Structure constants of short operators in planar N=4 SYM

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based on work with A. Georgoudis and A. Klemenchuk Sueiro

What controls structure constants?

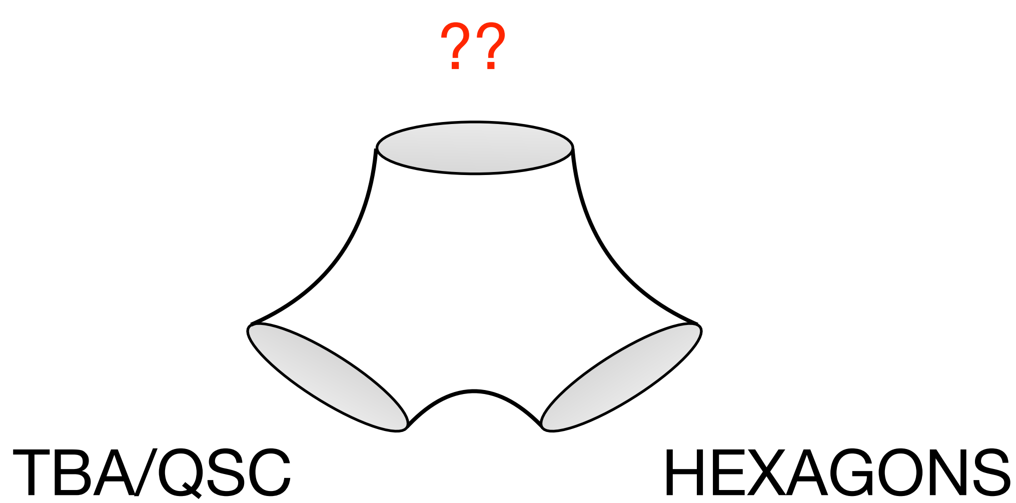
TBA/QSC: Overwhelming evidence that the spectrum of scaling dimensions of single-trace operators is described by the Quantum Spectral Curve - which upgrades the Thermodynamic Bethe Ansatz to deal with operators of any sort and make all symmetries manifest

HEXAGON: Growing evidence that correlation functions of single-trace operators may be built for all values of the ’t Hooft coupling (and order by order in 1/N) using hexagons

The former describes the punctures and the latter the planar geometry away from them

What controls structure constants?

Is there a unified picture for the spectrum and the correlation functions?



This might not be urgently needed for higher-point functions of protected operators

Structure constants more “interesting” for non-protected operators where inputs from the exact wave functions appear needed - wrapping issue

This talk : Evidence / conjecture that the two paths join when wrapping effects are becoming important

Remark: there are other ways, like combining spectral data with conformal bootstrap - see Fernando’s talk

see also [Cavaglia,Gromov,Julius,Preti] [Caron-Huot,Coronado,Trinh,Zahraee]

Plan

Conjecture for upgrading hexagon representation for structure constants using solution to spectral problem in the simple case of 2 BPS and 1 non-BPS operators

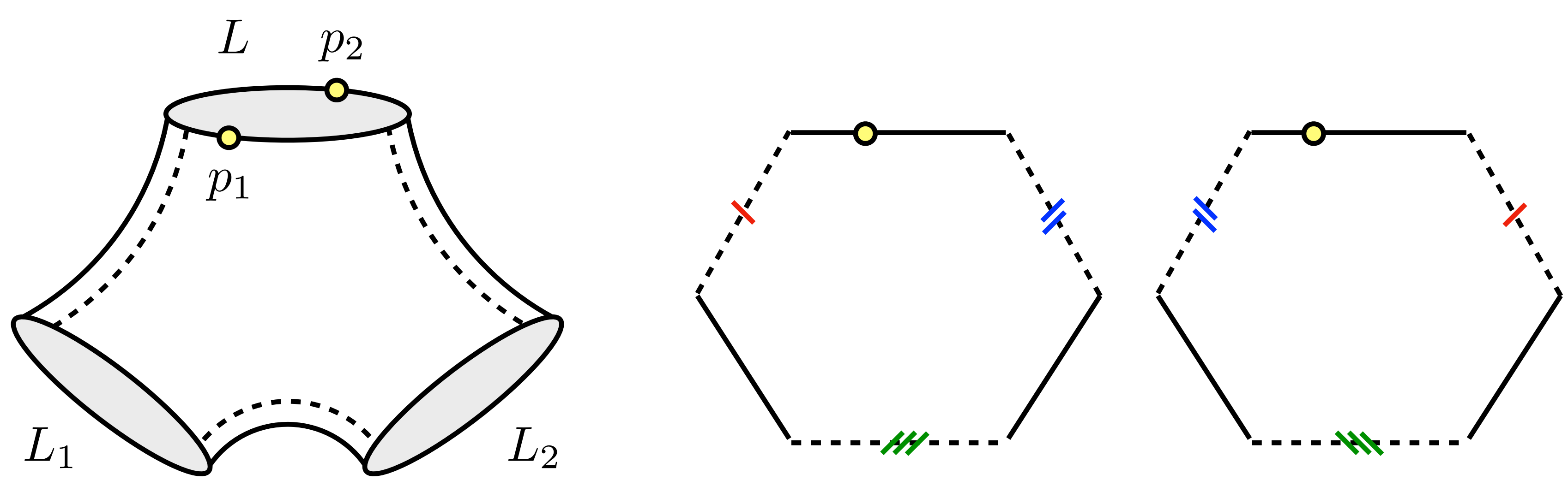
$$C^{\circ\circ\bullet} = \langle \mathrm{Tr}\, Z_1^{L_1} \mathrm{Tr}\, Z_2^{L_2} \mathrm{Tr}\, D^S Z^L \rangle$$

Checks at weak/strong coupling

Conjecture

Hexagons

The hexagon method relies on the idea that the pair of pants may be covered using hexagons - only two hexagons are needed for 3pt functions



Hexagons are bounded by 3 spin chains and 3 mirror cuts (seams)

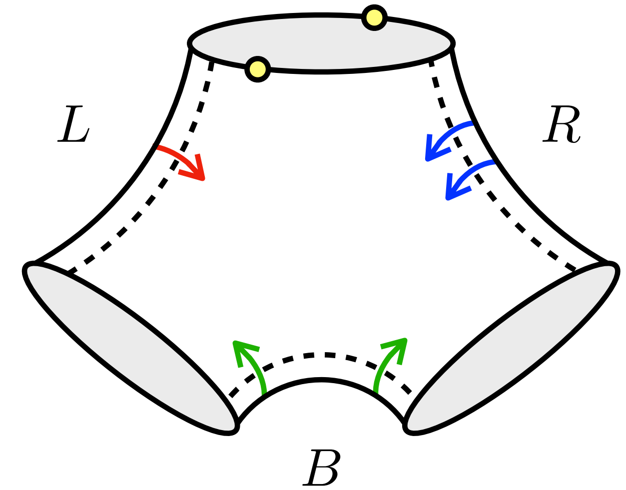
They are describe with form factors giving the amplitudes for absorption of excitations on all the edges

Excitations are magnons on the spin-chain operators or their mirror partner - describing the fluctuations of the open strings along the mirror cuts

The glue-back-together procedure entails summing over all the fluctuations of the latter open strings and also over all possible ways of distributing physical magnons on the two hexagons (will get back to that)

Hexagons

Sum over a complete basis of mirror states along each cut



$$C^{\circ\circ\bullet} = \mathcal{N} \times \sum_L \sum_R \sum_B e^{-\ell_L \mathcal{E}_L - \ell_R \mathcal{E}_R - \ell_B \mathcal{E}_B} |\mathcal{H}|^2$$

States are labelled by rapidities and spins

$$\sum = \sum_{N=0}^{\infty} \prod_{i=1}^{N} \sum_{a_i=1}^{\infty} \int \frac{du_i}{2\pi} \mu_{a_i}(u_i) \prod_{i<j} p_{a_i,a_j}(u_i,u_j)$$

They determine $\mathfrak{su}(2|2)^2$ rep. and energy of the magnon $\mathcal{E}_a(u) = \log\left(x^{[+a]}x^{[-a]}\right)$

Integrand follows from hexagon form factors after summing over all flavours

[BB,Goncalves,Komatsu,Vieira]

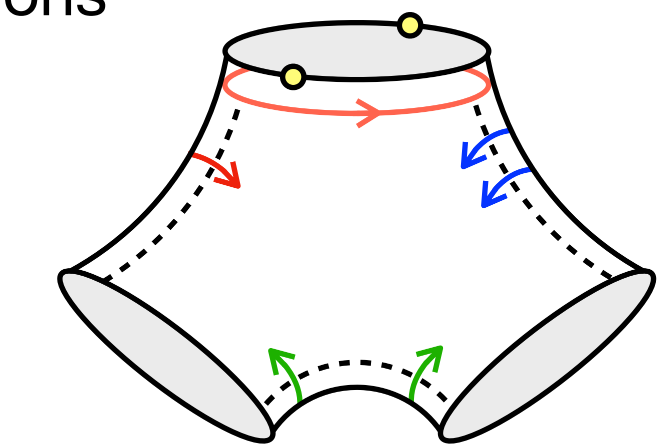
$$|\mathcal{H}|^2 = \prod_{i,j,k}^{N_L,N_R,N_B} \frac{\mathbb{W}^L_{a_i}(u_i)\mathbb{W}^R_{b_j}(v_j)\mathbb{W}^B_{c_k}(w_k)}{p_{a_i b_j}(u_i,v_j)}$$

The weights $\mathbb{W}_a(u)$ are given by $\mathfrak{su}(2|2)$ transfer matrices $T_a(u) = \mathrm{str}\, S_{a1}(u,\mathbf{z})$

E.g. $\mathbb{W}^R_a(u) = \dfrac{T_a(u)}{h_{1a}(\mathbf{z},u)}$ $\qquad$ $\mathbb{W}^L_a(u) = h_{a1}(u,\mathbf{z})\, T_a(u)$

Hexagons dead end?

This would be the end the day if it were not for wrapping corrections



They may be seen to arise from a singularity of the interaction between mirror magnons on L and R

$$p_{ab}(u,v) \sim \delta_{ab}\mu_a(u)(u-v)^2$$

These poles map to tiny loops winding around the excited operator and they remind us that finite-size corrections are not that easy to encode

Schematically, we are lacking a prescription for integrating the double poles and a way of introducing wrappings in the picture - the two are related as changing prescription may result in redefining what we mean by wrappings

Thee same type of corrections enter the energy levels so why not using TBA here?

Problem is that we are not calculating a free energy and it is not immediately obvious how to set up the TBA calculation on a pair of pants

Searching for wrappings

The long way to go

Start from something finite - like a 4pt function of protected operators - which may be calculated using hexagons in principle

[Fleury,Komatsu]
[Eden,Sfondrini]

Read off structure constants of interest by going to the OPE limit

Everything should be finite and physically regulated

Less risk of introducing artefacts of a badly devised regulator

But hard to follow all the way to the end - need to find appropriate mean field techniques (such as the ones developed for structure constants of determinant operators or studies of torus/cylinder partition functions)

[Jiang,Komatsu,Vescovi]
[Long,Kostov,Serban], [Kostov]

The lucky way to go

Extract as much as we can from such analyses at first orders (leading exponentially small corrections in the volume) and make an educated guess

Non-zero risk of making the wrong extrapolation

But as many tries as you want and conjectures may be tested in various regimes

Observations

Formula remains the same up to a "renormalisation" of the weights $\mathbb{W} = \mathbb{W}^{\text{asy}} + \delta\mathbb{W}$ (wrapping induced)

Extrapolating, the corrected weights obey the relations

$$\boxed{\begin{aligned} & e^{-L\mathcal{E}_a(u)}\mathbb{W}^L_a(u)\mathbb{W}^R_a(u) = \frac{Y_{a,0}}{1+Y_{a,0}} \\ & \frac{\mathbb{W}^L_a(u)}{\mathbb{W}^R_a(u)} = p_{a1}(u,\mathbf{z})\, e^{i\sum_b \fint \frac{dv}{2\pi} L_b(v)\partial_v \log p_{ba}(v,u)} \end{aligned}}$$

with $Y_{a,0}$ the Y functions solving the TBA equations and $L_a = \log\left(1+Y_{a,0}\right)$

Bonus: very pleasant form in terms of the super-conformal transfer matrices

[Gromov,Kazakov,Vieira]

$$Y_{a,s} = \mathbf{T}_{a,s+1}\mathbf{T}_{a,s-1}/\mathbf{T}_{a+1,s}\mathbf{T}_{a-1,s}$$

giving

$$\boxed{\mathbb{W}^L_a(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^-_{a,0}(u)} \qquad \mathbb{W}^R_a(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^+_{a,0}(u)}}$$

with $\mathbf{T}^{\pm}_{a,s} = \mathbf{T}_{a,s}(u \pm i/2)$

and with the T functions defined in the distinguished (GKLV) **T**-gauge [Gromov,Kazakov,Leurent,Volin]

Emerging philosophy

Asymptotic transfer matrices (weights) rely on symmetries preserved by a long ferromagnetic chain and the associated S-matrix describing scattering of magnons on the chain

Description is accurate for long operators but not for short operators (wrapping corrections)

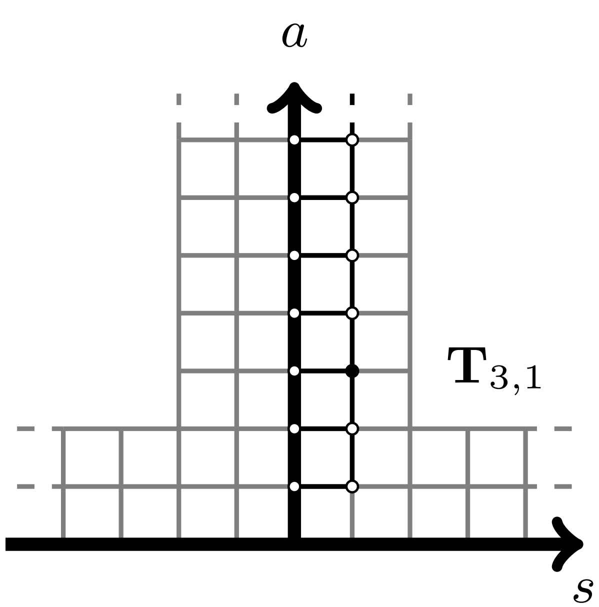
It appears natural somehow that wrapping corrections may be included in the formalism by means of transfer matrices associated to the *full* super-conformal algebra

Philosophy at work for the spectrum : T functions $\longrightarrow$ Y functions / TBA eqs

[Gromov,Kazakov,Vieira]

Main (defining) property: T functions obey the Hirota equation

$$\mathbf{T}^+_{a,s}\mathbf{T}^-_{a,s} = \mathbf{T}_{a+1,s}\mathbf{T}_{a-1,s} + \mathbf{T}_{a,s+1}\mathbf{T}_{a,s-1}$$



$\mathbf{T}_{a,s} = \mathbf{T}_{a,s}(u)$ depends on the spectral parameter and integers

The latter label (infinite-dimensional) representations of the super-conformal algebra (auxiliary spaces)

[Gromov,Kazakov,Leurent,Tsuboi]

Missing weight

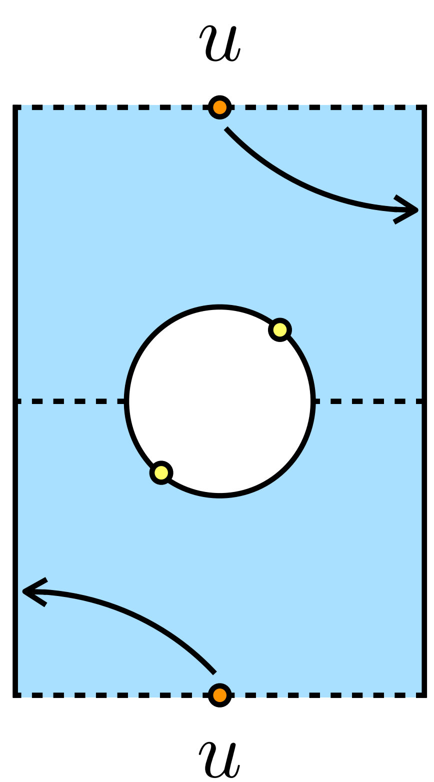
The corrections to the B-channel weights appear more complicated at first sight and not simply related to the T functions

A closer look reveals that they are transfer matrices but for a different family of representations

Namely, they are associated with finite-dimensional representations of the super-conformal algebra

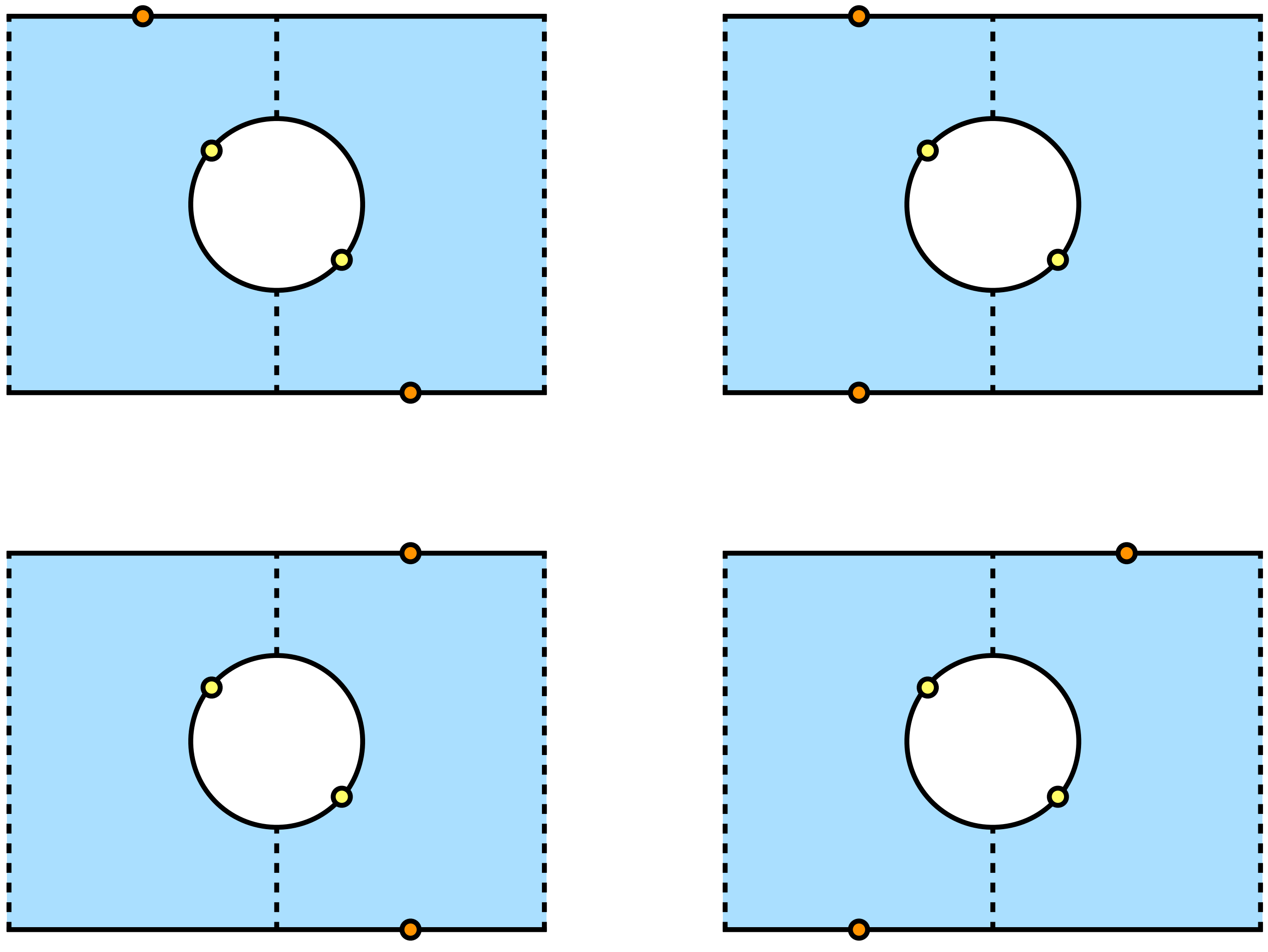
To see that, it helps thinking about these weights as diagonal form factors for the non-BPS operator and continue the argument of the weights to the physical kinematics

$$\mathbb{W}^B(u) =$$



Missing weight

In this kinematics the weight is given by the sum (for the different ways of putting a magnon on the bottom and top chains)



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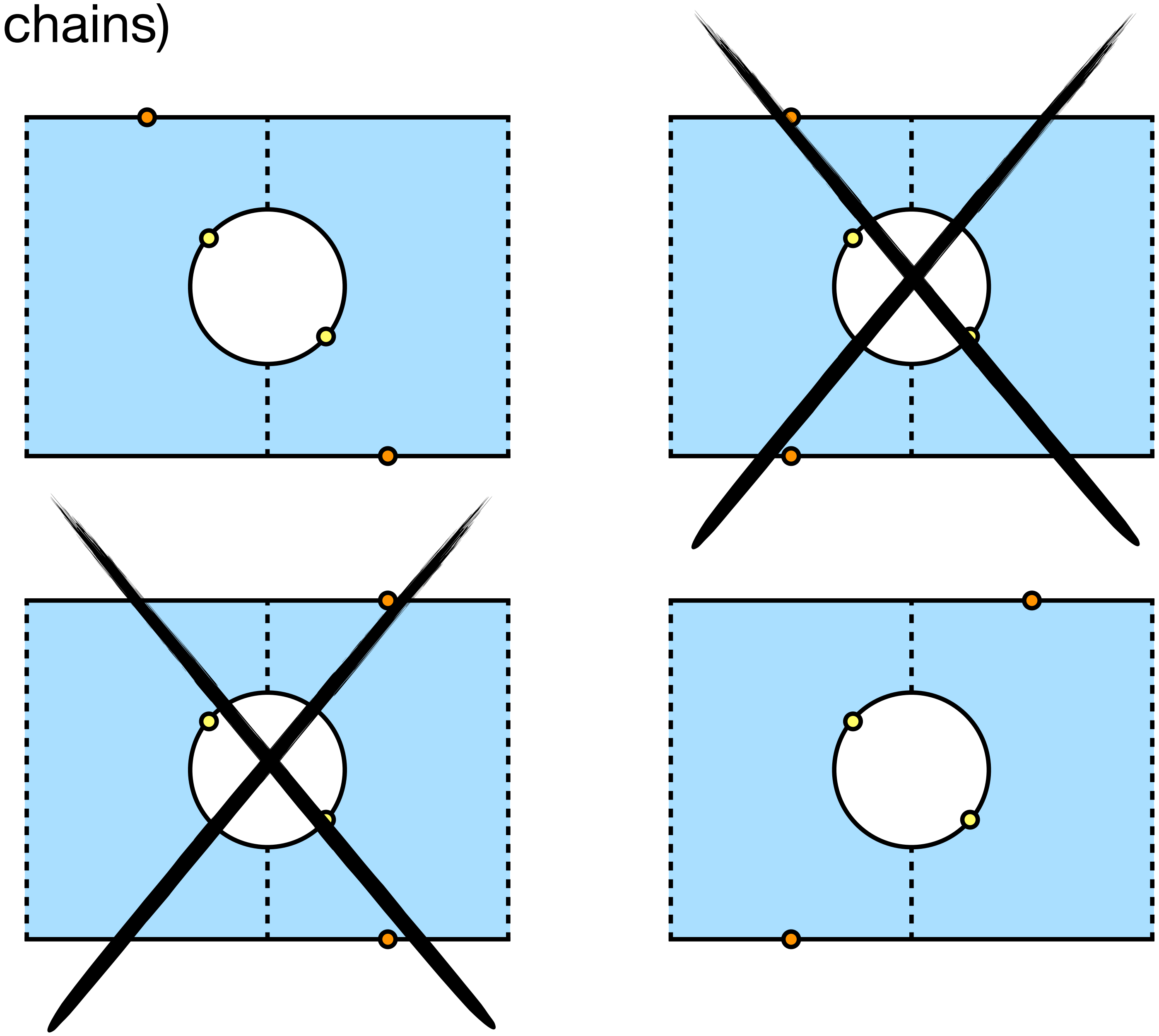
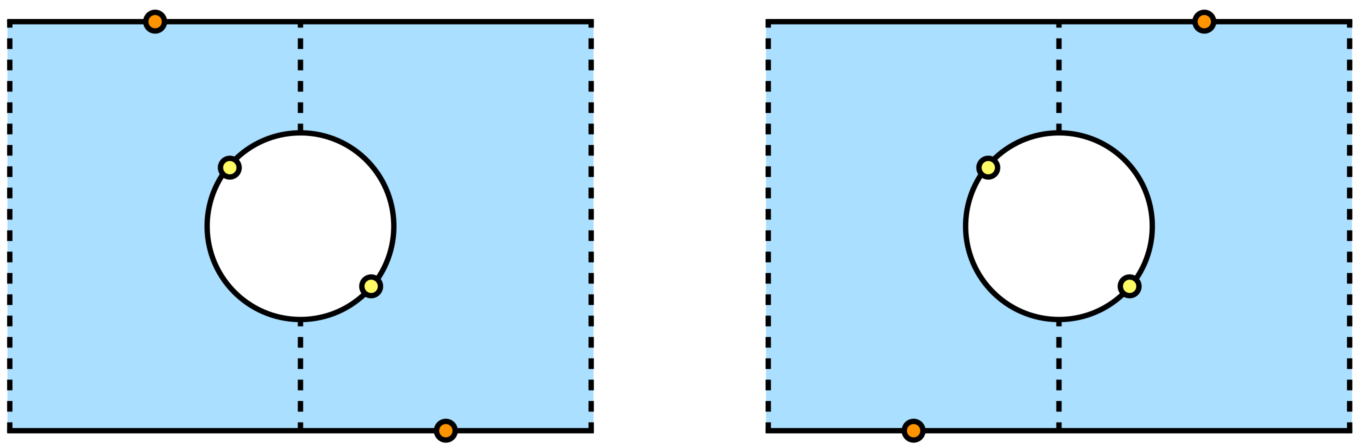


Two processes go to zero after identifying state at bottom and top and summing over the flavours

Missing weight

Only two terms remain



End result is of the type (for a=1)

$$\mathbb{W}^B(u) = \frac{e^{ipL/2}T(u)}{h(\mathbf{z},u)} + \frac{e^{-ipL/2}\bar{T}(u)}{h(u,\mathbf{z})}$$

which is a representation of the transfer matrix in the 4|4 of $\mathtt{su}(4|4)$ [Beisert]

This identification may be seen to remain when finite-size corrections are taken into account (for the leading exponentially small corrections at least)

Simplest solution to a little bootstrap of sort, since weight B should be regular and crossing symmetric

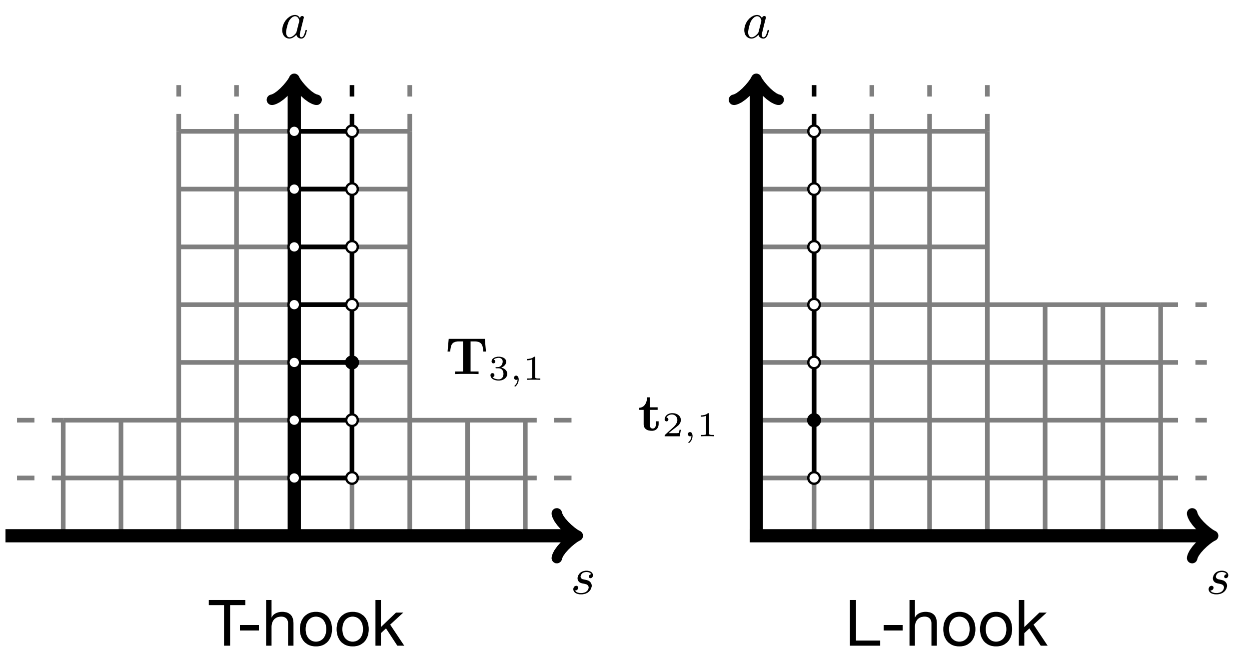
Conjecture summary

Uplifted weights

$$\mathbb{W}_a^L(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}_{a,0}^-(u)}$$

$$\mathbb{W}_a^R(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}_{a,0}^+(u)}$$

$$\mathbb{W}_a^B(u) = e^{-\frac{1}{2}L\mathcal{E}_a(u)}\mathbf{t}_{a,1}(u)$$



They come with a prescription for integrating the double poles

$$p_{ab}(u,v) \to p_{ab}(u+i0, v-i0)$$

Remark: transfer matrices in general are only defined up to gauge transformations

$$T_{a,s} \to g_1^{[a+s]} g_2^{[a-s]} g_3^{[s-a]} g_4^{[-a-s]} T_{a,s}$$

In our case the gauge is fixed and the T-t's are normalised as in GKLV

[Gromov,Kazakov,Leurent,Volin]
[Kazakov,Leurent,Volin]

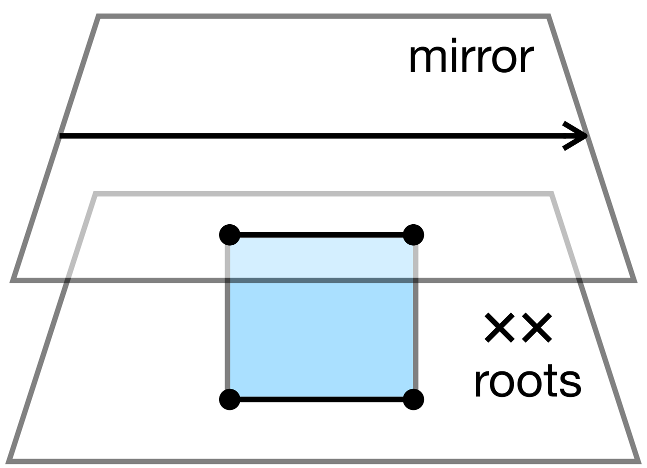
Bethe roots

[Dorey,Tateo]
[Bazhanov,Lukyanov,Zamolodchikov]
[Fioravanti,Mariottini,Quattrini,Ravanini]
[Pozsgay,Szecsenyi,Takacs]
[Jiang,Komatsu,Vescovi]

Formula embodies a sort of analytic continuation trick for excited states

Akin to excited TBA trick and procedures for generalised Leclair-Mussardo formula and for structure constants of determinant operators

Key observation : the weights have poles on the physical sheets at the locations of the Bethe roots



Contour of integration goes along the real mirror line and around the Bethe roots on the physical sheet

Not all weights have poles :

[Jiang,Komatsu,Kostov,Serban]

Bottom weight is regular (transfer matrices are regular according to GKLV)

Adjacent weights have simple poles but on different sheets - from zeros of $\mathbf{T}^{\pm}_{1,0}$

We can go for poles of either sheet (the two choices are equivalent)

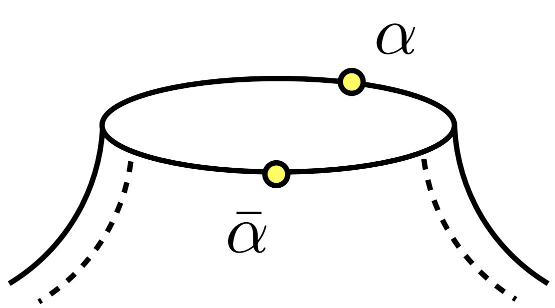
Contour integrals

The contour integrals produce a finite sum over the subsets of the set of roots
The mechanism is well known and was first put to work in [Jiang,Komatsu,Kostov,Serban]

This is the sum over partitions popularised by the tailoring approach to structure constants at weak coupling [Escobedo,Gromov,Sever,Vieira]

It describes the ways of distributing Bethe roots on the two hexagons

$$\sum \oint = \sum_{\alpha \subset \text{roots}} S_<(\alpha, \bar{\alpha})\; e^{ip(\alpha)\ell_R} \times$$



It yields $$\sum_{\alpha \subset \text{roots}} \frac{(-1)^{|\alpha|} e^{ip(\alpha)\ell_R}\, \mathcal{T}(\alpha)}{h(\bar{\alpha}, \alpha)}$$

with T the TBA-like dressing factor

$$\mathcal{T}(\alpha) = e^{\frac{i}{2}\Phi_\alpha - \frac{i}{2}\sum_a \int \frac{du}{2\pi} \log\left(1+Y_a(u)\right)\partial_u \log p_{a1}(u,\alpha)}$$

Remarks:
1) Nice to see the hexagon form factor coming out of the transfer matrices
2) The dressing agrees with the Luscher formula worked out in [BB,Fleury,Komatsu]

Normalisation

A proper normalisation $\mathcal{N}$ of the state is also needed

Drawing inspiration from recent results for structure constants of determinant operators / g-functions, we are led to set [Jiang,Komatsu,Vescovi]

$$\mathcal{N}^2 = \mathcal{D}^{-1} e^{-\frac{1}{2}\sum_{a,b}\int \frac{du\,dv}{(2\pi)^2} L_a(u) L_b(v) \partial_u \partial_v \log p_{ab}(u,v)}$$

where $L_a(u) = \log\left(1 + Y_a(u)\right)$ and with the Fredholm-like determinant

$$\log \mathcal{D} = -\sum_{N=1}^{\infty} \frac{1}{N} \prod_{i=1}^{N} \sum_{a_i=1}^{\infty} \int \frac{du_i}{2\pi} \frac{Y_{a_i,0}(u_i)}{1 + Y_{a_i,0}(u_i)} K^{\text{eff}}_{a_i,a_{i+1}}(u_i, u_{i+1})$$

The kernel K derives from the S-matrix and includes an average over the auxiliary Y functions

The contour of integration goes around the real line and around the zeros of $1 + Y_{1,0} = 0$

Checks

Gauge theory check

Consider a ratio of structure constants

$$R(\ell_B) = C^{\circ\circ\bullet} / \lim_{\ell_B \to \infty} C^{\circ\circ\bullet}$$

Thanks to factorised form of the conjecture, it is given by a single sum of mirror excitations

$$R(\ell_B) = 1 + \sum_{a=1}^{\infty} \int \frac{du}{2\pi} \mu_a(u) e^{-\ell_B \mathcal{E}_a(u)} \mathbb{W}_a^B(u) + \dots$$

(N-magnons integrals)

Threshold at weak coupling $g^2 \to 0$

N-magnons integral $= O(g^{2N(N+\ell_B)})$

1st wrapping $= O(g^{2(L+\ell_B+2)})$

Wrapping dominates for low enough L

Gauge theory check

Particular case $\langle \mathrm{Tr}\, Z_1^2\, \mathrm{Tr}\, Z_2^2\, \mathrm{Tr}\, D^2 Z^2 \rangle$ with $L = S = 2\ell_B = 1$

2-magnon integral kick in at 6 loops

Wrappings start at 5 loops

Field theory calculation reveals discrepancy with (naive) hexagon formula at 5 loops

[Chicherin,Georgoudis,Goncalves,Pereira]

$$\delta R / g^{10} = 972\zeta_3 - 2700\zeta_5 + 5355\zeta_7 - 2376\zeta_3\zeta_5 - 1512\zeta_9$$

With $\zeta_n = \sum_{a=1}^{\infty} a^{-n}$ the Riemann zeta function

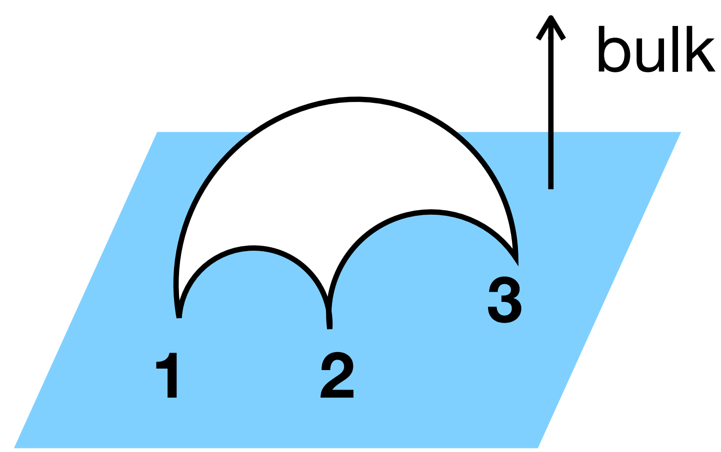
To check agreement, plug expressions for transfer matrix [Kazakov,Leurent,Volin]

$$\mathbf{t}_{a,1}(u) = -\sum_{j=1}^{4} \mathbf{Q}_j(u + ia/2)\tilde{\mathbf{Q}}^j(u - ia/2)$$

Apply perturbative QSC solvers to generate expressions at weak coupling [Marboe,Volin] [Gromov,Levkovich-Maslyuk,Sizov]

Holographic check

String Theory in $AdS_5 \times S^5$



Boundary data

2 protected operators producing point-like strings boosted along big circles in S_5

1 excited operator producing a classical string spinning in $AdS_5 \times S^5$

Strong coupling : string tension $\sqrt{\lambda}/2\pi \gg 1$

Different regimes depending on quantum numbers

Short string (flat-space limit) : $L, S = O(1) \;\Rightarrow\; \Delta = O(\lambda^{1/4})$ see Fernando's talk

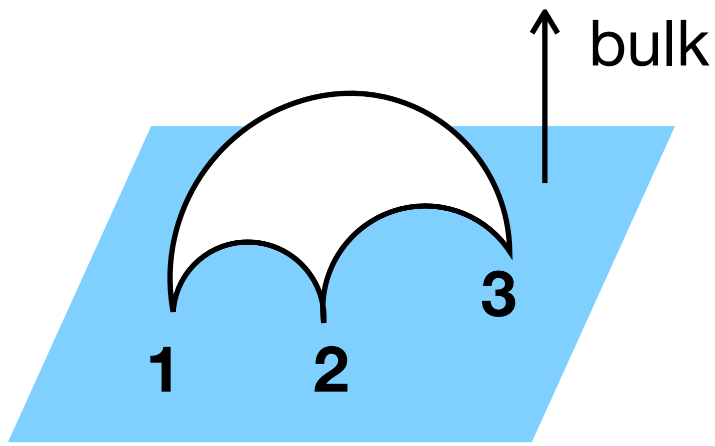
"CFT limit of TBA" : full of wrappings, very hard

Classical string : $L, S, \Delta = O(\sqrt{\lambda})$

More tractable: use classical world-sheet theory and its integrability (finite-gap solutions)

Holographic check

String Theory in $AdS_5 \times S^5$



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Classical limit

Area of minimal surface ending on 3 operators at boundary

$$\log C_{123} \approx -\mathrm{Area}$$

Integrability allows one to express the area in terms of the quasi-momenta of the classical curves for the punctures/states

[Kazama,Komatsu,Nishimura]

Holographic check

Boundary data

[Kazakov,Marshakov,Minhan,Zarembo]
[Beisert,Kazakov,Sakai,Zarembo]

Monodromy matrix $\Omega(x) \cong (e^{i\hat{p}_{1,2,3,4}}|e^{i\tilde{p}_{1,2,3,4}}) \in SU(4|4)$

With quasi-momenta depending on spectral parameter x

String formula

[Kazama,Komatsu,Nishimura]

$$\log C^{\circ\circ\bullet} = Real + Mirror$$

Real part comes from sum over partitions and Gaudin determinant (no wrappings)

[Gromov,Sever,Vieira]

Mirror part contains wrapping corrections

Several components matching with the various mirror channels and norm

Focus on ratio $$\log R_{\text{string}} = \int_{U^+} \frac{du(x)}{2\pi} \sum_{j=1}^{4} \left[\mathrm{Li}_2(\xi e^{i\tilde{p}_j}) - \mathrm{Li}_2(\xi e^{i\hat{p}_j})\right]$$

With $\xi = e^{-(L_1+L_2)\mathcal{E}(x)/2}$, $u(x) = g(x+1/x)$

NB: j=1,2 dominate at large L and j=3,4 represent wrappings

Holographic check

Hexagon formula

Use clustering method to re-sum the mirror series [Jiang,Komatsu,Kostov,Serban]

$$\log R_{\text{strong}} = \int_{U+} \frac{du(x)}{2\pi} \int_0^{\xi} \frac{dq}{q} \log \left[\sum_{a=0}^{\infty} q^a \mathbf{t}_{a,1}(x) \right]$$

With $\mathbf{t}_{a,0} = 1$

At strong coupling, the t's become characters of the group element Ω

[Gromov,Kazakov,Leurent,Tsuboi]

$$\sum_{a=0}^{\infty} q^a \mathbf{t}_{a,1} = \text{sdet}\left(1 - q\Omega(x)\right) = \prod_{j=1}^{4} \frac{1 - q e^{i\tilde{p}_j(x)}}{1 - q e^{i\hat{p}_j(x)}}$$

Hence

$$\log R_{\text{strong}} = -\int_{U+} \frac{du(x)}{2\pi} \text{str}\left[\text{Li}_2(\xi\,\Omega)\right]$$

In perfect agreement with the string formula

Conclusion

Conjecture for structure constants of short operators

Hexagon representation + solution to spectral problem

Similar to Leclair-Mussardo formula for finite-volume matrix
elements of local operators in integral QFTs

Lots of evidence: IR checks, agreements with gauge and string theory

Conjecture extends naturally to higher-rank sectors, for LR symmetric states

Proof?

Further evidence? Complete analysis of the conjecture at strong coupling for the remaining components (sum over adjacent channels - Fedholm determinant)

Can one re-sum the mirror series such as to obtain more concise representation?

Explore new regimes? Short strings?

Thank you!