta^4} \left(2\ell(\ell+2) - 3\right) f_0(\delta,\ell) = 0$$

Not at all obvious from the integrability perspective!

AdS Virasoro-Shapiro amplitude

- Moreover the structure of the equations is quite rigid!
- Combining crossing constraints with the transcendental structure of $\mathcal{N}=4$ SYM seems to be very powerful!
- We have a prediction for the leading Regge trajectory data...

$$\Delta(\ell) = \sqrt{2\ell+4}\lambda^{1/4} - 2 + \frac{3\ell^2+10\ell+16}{4\sqrt{2\ell+4}}\lambda^{-1/4} + \cdots$$

$$C^2(\ell) = f_0(\ell)(1+\frac{3\ell+23/4}{\sqrt{2\ell+4}}\lambda^{-1/4}+\left(R+2(\ell/2+1)^2\zeta_3\right)\lambda^{-1/2}+\cdots)$$

The dimension exactly agrees with the dimension at strong coupling of Konishi-like operators!

The OPE coefficients are new results!

Conclusions

- Explicit relation between the $1/\lambda$ expansion of the AdS VS amplitude and the planar CFT data of heavy operators/short strings at strong coupling.
- Non trivial constraints for the CFT data, not obvious from integrability!
- The rigidity of the equations + some mild assumptions, lead to new results!
- New connection between standard bootstrap techniques, localisation and integrability!

Computing the full AdS VS amplitude seems now doable, and the best way to package the CFT data of short strings!