Non-local geometry of Bethe Algebras and Monodromy bootstrap

Dmytro Volin

NORDITA

UPPSALA UNIVERSITET

IGST 2022

Based on

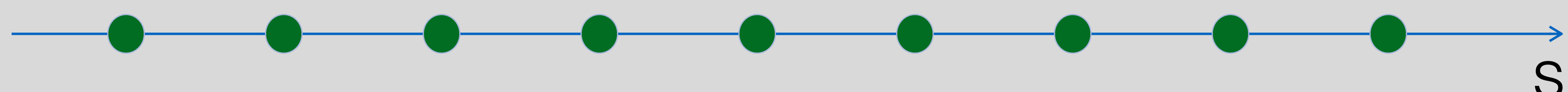
2008.10597 /w H.Shu & S.Ekhammar

2104.04539, 2109.06164, 22xx.xxxxx /w S.Ekhammar

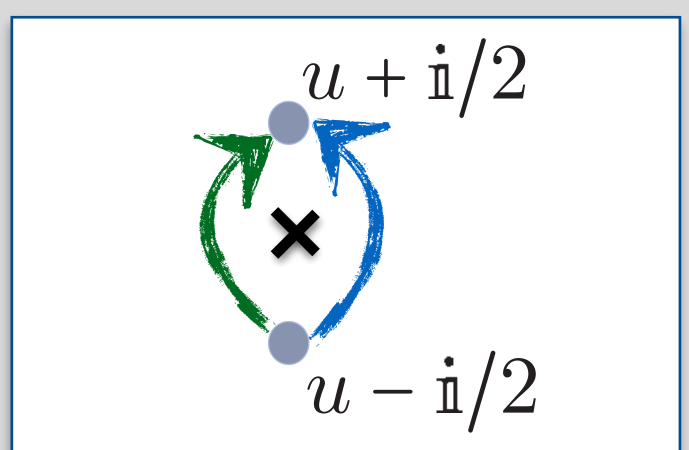
also earlier works…

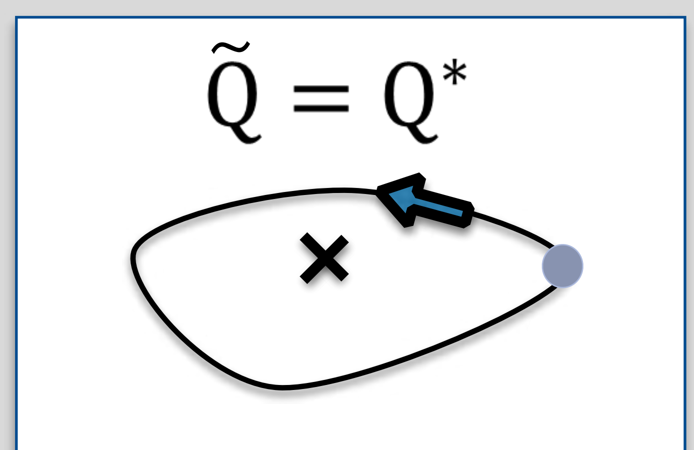
/w D.Chernyak, N.Gromov, V.Kazakov, S.Leurent, F.Levkovich-Maslyuk, C.Marboe, P.Ryan

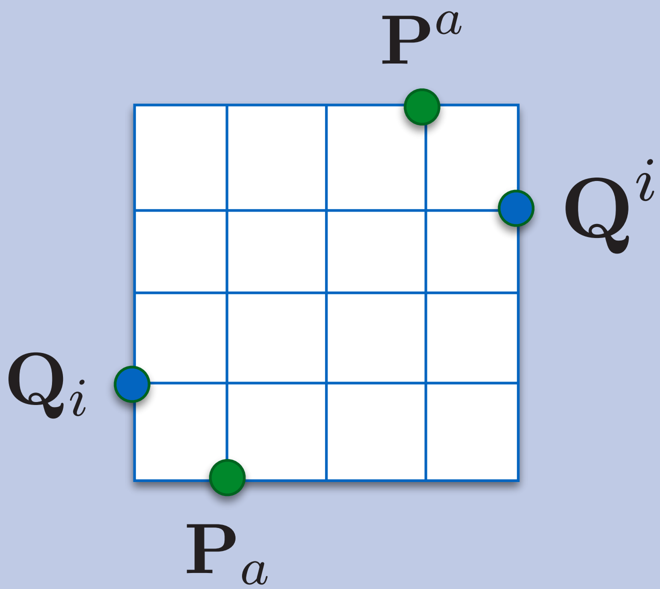
…and the IGST 2015 talk

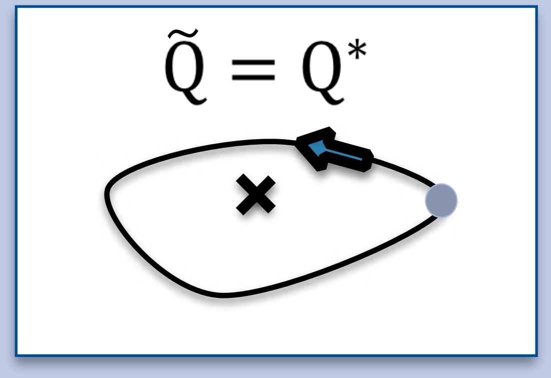


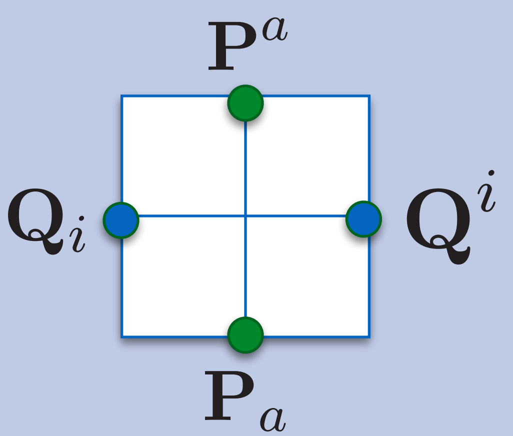
$$Y_s[u+\mathbb{i}/2]\,Y_s[u-\mathbb{i}/2] = (1+Y_{s+1}[u])(1+Y_{s-1}[u])$$











psl(4|4) Q-system

$$\tilde{\mathbf{P}}_a = \mu_{ab}\,\mathbf{P}^b$$

$\mathrm{AdS}_5/\mathrm{CFT}_4$ QSC

[Gromov, Kazakov, Leurent, D.V.'13]

sl(2|2) Q-system

$$\tilde{\mathbf{P}}_a = \frac{\mu}{F}\epsilon_{ab}\mathbf{P}^b$$

Hubbard QSC

[D.V.'15]
[Ekhammar, D.V.'21]

GOAL: given a symmetry algebra $\mathfrak{g}$, apply monodromy bootstrap to get an AdS/CFT-type QSC

PROBLEM: what is Q-system for given $\mathfrak{g}$?

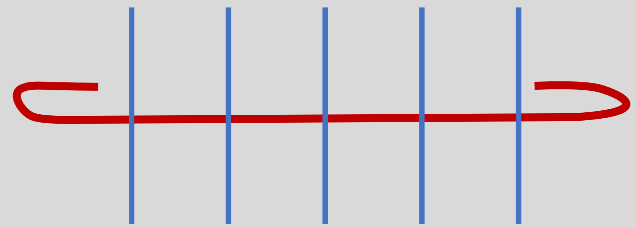
THIS TALK:

- Q-system when $\mathfrak{g}$ - any (non-supersymmetric) simple Lie algebra
- Monodromy bootstrap to conjecture $\mathrm{AdS}_3/\mathrm{CFT}_2$ QSC
- Bonus spin-offs

\- solving T-systems based on any Dynkin diagrams
\- alternatives to Bethe equations
\- conjectures about rigorous completeness theorems

Part Ia
Q-systems

Main example: rational spin chains - representations of Yangian $Y(\mathfrak{g})$

$$T(u) = Tr\ M(u) =$$ 

$$[T(u), T(v)] = 0$$

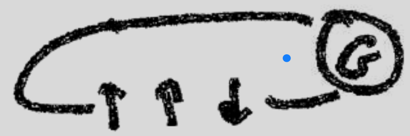
set of commuting quantities

Bethe Algebra

- For finite-dimensional Hilbert spaces commuting quantities always exist and hence their existence is a useless feature to test integrability.
- Integrability is hidden in existence of commuting operators - **functions of spectral parameter** which satisfy certain functional relations

- Integrability is hidden in existence of commuting operators - **functions of spectral parameter** which satisfy certain functional rhelations

SU(2) example



$\#\uparrow = M$

$\#\downarrow = L-M$

$G = \begin{pmatrix} x & \\ & 1/x \end{pmatrix}$

Baxter Q-operators $\binom{L}{M}\times\binom{L}{M}$ matrices

$Q = x^{-iu} \cdot$ polynomial in u of degree M

$\bar{Q} = x^{+iu} \cdot$ p lynomial in u of degree $L-M$

$$W(Q,\bar{Q}) = \begin{vmatrix} Q(u+i/2) & Q(u-i/2) \\ \bar{Q}(u+ & \bar{Q}(u-i/2) \end{vmatrix} = u^L$$

Wronskian Bethe equations

Bonus 1: $\left.\begin{array}{l} \bullet \text{ Multiplicity-free solutions only} \\ \bullet \ \binom{L}{M} \text{ solutions} \end{array}\right\} \leftarrow$ rigorous completeness

Bonus 2: Easy to get transfer matrices

$$T_s = \begin{vmatrix} Q(u+\frac{i}{2}(s-1)) & Q(u-\frac{i}{2}(s-1)) \\ \bar{Q}(u+\frac{i}{2}(s-1)) & \bar{Q}(u-\frac{i}{2}(s-1)) \end{vmatrix}$$

- We want to do something similar for any simple Lie algebras

$$T_s = \begin{vmatrix} Q(u+\tfrac{i}{2}(s-1)) & Q(u-\tfrac{i}{2}(s-1)) \\ \bar{Q}(u+\tfrac{i}{2}(s-1)) & \bar{Q}(u-\tfrac{i}{2}(s-1)) \end{vmatrix}$$

$$W(\mathbb{Q}, \bar{\mathbb{Q}}) = \begin{vmatrix} \mathbb{Q}(u+\tfrac{i}{2}) & \mathbb{Q}(u-\tfrac{i}{2}) \\ \bar{\mathbb{Q}}(u+\tfrac{i}{2}) & \bar{\mathbb{Q}}(u-\tfrac{i}{2}) \end{vmatrix} = u^L$$

$$\mathbb{Q}_{(1)} = \begin{pmatrix} \mathbb{Q} \\ \bar{\mathbb{Q}} \end{pmatrix}$$

$$T_s = \mathbb{Q}_{(1)}^{[s+1]} \wedge \mathbb{Q}_{(1)}^{[s-1]}$$

$$Q_{(1)} = \sigma \cdot \mathbb{Q}_{(1)} \qquad \sigma^+ \sigma^- = \frac{1}{u^L}$$

$$Q_{(1)}^{+} \wedge Q_{(1)}^{-} = 1$$

$$W(Q, \bar{Q}) = \begin{vmatrix} Q(u+\tfrac{i}{2}) & Q(u-\tfrac{i}{2}) \\ \bar{Q}(u+\tfrac{i}{2}) & \bar{Q}(u-\tfrac{i}{2}) \end{vmatrix} = 1$$

Q-systems reflect quantum mechanical principles

$$0 = p^2 + 2(V-E) = \left(p + \sqrt{2(E-V)}\right)\left(p - \sqrt{2(E-V)}\right)$$

S_2 symmetry

↓ quantisation

$$0 = \left(-\hbar^2 \frac{d^2}{dx^2} + 2(V-E)\right)\Psi$$

- GL_2 symmetry (allowing superpositions)
- Projective representations (phase is irrelevant)

$$0 = \det(\lambda - M(u)) = \prod_{i=1}^{N} (\lambda - \lambda_i)$$

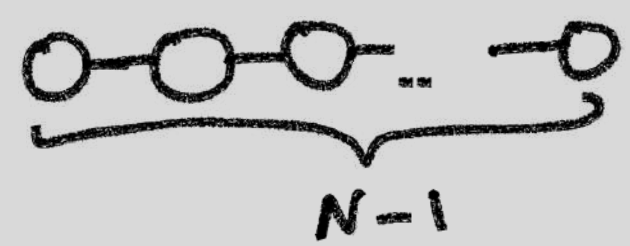
S_N

↓ quantisation

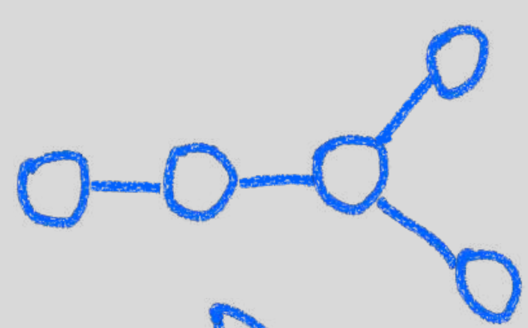
$$0 = \det\left(1 - M e^{-\hbar \partial_u}\right) Q$$

sl_N (PGL_N)

A_{N-1}



Coxeter group W ←



D_n

→ Lie algebra $\mathfrak{g} = so(2n)$

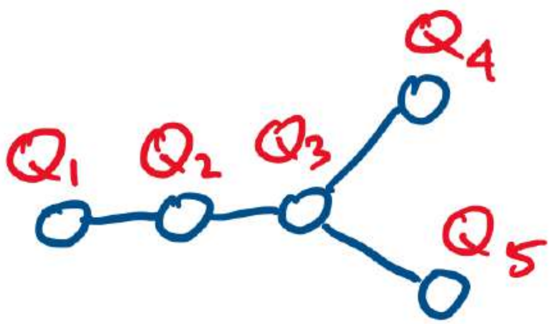
Part Ib

A,D,E

$Q\bar{Q}$-system and Bethe equations

$sl_2 \equiv A_1$

$$W(Q,\bar{Q}) = \begin{vmatrix} Q(u+i/2) & Q(u-i/2) \\ \bar{Q}(u+i/2) & \bar{Q}(u-i/2) \end{vmatrix} = 1$$



$$W(Q_a,\bar{Q}_a) = \prod_{b\sim a} Q_b$$

$Q\bar{Q}$-system

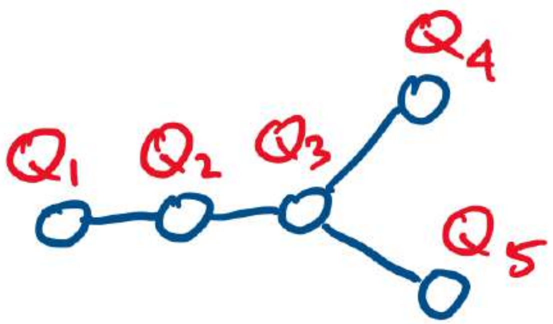
$a = 1,2,\ldots,r$ - nodes of Dynkin diagram

$b \sim a$ - nodes adjacent to a

$Q\bar{Q}$-system and Bethe equations

$sl_2 \equiv A_1$

$$W(Q,\bar{Q}) = \begin{vmatrix} Q(u+i/2) & Q(u-i/2) \\ \bar{Q}(u+i/2) & \bar{Q}(u-i/2) \end{vmatrix} = 1$$



$$W(Q_a,\bar{Q}_a) = \prod_{b\sim a} Q_b$$

$Q\bar{Q}$-system

$a = 1,2,\ldots,r$ - nodes of Dynkin diagram

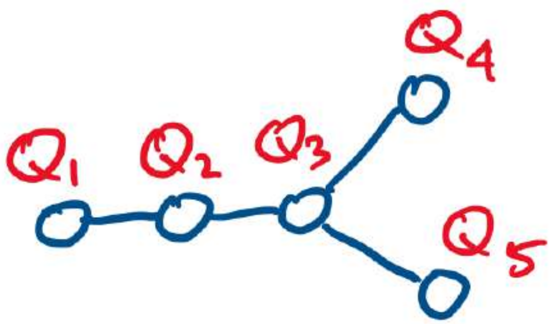
$b \sim a$ - nodes adjacent to a

$$\begin{vmatrix} Q_a^{[2]} & Q_a \\ \bar{Q}_a^{[2]} & \bar{Q}_a \end{vmatrix} = \prod_{b\sim a} Q_b^{+} \qquad \begin{vmatrix} Q_a & Q_a^{[-2]} \\ \bar{Q}_a & \bar{Q}_a^{[-2]} \end{vmatrix} = \prod_{b\sim a} Q_b^{-}$$

$Q\bar{Q}$-system and Bethe equations

$\mathfrak{sl}_2 \equiv A_1$

$$W(Q,\bar{Q}) = \begin{vmatrix} Q(u+\tfrac{i}{2}) & Q(u-\tfrac{i}{2}) \\ \bar{Q}(u+\tfrac{i}{2}) & \bar{Q}(u-\tfrac{i}{2}) \end{vmatrix} = 1$$



$$W(Q_a,\bar{Q}_a) = \prod_{b\sim a} Q_b$$

$Q\bar{Q}$-system

$a = 1,2,\ldots,r$ - nodes of Dynkin diagram

$b \sim a$ - nodes adjacent to a

$$\begin{vmatrix} Q_a^{[2]} & Q_a \\ \bar{Q}_a^{[2]} & Q_a \end{vmatrix} = \prod_{b\sim a} Q_b^{+} \qquad \begin{vmatrix} Q_a & Q_a^{[-2]} \\ Q_a & \bar{Q}_a^{[-2]} \end{vmatrix} = \prod_{b\sim a} Q_b^{-}$$

divide one by another

$$\frac{Q_a^{[2]}}{Q_a^{[-2]}} \prod_{b\sim a} \frac{Q_b^{-}}{Q_b^{+}} = -1$$

(at zeros of Q_a)

$$Q_a = \text{source} \times \prod_{i=1}^{M_a} (u - u_{a,i})$$

$$\prod_{b=1}^{r} \prod_{j=1}^{M_b} \frac{u_{a,i} - u_{b,j} + i C_{ab}}{u_{a,i} - u_{b,j} - i C_{ab}} = \text{source}$$

Conventional Bethe equations

Mutation game

Mutation s_a at Dynkin node **a** replaces Q_a with $\bar{Q}_a$ and updates accordingly $Q\bar{Q}$-system

Mutation is also known as:

…

Going beyond equator

[Bosonic] duality transformations

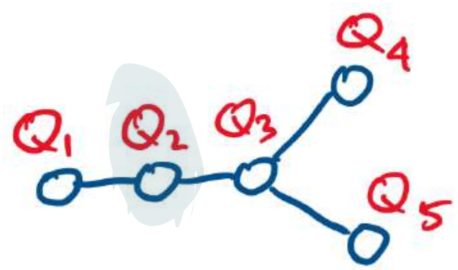
Reproduction [of Population]

Backlund transform

Weyl transform

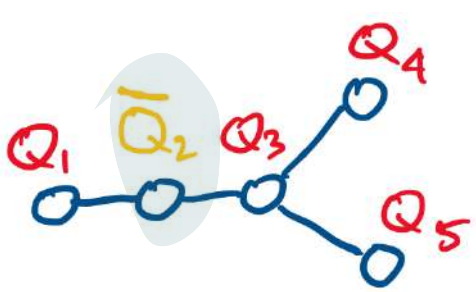
…

$$W(Q_a,\bar{Q}_a)=\prod_{b\sim a} Q_b$$



$$W(Q_1,\bar{Q}_1)=Q_2$$
$$W(Q_2,\bar{Q}_2)=Q_1Q_3$$
$$W(Q_3,\bar{Q}_3)=Q_2Q_4Q_5$$
$$W(Q_4,\bar{Q}_4)=Q_3$$
$$W(Q_5,\bar{Q}_5)=Q_3$$

$$\Big\downarrow s_2$$

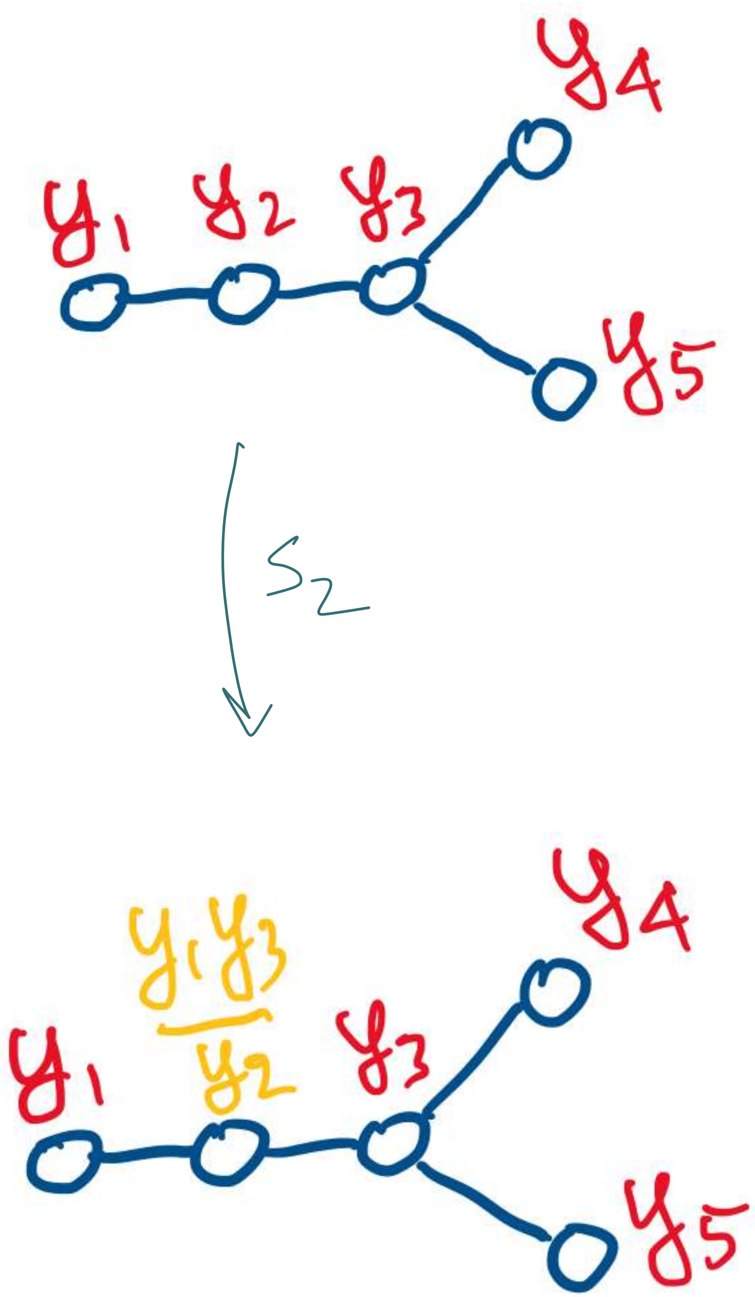


$$W(Q_1,*)=\bar{Q}_2$$
$$W(\bar{Q}_2,*)=Q_1Q_3$$
$$W(Q_3,*)=\bar{Q}_2Q_4Q_5$$
$$W(Q_4,\bar{Q}_4)=Q_3$$
$$W(Q_5,\bar{Q}_5)=Q_3$$

Mutation game in character (classical) limit

$$Q_a = y_a^{-iu} \times \text{const}$$

$$W(Q_a, \overline{Q}_a) = \prod_{b \sim a} Q_b$$



$$y_a = \frac{\prod_{b \sim a} y_b}{y_a}$$

Weyl group action
in theory of characters

$$\chi_\lambda = \mathrm{Tr}_\lambda\Big(\prod_a y_a^{h_a}\Big)$$

Mutation game in character (classical) limit

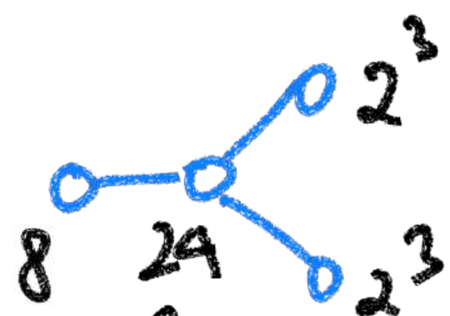
How many different tuples $(y_1, y_2, \ldots, y_r)$ will you get by repeatedly doing mutations?

= Number of elements in Weyl group W

How many different y_a inside $(y_1, y_2, \ldots, y_r)$ will you get by repeatedly doing mutations?

= Length of Weyl orbit of a-th fundamental weight

$\mathfrak{sl}(4)$ $|W| = 4!$



$\binom{8}{2} - 4$

$\mathfrak{so}(8)$ $|W| = 2^3 \cdot 4!$

Full (quantum) mutation game

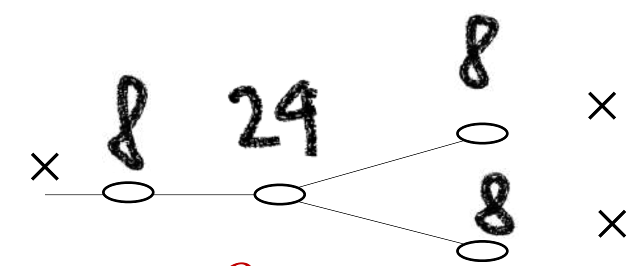
$$W(Q_a,\bar{Q}_a) = \prod_{b\sim a} Q_b$$

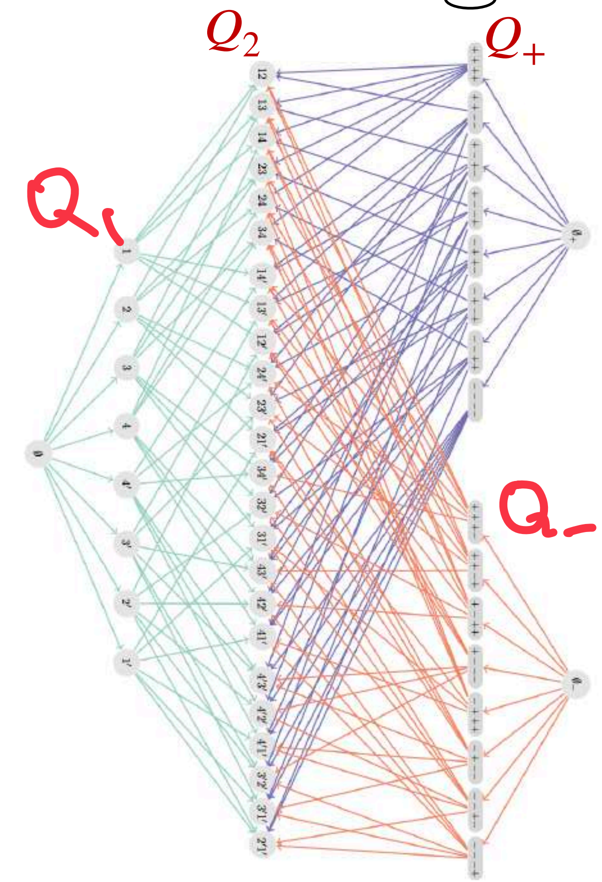
$\bar{Q}_a + \alpha Q_a$ is solution for any α

We might have a principle to uniquely pick solution (e.g. pure twist in asymptotic)

$$\bar{Q}_a = \bar{y}_a^{-iu}\cdot f(u) + \alpha \cancel{Q_a}$$

$$\cancel{Q_a} = y_a^{-iu}\cdot h(u)$$

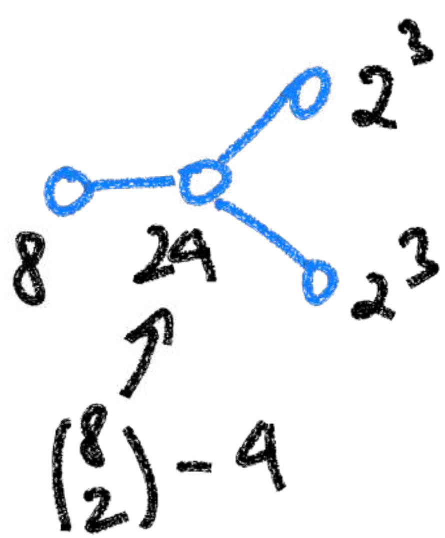




Hasse for D_4

Image from [Ferrando, Frassek, Kazakov `20]

Then mutation consistently works exactly under the same combinatorics as the character solution [Nontrivial statement]



Full (quantum) mutation game

Generically, there is no particular criterium to pick a solution. We then play mutations by picking some solution randomly every time

$$W(Q,\bar{Q}) = 1$$

$$Q$$
$$\downarrow s$$
$$\alpha Q + \bar{Q} \quad \leftarrow \text{any } \alpha$$
$$\downarrow s$$
$$\alpha' Q + \beta' \bar{Q} \quad \leftarrow \text{any } [\alpha':\beta'] \neq [\alpha:1]$$
$$\downarrow s$$
$$\alpha'' Q + \beta'' \bar{Q} \quad \leftarrow \text{any } [\alpha'':\beta''] \neq [\alpha':\beta']$$
$$\downarrow s$$
$$\cdots$$

$$2 = \dim\left(\mathrm{span}\{Q, s\cdot Q, s^2\cdot Q, \ldots,\}\right)$$

Analogy

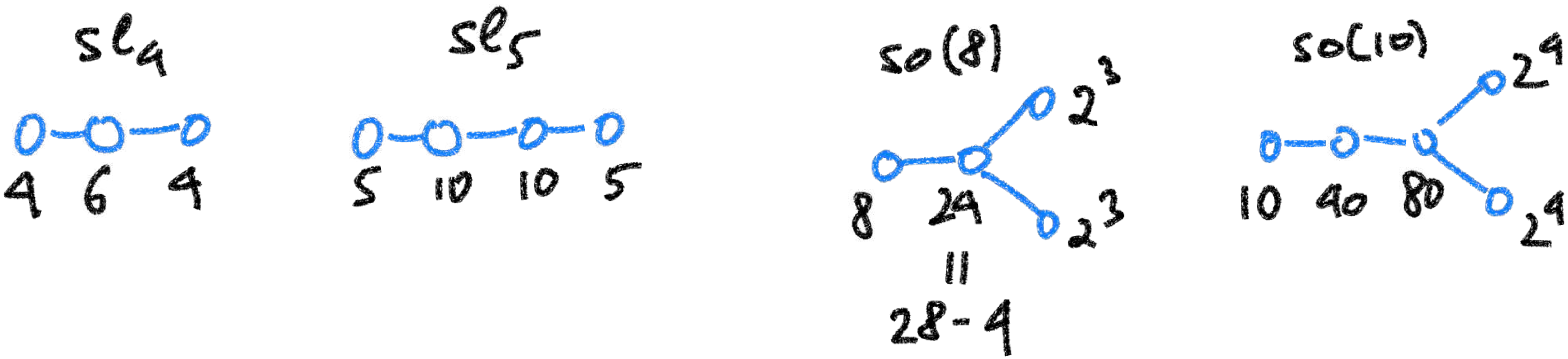
Q - small solution

$\bar{Q}$ - large solution

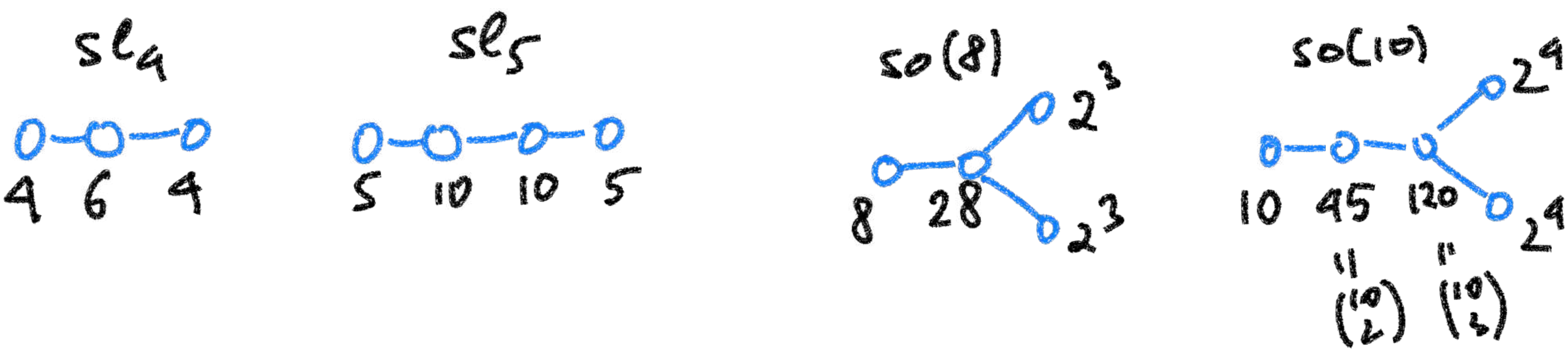
Mutation - crossing to next Stokes sector

Full (quantum) mutation game

How many different y_a inside $(y_1, y_2, \ldots, y_r)$ will you get by repeatedly doing mutations?

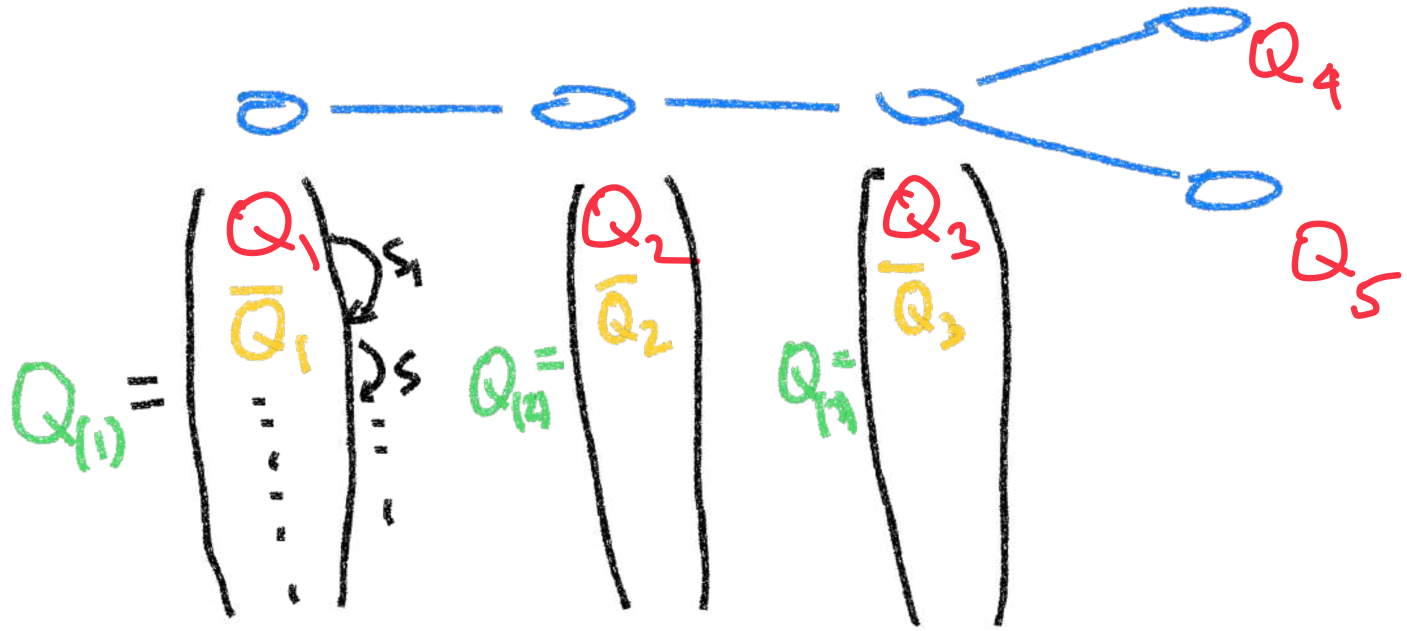


Pick all Q_a that emerge after repeated mutations of $(Q_1, Q_2, \ldots, Q_r)$. What is dimension of their linear span?



These are not lengths of a-th Weyl orbits any longer but dimensions of a-th fundamental representations of the symmetry algebra!

Full (quantum) mutation game



$$W(Q_a, \bar{Q}_a) = \prod_{b \sim a} Q_b$$

$$Q^{+}_{(a)} \wedge Q^{-}_{(a)} = \bigotimes_{b \sim a} Q_{(b)}$$

First time these equations in [Masoero,Raimondo,Valeri'15], also [Sun'12], but their derivation and usage was rather the opposite way around

Consistency of the above relations is a nontrivial property by itself. One can think about it as Yang-Baxter equation realised on the level of Bethe algebra.

Fused flag

$Q_{(1)} \otimes Q_{(2)} \ldots \otimes Q_{(r)}$ are described by a new non-local Geometric structure which we call fused flag

- Flag manifold (whose points are flags) is, by definition, the space G/B

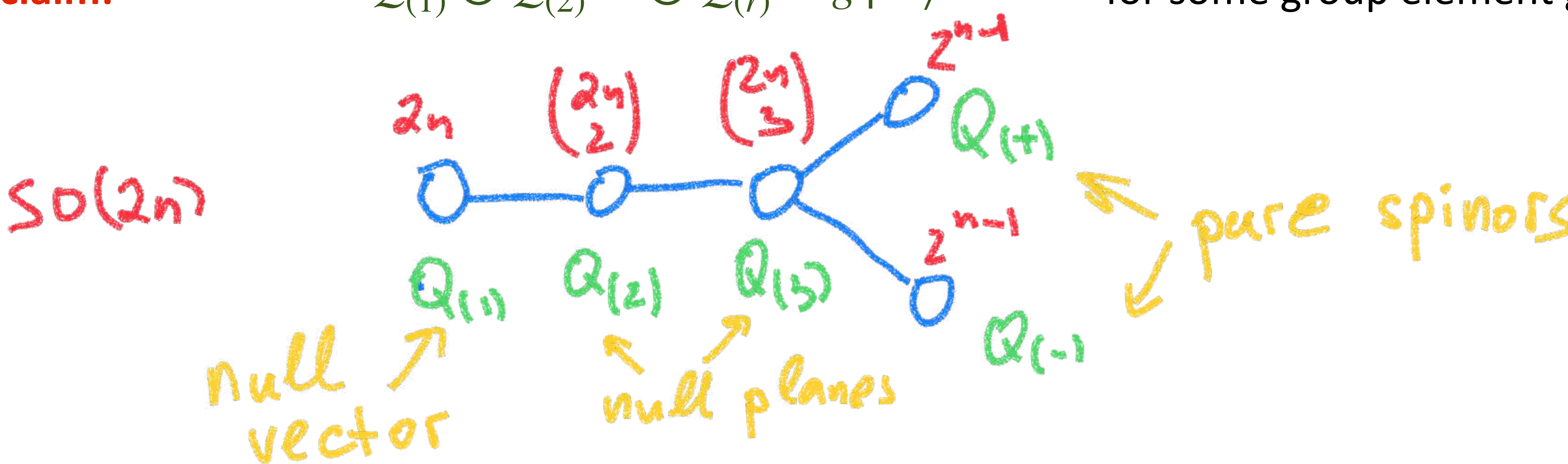
- Homogeneous space G/B can be parameterised by an order parameter - object that transforms under G and has B as stability group

- $Q_{(1)} \otimes Q_{(2)} \ldots \otimes Q_{(r)} \in L(\omega_1) \otimes L(\omega_2) \ldots \otimes L(\omega_r) \equiv L_{\text{big}} \simeq \mathbb{C}^d$

- $|\Omega\rangle \equiv |\text{HWS}\rangle_1 \otimes |\text{HWS}\rangle_2 \ldots \otimes |\text{HWS}\rangle_r$ is (projectively = up to normalisation) invariant under B-action

- G-orbit of $|\Omega\rangle$ in $\mathbb{P}(L_{\text{big}}) = \mathbb{CP}^{d-1}$ is precisely G/B then

Rough claim: $Q_{(1)} \otimes Q_{(2)} \ldots \otimes Q_{(r)} = g\,|\Omega\rangle$ for some group element g



Fused flag

Rough claim: $Q_{(1)} \otimes Q_{(2)} \ldots \otimes Q_{(r)} = g\,|\,\Omega\rangle$ for some group element g

Exact claim:

[Shu,Ekhammar, D.V.'20]

$$Q_{(1)}(u+ip_1) \otimes Q_{(2)}(u+ip_2) \ldots \otimes Q_{(r)}(u+ip_r) = g(u;\mathbf{p})\,|\,\Omega\rangle$$

for any $\mathbf{p}$ such that $p_a - p_b = \pm 1$ if $a \sim b$

Derivation is using ODE/IM techniques of [Masoero,Raimondo,Valeri'15], [Sun'12]

Remark: $g(u,\mathbf{p})$ is a finite-difference oper for the Coxeter element defined by $\mathbf{p}$.

This establishes connection to (G,q)-oper in [Frenkel, Koroteev, Sage, Zeitlin'20], sl_N case of this equivalence is [Kazakov, Leurent, D.V.'16] vs [Koroteev,Zeitlin'18]

Part Ic
Application of Q-system

$$T_s = \begin{vmatrix} \mathbb{Q}(u+\tfrac{i}{2}(s-1)) & \mathbb{Q}(u-\tfrac{i}{2}(s-1)) \\ \bar{\mathbb{Q}}(u+\tfrac{i}{2}(s-1)) & \bar{\mathbb{Q}}(u-\tfrac{i}{2}(s-1)) \end{vmatrix}$$

$$W(\mathbb{Q}, \bar{\mathbb{Q}}) = \begin{vmatrix} \mathbb{Q}(u+{}^{i}\!/_{2}) & \mathbb{Q}(u-{}^{i}\!/_{2}) \\ \bar{\mathbb{Q}}(u+{}^{i}\!/_{2}) & \bar{\mathbb{Q}}(u-{}^{i}\!/_{2}) \end{vmatrix} = u^L$$

$$\mathbb{Q}_{(1)} = \begin{pmatrix} \mathbb{Q} \\ \bar{\mathbb{Q}} \end{pmatrix}$$

$$\boxed{T_s = \mathbb{Q}_{(1)}^{[s+1]} \wedge \mathbb{Q}_{(1)}^{[s-1]}}$$

$$Q_{(1)} = \sigma \cdot \mathbb{Q}_{(1)} \qquad \sigma^+ \sigma^- = \frac{1}{u^L}$$

$$Q_{(1)}^{+} \wedge Q_{(1)}^{-} = 1 \qquad W(Q, \bar{Q}) = \begin{vmatrix} Q(u+{}^{i}\!/_{2}) & Q(u-{}^{i}\!/_{2}) \\ \bar{Q}(u+{}^{i}\!/_{2}) & \bar{Q}(u-{}^{i}\!/_{2}) \end{vmatrix} = 1$$

Completeness conjectures

$$Q_{(a)} = \sigma_a \, \mathbb{Q}_{(a)}$$

$\nwarrow$ Baxter polynomials

$$\mathbb{Q}_{(a)} = \begin{pmatrix} \mathbb{Q}_a \\ \overline{\mathbb{Q}}_a \\ \vdots \\ \vdots \end{pmatrix}$$

$$W(Q_a, \bar{Q}_a) = \prod_{b \sim a} Q_b \Rightarrow W(\mathbb{Q}_a, \overline{\mathbb{Q}}_a) = P_a \prod_{b \sim a} \mathbb{Q}_b$$

P_a – Drinfeld polynomials

$$\left.\begin{aligned} P_1 &= u^L \\ P_{a \neq 1} &= 1 \end{aligned}\right\} \text{spin chain in vector}$$

Conjecture: the above $Q\bar{Q}$-system has the right number of solutions for generic values of parameters (but can have more for special points including somtimes simple chains like homogeneous vector etc)

$$\mathbb{Q}^{+}_{(a)} \wedge \mathbb{Q}^{-}_{(a)} = P_a \bigotimes_{b \sim a} \mathbb{Q}_{(b)}$$

Conjecture: the fully extended Q-system (above relations) has the right number of solutions always

Wronskian Bethe equations

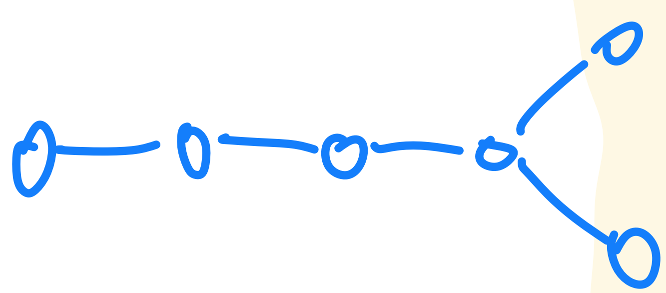
$sl(N)$

$$Q_{(N-1)} = \begin{pmatrix} Q^1 \\ Q^2 \\ \vdots \\ Q^N \end{pmatrix}$$

$$\boxed{W(Q^1, \dots, Q^N) = \prod_{\alpha=1}^{L} (u - \theta_\alpha)}$$

Theorem: The right number of solutions with no two solutions ever coincide if inhomogeneities are real **[Mukhin, Tarasov, Varchenko'12] [Chernyak,Leurent,D.V.'20]**

$so(2N)$



Pure spinor

Q - system

$$W(\Psi_1, \dots, \Psi_N) = \Psi_\emptyset^{[N-2]_D}$$

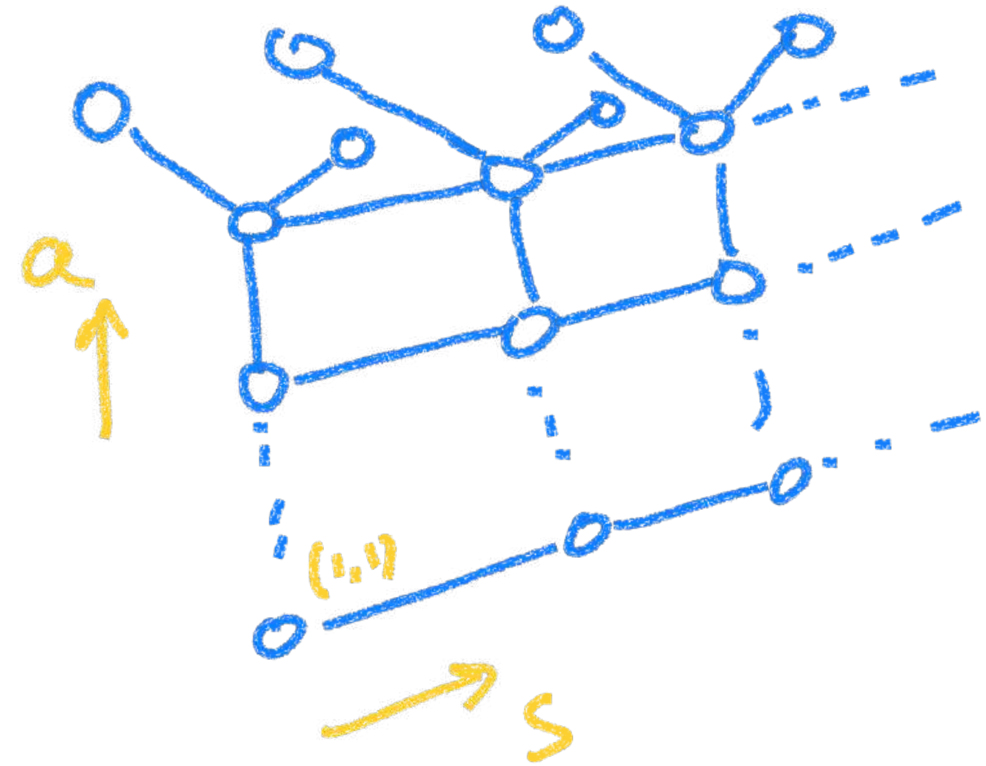
$$W(\Psi_a, \Psi_b) = W(\Psi_\emptyset, \Psi_{ab})$$

L	V	$\wedge^2 V$	Ψ_-	Ψ_+
2	3(0.9s)	10(11.5s)	3(2.0s)	3(2.4s)
3	7(2.0s)	68(95.0s)	7(5.0s)	7(11.0s)
4	26(11.2s)	631(1177s)	26(29.6s)	26(120s)
5	85(28.0s)	-	85(78s)	85(322s)
6	365(79s)	-	365(1435s)	365(2278s)
7	1456(1483s)	-	-	-

L	V	$\wedge^2 V$	$\wedge^3 V$	Ψ_-	Ψ_+
2	3(2.4s)	9(21.1s)	20(108s)	3(8s)	3(18.5s)
3	7(4.7s)	60(176s)	-	9(22.8s)	9(136s)
4	25(21.0s)	-	-	42(325s)	42(1571s)
5	82(215s)	-	-	-	-

Solution of Hirota equations

$$T^{+}_{a,s}T^{-}_{a,s} = T_{a,s+1}T_{a,s-1} + \prod_{b\sim a} T_{b,s}$$



$$T_{a,s} = \left\langle Q_{(a)}^{[s+\frac{h}{2}]} , Q_{(a^*)}^{[-s-\frac{h}{2}]} \right\rangle$$

[Ferrando, Frassek, Kazakov `20]
[Ekhammar, Shu, Volin'20]

Bonus from flag fusion:

$$T_{a,0} = \left\langle Q_{(a)}^{[\frac{h}{2}]} , Q_{(a^*)}^{[-\frac{h}{2}]} \right\rangle = 1$$

$$\left\langle Q_{(a)}^{[s+\frac{h}{2}]} , Q_{(a^*)}^{[-s-\frac{h}{2}]} \right\rangle = 0,$$

$$s = -1, -2, \dots, -\frac{h}{2}$$

Can be used as a version of Bethe equations

Part Id
Non-simply laced case

$Q\bar{Q}$-system and Bethe equations

$$W(Q_a, \bar{Q}_a) = \prod_{b \sim a} Q_b$$

$$Q_a = \text{source} \times \prod_{i=1}^{M_a} (u - u_{a,i})$$

$$\prod_{b=1}^{r} \prod_{j=1}^{M_b} \frac{u_{a,i} - u_{b,j} + i\, C_{ab}}{u_{a,i} - u_{b,j} - i\, C_{ab}} = \text{source}$$

$B_3 \simeq so(7)$ spin chain

$$\text{source} = \prod_{b=1}^{r} \prod_{j=1}^{M_b} \frac{u_{a,i} - u_{b,j} + i(\alpha_a, \alpha_b)}{u_{a,i} - u_{b,j} - i(\alpha_a, \alpha_b)}$$

$$(\alpha_a, \alpha_b) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$W(Q_1, \bar{Q}_1) = Q_2$$

$$W(Q_2, \bar{Q}_2) = Q_1 Q_3$$

$$W_{1/2}(Q_3, \bar{Q}_3) = Q_2^{[+1/2]} Q_2^{[-1/2]}$$

Playing mutation game

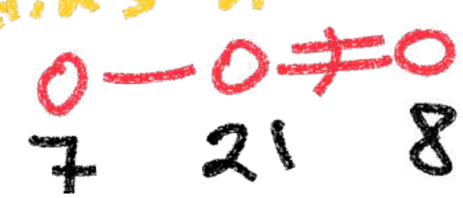
$B_3 \simeq so(7)$ spin chain

$$w(Q_1,\overline{Q}_1)=Q_2$$
$$w(Q_2,\overline{Q}_2)=Q_1Q_3$$
$$w_{1/2}(Q_3,\overline{Q}_3)=Q_2^{[+1/2]}Q_2^{[-1/2]}$$

Character ansatz reproduces Weyl group all right
(it is not sensible to fancy shifts of spectral parameter, simply power counting)

Below are the numbers for dimension of linear span of mutated Q-functions

Compare to dim's of represent.



Langlands duality

$B_3 \simeq so(7)$ spin chain

$$Q_{(\bar{1})} = \sigma^*_{\text{eqv}} \cdot Q_{(1)} = \sigma^* Q^{[-1]}_{(1)}$$
$$Q_{(\bar{2})} = \sigma^*_{\text{eqv}} \cdot Q_{(2)} = \sigma^* Q^{[-1]}_{(2)}$$

$$W(Q_1, \bar{Q}_1) = Q_2$$
$$W(Q_2, \bar{Q}_2) = Q_1 Q_3$$
$$W_{1/2}(Q_3, \bar{Q}_3) = Q_2^{[+1/2]} Q_2^{[-1/2]}$$

σ - outer autmorophism based on reflection of A_5 Dynkin diagram

$$Q^+_{(1)} \wedge Q^-_{(\bar{1})} = Q_{(2)}$$
$$Q^+_{(2)} \wedge Q^-_{(\bar{2})} = Q_{(1)} \otimes Q_{(3)}$$
$$Q_{(3)} \wedge \sigma^*_{\text{ev}} Q_{(3)} = Q_{(2)} \otimes \sigma^*_{\text{ev}} Q_{(2)}$$

Derived from ODE/IM by
[Masoero,Raimondo, Valeri ‘16]
based on twisted affine algebra $A_5^{(2)}$

Hidden message behind: Yangian $Y(B_3)$ is a limit of quantum affine $U_q(B_3^{(1)})$, and

$B_3^{(1)}$ and $A_5^{(2)}$ are Langlands-dual Kac-Moody algebras

Langlands duality

$B_3 \simeq so(7)$ spin chain

$$Q^+_{(1)} \wedge Q^-_{(1)} = Q_{(2)}$$
$$Q^+_{(2)} \wedge Q^-_{(2)} = Q_{(1)} \otimes Q_{(3)}$$
$$Q_{(3)} \wedge \sigma^*_{\mathrm{ev}} Q_{(3)} = Q_{(2)} \otimes \sigma^*_{\mathrm{ev}} Q_{(2)}$$

$$W(Q_1, \overline{Q}_1) = Q_2$$
$$W(Q_2, \overline{Q}_2) = Q_1 Q_3$$
$$W_{1/2}(Q_3, \overline{Q}_3) = Q_2^{[+1/2]} Q_2^{[-1/2]}$$

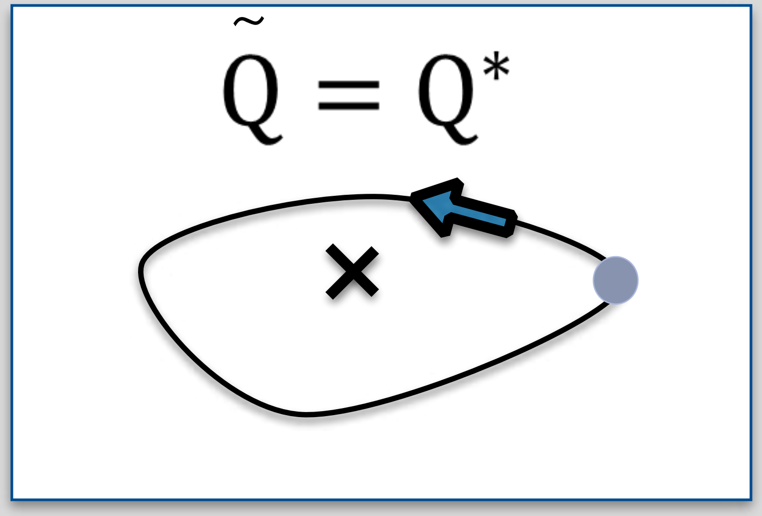
σ - outer autmorophism based on reflection of A_5 Dynkin diagram

so(7) spin chain is described by Q-system who transforms covariantly under action of $A_5^{(2)}$. Its zero-level subalgebra is sp(6)

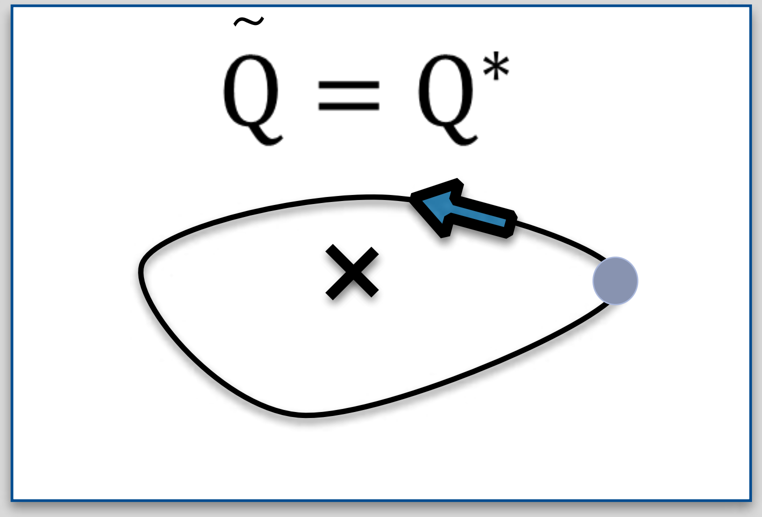
Example of T-functions computation:

$$T_{1,s} = \omega^{ab} V_a^{[s+r-\frac{1}{2}]} V_b^{[-s-r+\frac{1}{2}]}$$

Part II
Monodromy bootstrap



Q^* means applying some symmetry transformation on Q-system



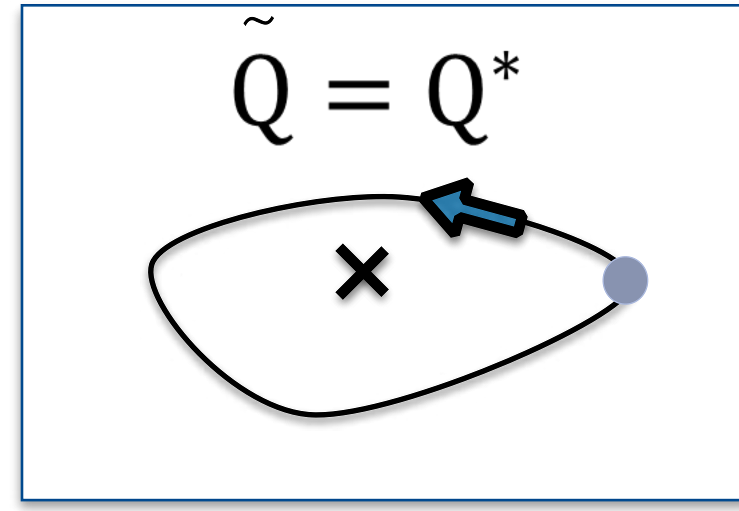
Symmetries:

- Covariant action of (Langlands dual in principle) symmetry group. Can consider matrices that are i-periodic in spectral parameter

- Potential rescalings, due to projectivity - gauge transformations

- Outer automorphisms existing in the model

$\tilde{Q} = Q^{\gamma}$ means performing analytic continutation around branch point

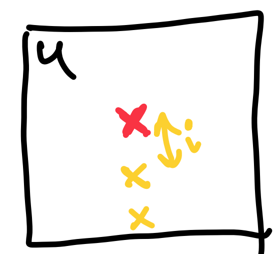


... but this is extremely subtle

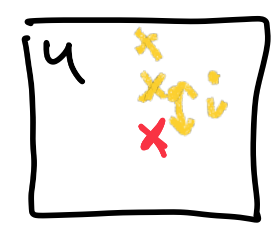
$$W(Q_a, \bar{Q}_a) = \prod_{b \sim a} Q_b$$

Two ways to solve:

$$\bar{Q}_a = -Q_a \sum_{n=0}^{\infty} \left(\frac{\prod_{b\sim a} Q_b}{Q_a^+ Q_a^-} \right)^{[2n+1]}$$

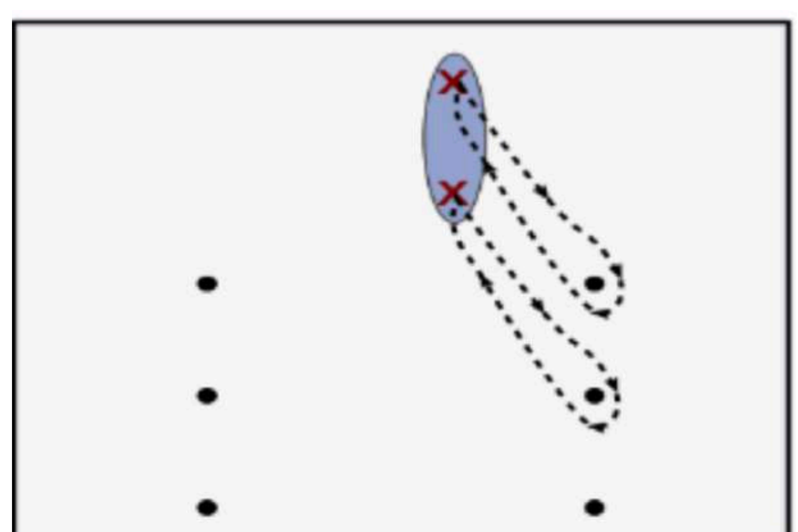


$$\bar{Q}_a = Q_a \sum_{n=0}^{\infty} \left(\frac{\prod_{b\sim a} Q_b}{Q_a^+ Q_a^-} \right)^{[-2n-1]}$$

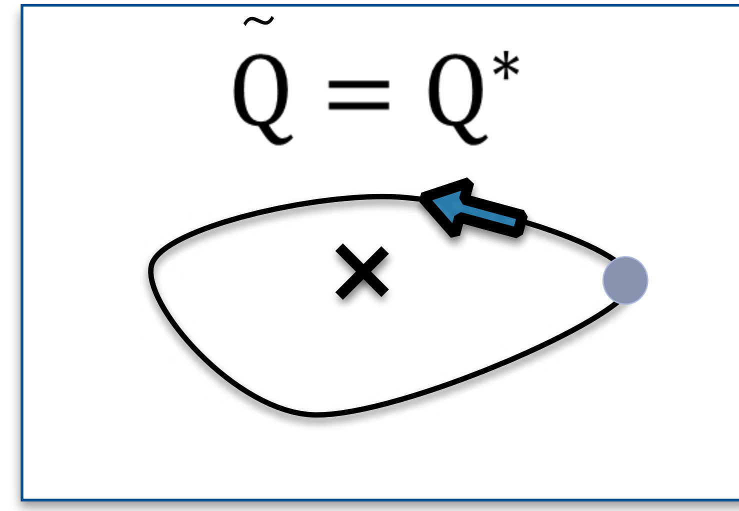


... Infinite ladders of branch points separated by i

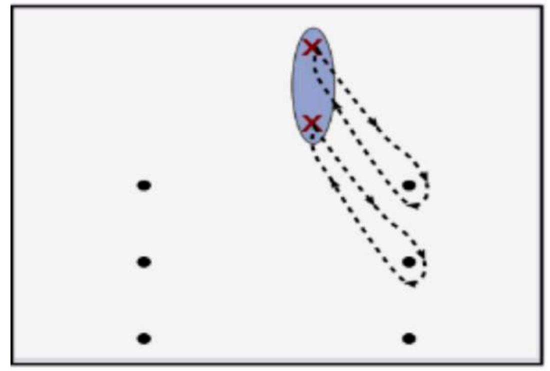
Naive continuation will fail



$\tilde{Q} = Q^{\gamma}$ means performing analytic continutation around branch point

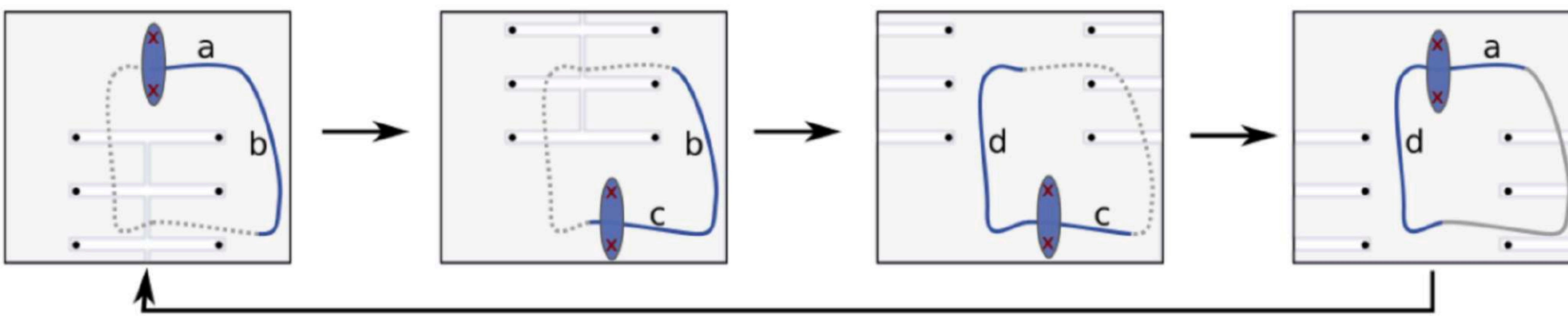


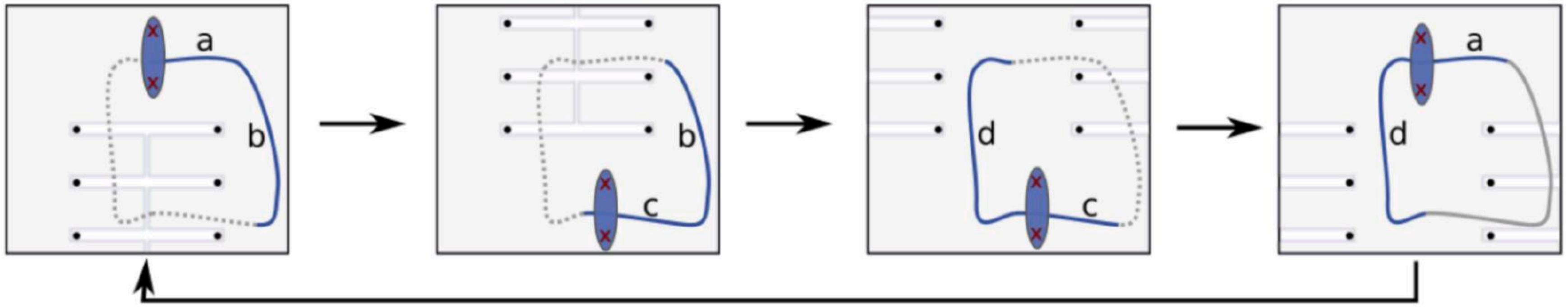
Naive continuation will fail

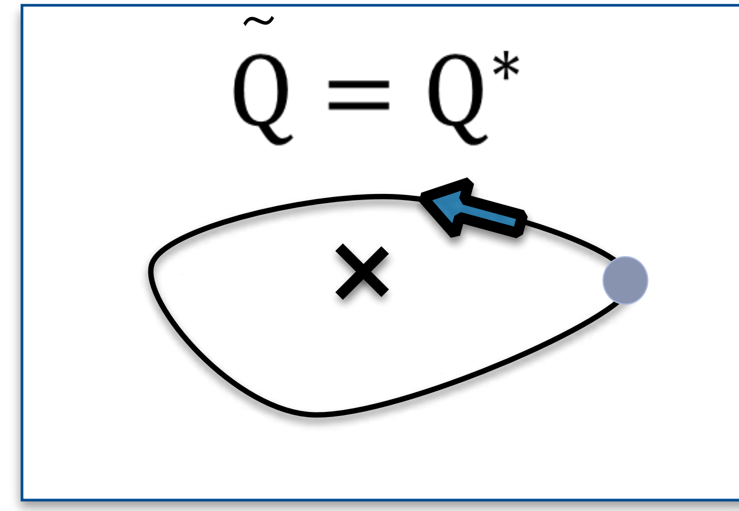


Idea: we shall use symmetries while doing continuation to ensure that every time we are in a situation when no ladder of branch cuts is present.

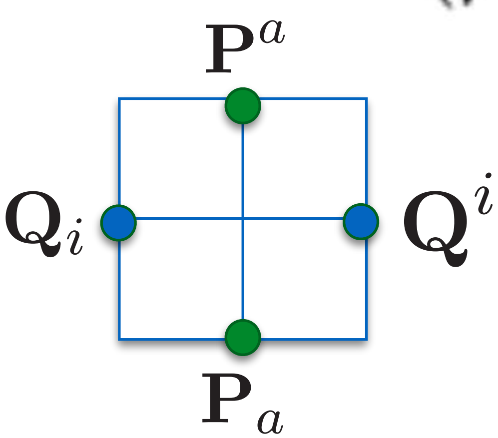
$$(\mu_{\check{h}})^{-1} \cdot (\omega_{\hat{h}}) \cdot \mathcal{Q} = \mathcal{Q}^*$$







$$(\mu_{\check{h}})^{-1} \cdot (\omega_{\hat{h}}) \cdot \mathcal{Q} = \mathcal{Q}^*$$



Type-B model:

$\mathcal{Q}^*$ is Hodge duality

$\mathbf{P} \in V$ (vector whose components are P_a)

$\mathbf{P}^* \in V^*$ (vector whose components are P^a)

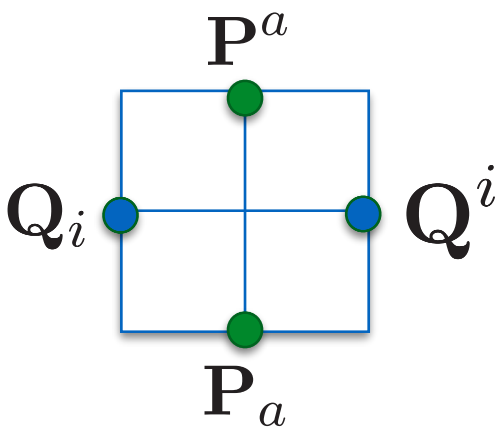
$\mu \in V \otimes V$ (matrix with two lower indices)

$$\widetilde{\mathbf{P}} \otimes \widetilde{\mathbf{P}}^* = -\mu\, \mathbf{P}^* \otimes \mathbf{P}\, \mu^{-1} \times (-1)^{p_\mu + p_\omega} ,$$

$$\overset{*}{\mu} = \left(1 + \frac{1}{F} \mathbf{P} \otimes \mathbf{P}^*\right) \mu \left(1 + \frac{1}{F} \mathbf{P}^* \otimes \mathbf{P}\right) ,$$

$$\mathrm{Tr}\, \mathbf{P} \otimes \mathbf{P}^* = \frac{1}{F} - F ,$$

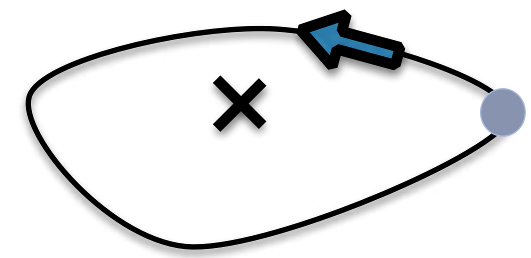
$$\widetilde{F}\, F = 1 \times (-1)^{p_\mu + p_\omega} .$$



Type-B model:

Q^* is Hodge duality

$$\tilde{\mathrm{Q}} = \mathrm{Q}^*$$



$$\tilde{\mathbf{P}} \otimes \tilde{\mathbf{P}}^* = -\mu \mathbf{P}^* \otimes \mathbf{P}\, \mu^{-1} \times (-1)^{\mathrm{p}_\mu + \mathrm{p}_\omega} ,$$
$$\tilde{\mu} = \left(1 + \frac{1}{F} \mathbf{P} \otimes \mathbf{P}^*\right) \mu \left(1 + \frac{1}{F} \mathbf{P}^* \otimes \mathbf{P}\right) ,$$
$$\mathrm{Tr}\, \mathbf{P} \otimes \mathbf{P}^* = \frac{1}{F} - F ,$$
$$\tilde{F} F = 1 \times (-1)^{\mathrm{p}_\mu + \mathrm{p}_\omega} .$$

$\mathbf{P} \in V$ (vector whose components are P_a)

$\mathbf{P}^* \in V^*$ (vector whose components are P^a)

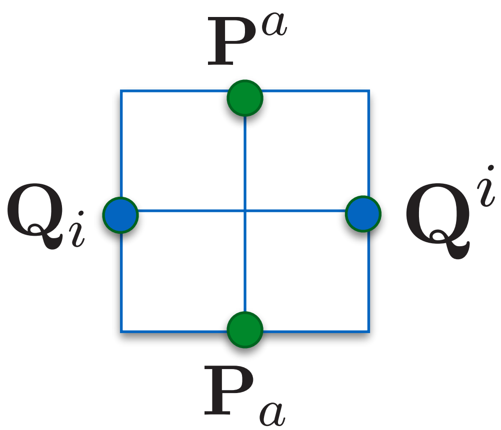
$\mu \in V \otimes V$ (matrix with two lower indices)

$F^2 = 1$ is zero central charge condition: $\mathfrak{sl}(2|2) \to \mathfrak{psl}(2|2)$

Theorem:
If one requires square root cut condition
then $F^2 \neq 1$ and $\mu_{ab} = \mu \epsilon_{ab}$

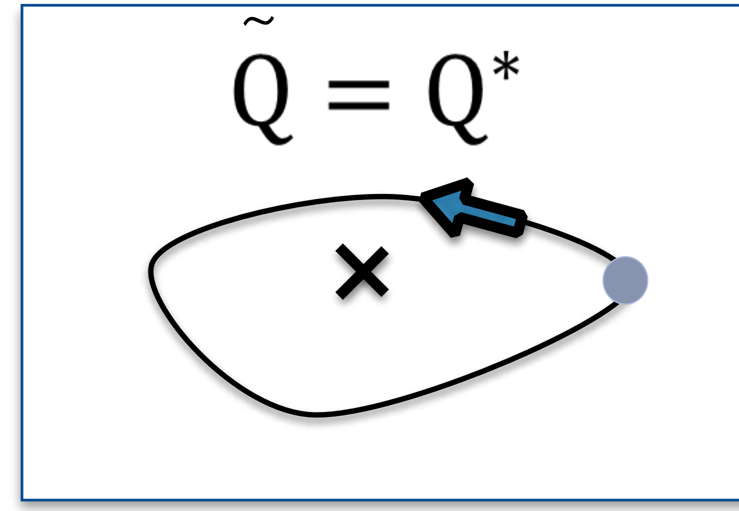
$$\tilde{\mathbf{P}}_a = \frac{\mu}{F} \epsilon_{ab} \mathbf{P}^b , \quad \tilde{\mathbf{P}}^a = -\frac{F}{\mu} \epsilon^{ab} \mathbf{P}_b , \quad \mu - \tilde{\mu} = \epsilon^{ab} \mathbf{P}_a \tilde{\mathbf{P}}_b , \quad \mathbf{P}^a \mathbf{P}_a = \frac{1}{F} - F \qquad \frac{\mu}{\tilde{\mu}} = \frac{F}{\tilde{F}} = F^2$$

This is QSC for Hubbard model - depending on analytic ansatz, can derive Bethe equations or TBA. Can be mapped to the Riemann-Hilbert problems derived from TBA in [Cavaglia, Cornagliotto, Mattelliano, Tateo'15]



Type-A model:

$$Q^* = Q$$



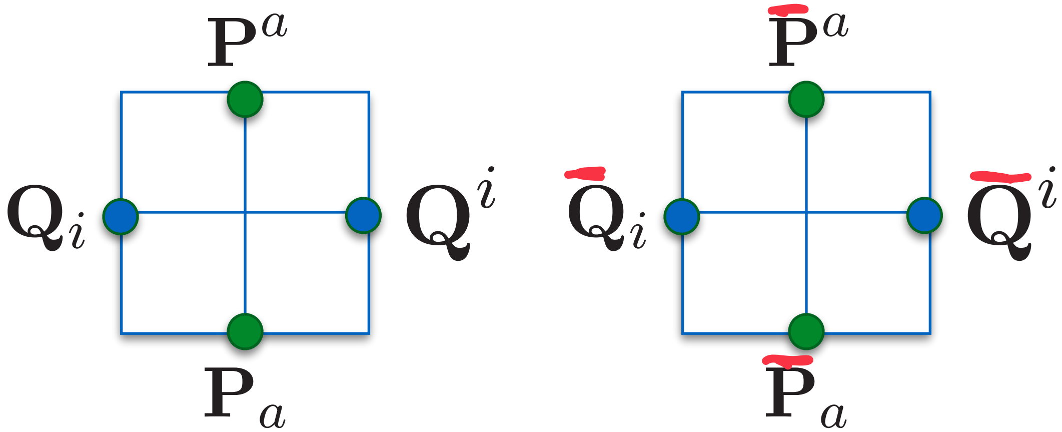
$$\begin{aligned}
\widetilde{\mathbf{P}} \otimes \widetilde{\mathbf{P}}^* &= \mu\, \mathbf{P} \otimes \mathbf{P}^*\, \mu^{-1}\,, \\
\tilde{\mu} &= \left(1 + \frac{1}{F}\mathbf{P} \otimes \mathbf{P}^*\right) \mu\, (1 - F\,\mathbf{P} \otimes \mathbf{P}^*) \\
\mathrm{Tr}\, \mathbf{P} \otimes \mathbf{P}^* &= \frac{1}{F} - F\,, \\
\widetilde{F}/F &= \det \mu = 1\,.
\end{aligned}$$

$\mathbf{P} \in V \qquad \mathbf{P}^* \in V^*$

$\mu \in V \otimes V^*$

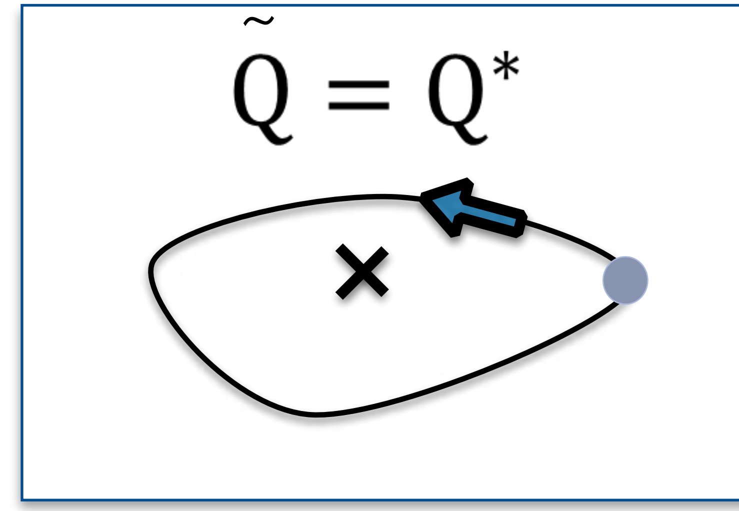
Theorem: If one requires square root cut condition then there is no cuts at all

Conjecture: $F^2 = 1$ case offers QSC for AdS_2/CFT_1

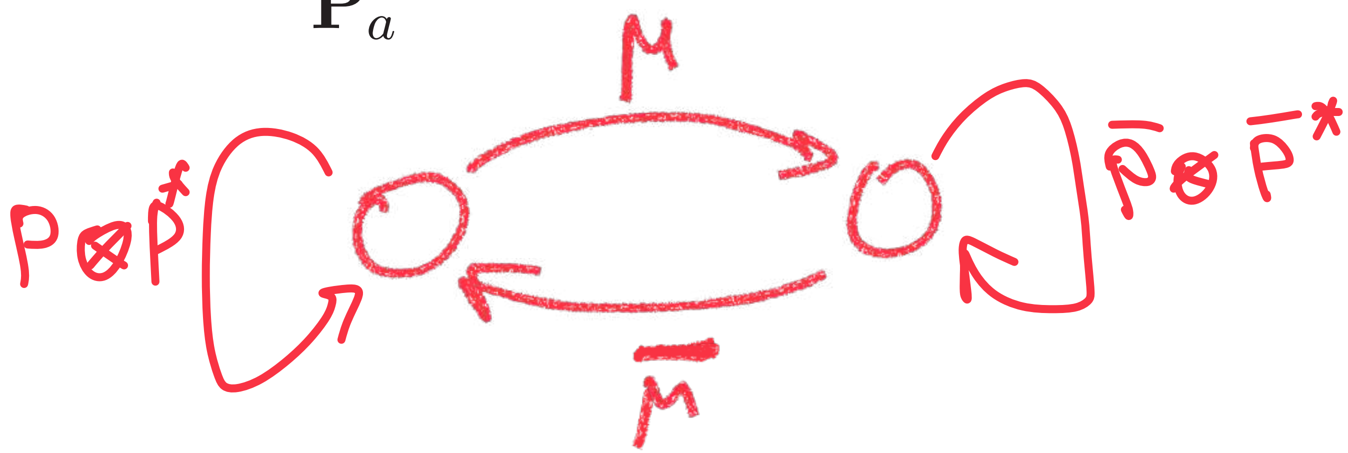


Type-C model:

$$\widetilde{Q} \simeq \bar{Q}^*\,,\ \widetilde{\bar{Q}} \simeq Q^*$$



Quiver describing: Type-c



$$\widetilde{\mathbf{P}}\otimes\widetilde{\mathbf{P}}^* = -\bar{\mu}\,\bar{\mathbf{P}}^*\otimes\bar{\mathbf{P}}\,\bar{\mu}^{-1}\times(-1)^{\mathrm{p}_{\bar{\mu}}+\mathrm{p}_{\omega}}\,,\quad \widetilde{\bar{\mathbf{P}}}\otimes\widetilde{\bar{\mathbf{P}}}^* = -\,\mu\,\mathbf{P}^*\otimes\mathbf{P}\,\mu^{-1}\times(-1)^{\mathrm{p}_{\mu}+\mathrm{p}_{\bar{\omega}}}$$

$$\overset{*}{\tilde{\mu}} = \left(1+\frac{1}{\bar{F}}\bar{\mathbf{P}}\otimes\bar{\mathbf{P}}^*\right)\mu\left(1+\frac{1}{F}\mathbf{P}^*\otimes\mathbf{P}\right),\quad \overset{*}{\tilde{\bar{\mu}}} = \left(1+\frac{1}{F}\mathbf{P}\otimes\mathbf{P}^*\right)\bar{\mu}\left(1+\frac{1}{\bar{F}}\bar{\mathbf{P}}^*\otimes\bar{\mathbf{P}}\right)$$

$$\mathrm{Tr}\,\mathbf{P}\otimes\mathbf{P}^* = \frac{1}{F}-F\,,\quad \mathrm{Tr}\,\bar{\mathbf{P}}\otimes\bar{\mathbf{P}}^* = \frac{1}{\bar{F}}-\bar{F}\,,$$

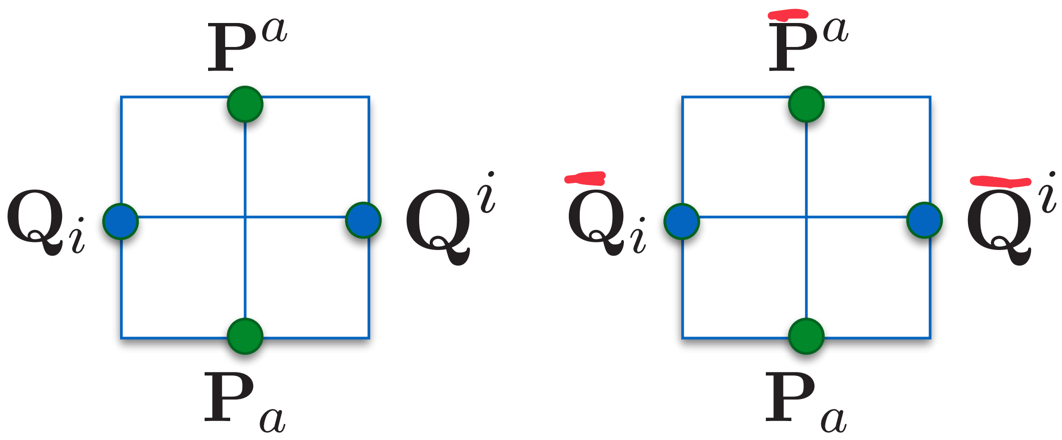
$$\widetilde{F}\,\bar{F}\times(-1)^{\mathrm{p}_{\bar{\mu}}+\mathrm{p}_{\omega}} = F\,\widetilde{\bar{F}}\times(-1)^{\mathrm{p}_{\mu}+\mathrm{p}_{\bar{\omega}}} = 1\,.$$

Using matrices $\Pi = 1+\frac{1}{F}\mathbf{P}\otimes\mathbf{P}^*$, $\bar{\Pi} = 1+\frac{1}{\bar{F}}\bar{\mathbf{P}}\otimes\bar{\mathbf{P}}^*$, the QSC equations become

$$\widetilde{\Pi} = \bar{\mu}\,\bar{\Pi}^{-\mathrm{T}}\,\bar{\mu}^{-1}\,,\qquad \widetilde{\bar{\Pi}} = \mu\,\Pi^{-\mathrm{T}}\,\mu^{-1}$$

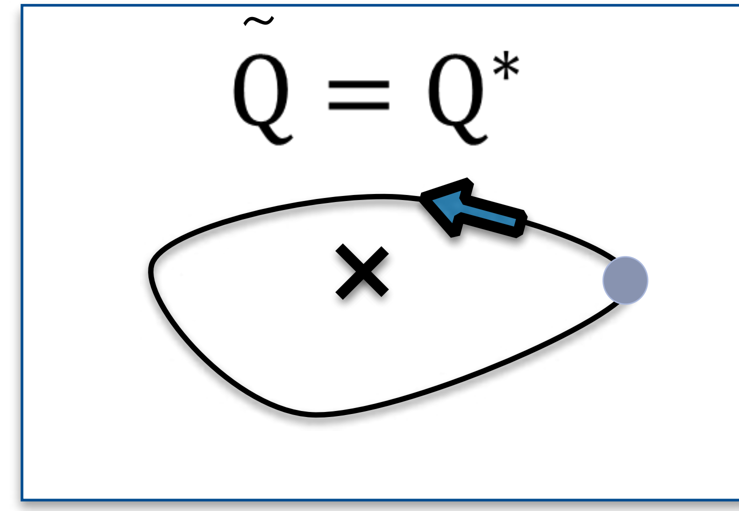
$$\overset{*}{\tilde{\mu}} = \bar{\Pi}\,\mu\,\Pi^{\mathrm{T}}\,,\qquad \overset{*}{\tilde{\bar{\mu}}} = \Pi\,\bar{\mu}\,\bar{\Pi}^{\mathrm{T}}\,,$$

Conjecture: $F^2=\bar{F}^2=1$ case offers QSC for AdS/CFT integrable system with $AdS_3\times S^3\times T^4$ background supperted by RR-flux. An equivalent proposal is in **[Cavaglia, Gromov, Torrielli, Stefanski'21]**



Type-D model:

$$\tilde{Q} \simeq \bar{Q}^* \,, \; \tilde{\bar{Q}} \simeq Q$$



It is based on 4th-order outer automorphism

$$\tilde{\mathbf{P}} \otimes \tilde{\mathbf{P}}^* = -\,\bar{\mu}\,\bar{\mathbf{P}}^* \otimes \bar{\mathbf{P}}\,\bar{\mu}^{-1} \times (-1)^{\mathrm{p}_{\bar{\mu}}+\mathrm{p}_{\omega}} \,, \quad \tilde{\bar{\mathbf{P}}} \otimes \tilde{\bar{\mathbf{P}}}^* = \mu\,\mathbf{P} \otimes \mathbf{P}^*\,\mu^{-1} \times (-1)^{\mathrm{p}_{\mu}+\mathrm{p}_{\bar{\omega}}} \,,$$

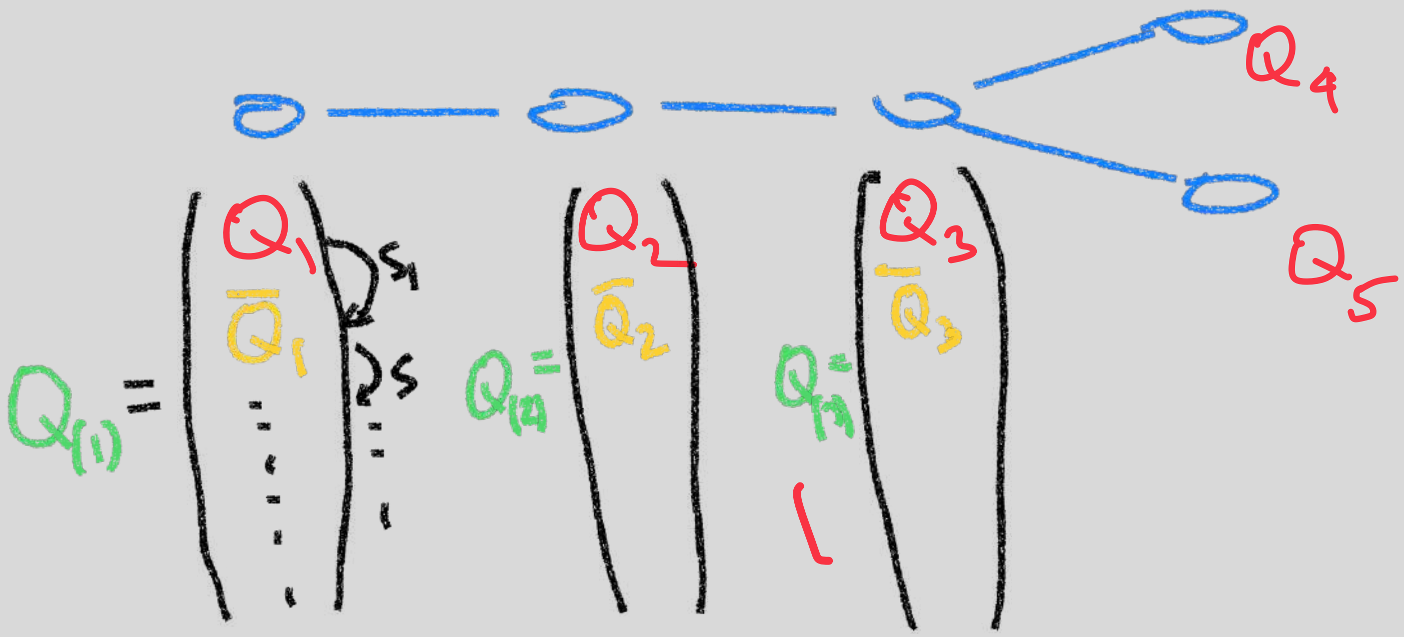
$$\tilde{\mu} = \left(1 + \frac{1}{\bar{F}}\,\bar{\mathbf{P}} \otimes \bar{\mathbf{P}}^*\right) \mu \,\left(1 - F\,\mathbf{P} \otimes \mathbf{P}^*\right) \,, \quad \tilde{\bar{\mu}} = \left(1 + \frac{1}{F}\,\mathbf{P} \otimes \mathbf{P}^*\right) \bar{\mu} \,\left(1 + \frac{1}{\bar{F}}\,\bar{\mathbf{P}}^* \otimes \bar{\mathbf{P}}\right) \,,$$

$$\mathrm{Tr}\,\mathbf{P} \otimes \mathbf{P}^* = \frac{1}{F} - F \,, \quad \mathrm{Tr}\,\bar{\mathbf{P}} \otimes \bar{\mathbf{P}}^* = \frac{1}{\bar{F}} - \bar{F} \,,$$

$$\tilde{F}\bar{F} \times (-1)^{\mathrm{p}_{\bar{\mu}}+\mathrm{p}_{\omega}} = F/\tilde{\bar{F}} \times (-1)^{\mathrm{p}_{\mu}+\mathrm{p}_{\bar{\omega}}} = 1 \,.$$

Conclusions 1

- We propose to extend Q-functions (whose zeros are Bethe roots) to a much larger set of Q-functions. It has covariance w.r.t. to action of Langlands dual of the symmetry algebra, and this covariance leads to numerous insights and *applications*



All properties of Q-system are summarised into fused flag condition and we speculate that this can be a view to define integrability, especially when quantum algebra is unknown.

$$Q_{(1)}(u+ip_1)\otimes Q_{(2)}(u+ip_2)\ldots\otimes Q_{(r)}(u+ip_r) = g(u;\mathbf{p})\,|\Omega\rangle$$

for any $\mathbf{p}$ such that $p_a - p_b = \pm 1$ if $a \sim b$

Related studies were done by many research groups motivated by quite different reasons, faithful citation of literature is difficult.
Below is quite incomplete account for development of Baxter Q-systems

Bethe equations

$$-1=\prod_{b=1}^{r}\frac{Q_b^{[+c_{ab}]}}{Q_b^{[-c_{ab}]}}$$

[Ogievetsky,Wiegmann’86]

QQ system:

$$W\left(Q_a, Q^2_{(a)}\right)=\prod_{b\sim a}Q_b$$

[Pronko, Stroganov`98]
[Voros`99]

[Masoero, Raimondo, Valeri`15]

[Frenkel, Hernandez’16]

Q system on Weyl orbit:

$$W\left(Q^{s(1)}_{(a)}, Q^{s(2)}_{(a)}\right)=\pm_s\prod_{b\sim a}Q^{s(1)}_{(b)}$$

[Pronko, Stroganov`00][Bazhanov,Hibberd,Khoroshkin’01]
[Tsuboi’09]

[Mukhin,Varchenko’05]

[Masoero, Raimondo`18]
[Ferrando, Frassek, Kazakov `20]
[Koroteev, Zeitlin’21]

Extended Q system:

$$Q_{(a)}(u+\frac{\hbar}{2}p_a)=G\left[p\right](u)\,|\,HWS\rangle$$

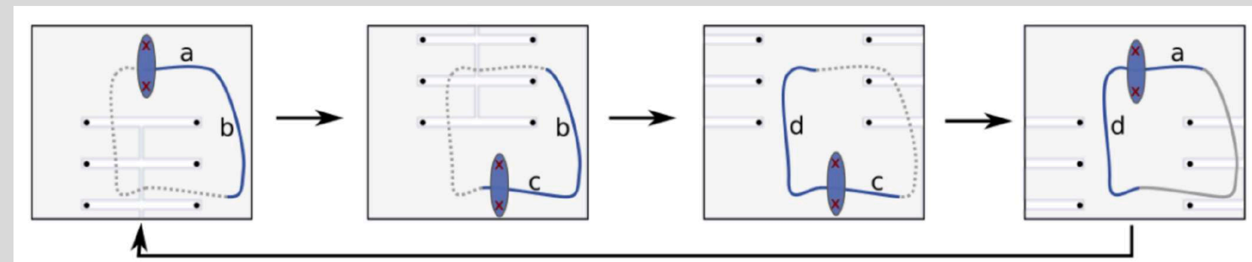
Generalised Plücker coordinates (terminology of [Fomin,Zelevinsky’98])

Extended Plücker coordinates

Conclusions 2

- We make precise meaning of the monodromy bootstrap idea and applied it for the first time in a situation of AdS3/CFT2 integrability where no alternative derivation of QSC was available

$$(\mu_{\check{\mathrm{h}}})^{-1} \cdot (\omega_{\hat{\mathrm{h}}}) \cdot \mathcal{Q} = \mathcal{Q}^*$$



- We are typically forced by crossing equation to take non-idempotent representative of outer autmorphism - square root cut of AdS5/CFT4 turns out to be a luxury, generically they are impossible.

- Monodromy bootstrap offers a way to construct QSC's for a variety of algebras and we expect that AdS/CFT-type integrable systems won't be restricted to isolated points as they are now.

- Preliminary: Using SL(N|N) instead of SL(2|2) seem to work and get equivalent equation

- Future goals:
  - OSP case, extended Q-system is probably guessable from mutation game (work in progress)
  - D(2,1|a) case
  - Queer Lie superalgebras

- AdS3/CFT2 with flux may require Q-system as infinite-dimensional representation of affine Lie algebra. Mutation game cannot be bruteforced on computer then and more intelegent action may be required.

The obtained answers are universal algebro-geometric structures

… Many open directions.

If the topic is interesting for your research:

- e-mail us and let us discuss
- apply for the workshop:

- XXX (Yangian, twisted Yangian)
- XXZ (quantum [twisted] affine)
- Elliptic (?)
- AdS/CFT
- TBA
- ODE/IM

…..

- q-characters
- [finite difference] opers
- [equivariant quantum] cohomology rings
- Bethe/gauge

…..

- 4dCS

``Geometric and Representation-Theoretic Aspects of Quantum Integrability"

August 29-October 21, 2022 @Simons Center

[organisers: P.Koroteev, E.Pomoni, B.Vicedo, D.Volin, A.Zeitlin]