On the UV behaviour of Integrable theories

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What is the space of (integrable) QFTs?

Different integrable Models

$S(\theta)$

$\xrightarrow{\text{TBA}}$

Possible UV behaviours

$E_0(R)$

Outline

- Motivation
  - Irrelevant deformations
  - S-matrix bootstrap
- Set-up and numerical method
  - Thermodynamic Bethe Ansatz
  - Pseudo-arclength continuation
- Results
  - Elliptic sinh-Gordon
  - k resonances
  - "Bosonic minimal models"
- Final Remarks

Motivation

Irrelevant deformations

[Smirnov, Zamolodchikov '16; Cavaglia, Negro, Szecsenyi, Tateo '16]

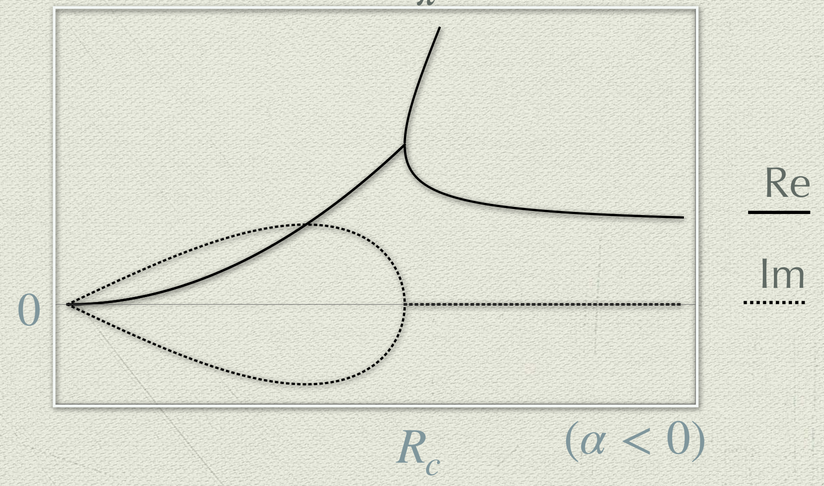
- $T\bar{T}$ operator $\quad \mathscr{A}_0 \to \mathscr{A}_0 + \alpha \int T\bar{T} \qquad T\bar{T}(z) = \lim_{z \to z'} T(z)\bar{T}(z') - \Theta(z)\bar{\Theta}(z')$

- Solvable $\quad E(\alpha, R) = E\left(0, R - \alpha E(\alpha, R)\right)$

  e.g. initial theory is CFT: $\quad E(0,R) \sim -\frac{\pi}{6}\frac{c}{R}$

  $$E(\alpha, R) \sim \frac{R}{2\alpha}\left(1 \mp \sqrt{1 + \frac{2\pi\alpha c}{3R^2}}\right)$$

$$\tilde{c}(\alpha, R) = -\frac{6R}{\pi} E(\alpha, R)$$



- Hagedorn temperature $(\alpha < 0) \quad T_H = R_c^{-1}$

- Interpretation?

- S-matrix definition $\quad S_0(\theta) \to S_0(\theta)\, e^{i\alpha m^2 \sinh(\theta)}$

Irrelevant deformations

- Integrable theories ∞ currents
- Generalized $T\bar{T}$ deformations: $T\bar{T}^{(s)} = T_{s+1}\bar{T}_{s+1} - \Theta_{s-1}\bar{\Theta}_{s-1}$
- S-matrix deformation $S_0(\theta) \to S_0(\theta)\,\Phi(\theta)$

$\Phi(\theta)\Phi(-\theta) = 1$

$\Phi(\theta) = \Phi(i\pi - \theta)$

CDD factors [Camilo, Fleury, Lencses, Negro, Zamolodchikov '21]

$$\Phi(\theta) = \boxed{\left[\prod_{j=1}^{k} \frac{\sinh(\theta) - \sinh(\theta_j)}{\sinh(\theta) + \sinh(\theta_j)}\right]} \left[\exp\left(-i \sum_s a_s \sinh(s\theta)\right)\right]$$

resonances

entire [Hernandez-Chifflet, Negro, Sfondrini '19]

S-matrix Bootstrap

- Explore space of consistent S-matrices (unitarity, crossing, analyticity)

  → Bounds on space of couplings

- In 2D integrable theories at special points. E.g. monolith

[LC, He, Kruczenski, Vieira '19]

- "$O(N)$ periodic Yang-Baxter" model

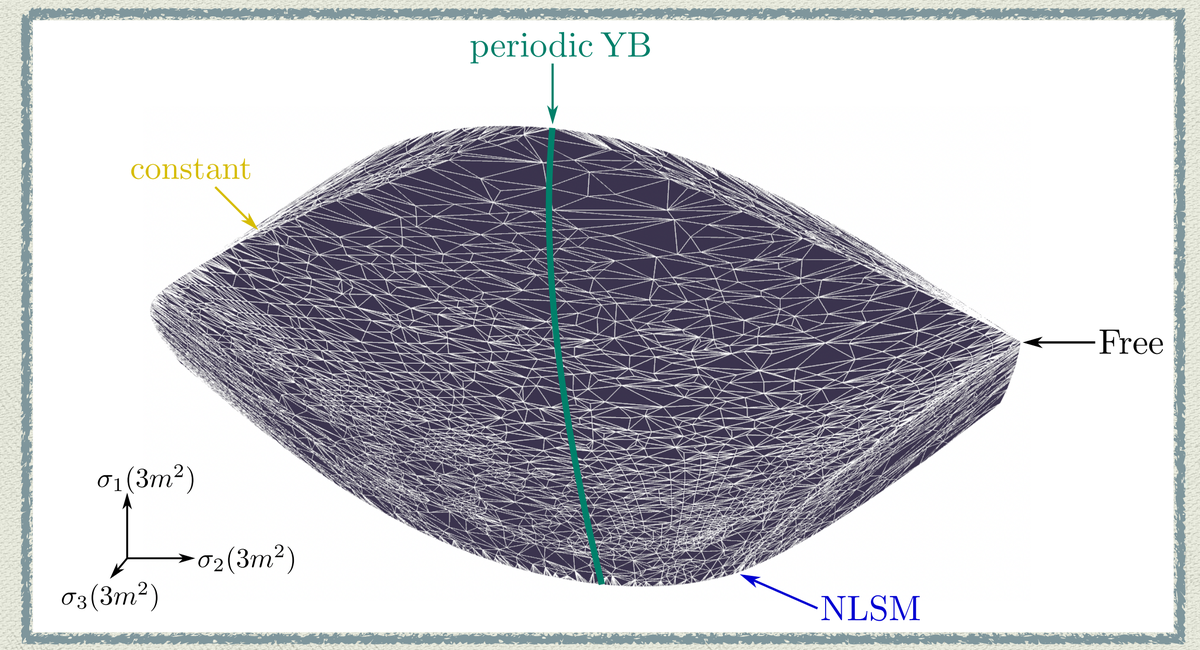
  [Hortacsu, B. Schroer and H. J. Thun '79]

  $S^{pYB}(\theta) = S^{pYB}(\theta + \tau)$

  ∞ resonances

- Generically, optimal S-matrices have ∞ resonances

- Physical? UV nature?



Allowed O(N) S-matrices with no bound states

Set-up and numerical method

Thermodynamic Bethe Ansatz

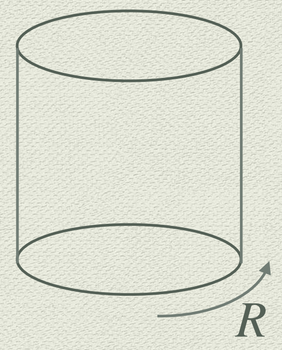
- TBA equations: (single stable particle of mass m)

$$\epsilon(\theta) = r\cosh\theta \mp \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \boxed{\varphi(\theta-\theta')} \ln\left[1 \pm e^{-\epsilon(\theta')}\right]$$

$r = mR$

$\varphi(\theta) = -\,i\,\partial_\theta \ln S(\theta)$

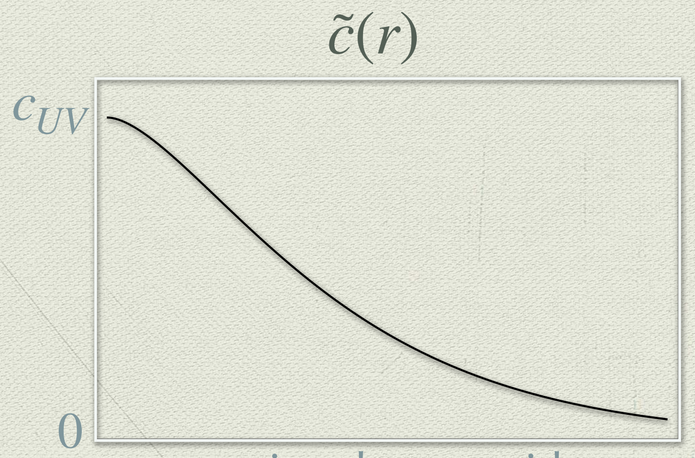
$S(0) = \mp 1$



- Ground state energy / Effective central charge

$$E_0(R) = -\frac{\pi}{6}\frac{\tilde{c}(r)}{R} \qquad \tilde{c}(r) = \pm\frac{6}{\pi}\int_{-\infty}^{\infty} r\cosh(\theta)\ln\left[1 \pm e^{-\epsilon(\theta)}\right]\frac{d\theta}{2\pi}$$

$$\lim_{r\to 0} \tilde{c}(r) = c_{UV} - 12(\Delta_{\min} + \bar{\Delta}_{\min})$$



massive theory with conventional UV fixed point

Thermodynamic Bethe Ansatz

- Possible asymptotics $r \to \infty$

$$\begin{cases} \epsilon(\theta) \sim r\cosh(\theta) & \text{usual} \\ \epsilon(\theta) \sim -\, r f(\theta) & f(\theta) > 0 \ \text{ for } \theta \in \Theta \\ \left(\epsilon(\theta) \sim r^{-1}\cosh(\theta) \quad \text{only } T\bar{T}\right) \end{cases}$$

[Camilo, Fleury, Lencses, Negro, Zamolodchikov '21]

$$\boxed{|\varphi|_1 = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\varphi(\theta) > 1}$$

resonances - #bound states

condition for second branch $\tilde{c} \sim r^2$

- UV $(r \to 0)$: if plateau solutions $\to$ "Dilog trick"

$$\tilde{c}(0) = \frac{6}{\pi^2} \times \begin{cases} \mathrm{L}\left(\dfrac{1}{1+e^{\epsilon}}\right) & \text{fermionic} \\ \mathrm{L}\left(e^{-\epsilon}\right) & \text{bosonic} \end{cases}$$

$$\mathrm{L}(x) = \mathrm{Li}_2(x) + \frac{1}{2}\ln(x)\ln(1-x)$$

Numerical method

- Iterative solutions

For each r : $$\epsilon^{(k+1)}(\theta) = r\cosh\theta \mp \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\,\varphi(\theta-\theta')\,\ln\left(1 \pm e^{-\epsilon^{(k)}(\theta')}\right)$$

Might stop converging at some r (not necessarily a sign for turning point)

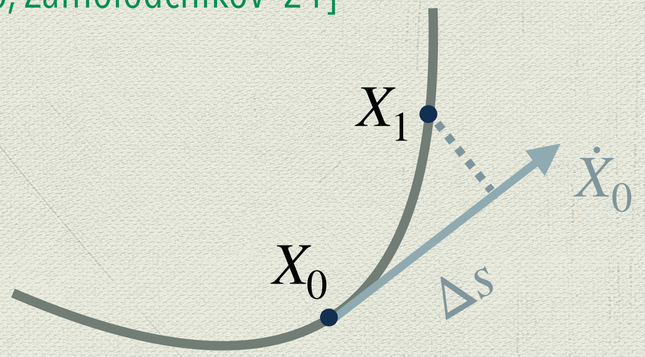
- Pseudo-arclength continuation

[see e.g. Allgower, Georg '12]
[Camilo, Fleury, Lencses, Negro, Zamolodchikov '21]

$$\{\epsilon, r\} \to \{\epsilon(s), r(s)\} = X(s)$$

Use arclength s instead of r to parametrize solution curve

$$H(X_1) = 0$$
$$(X_1 - X_0)\cdot \dot{X}_0 = \Delta s$$
(predictor / corrector)



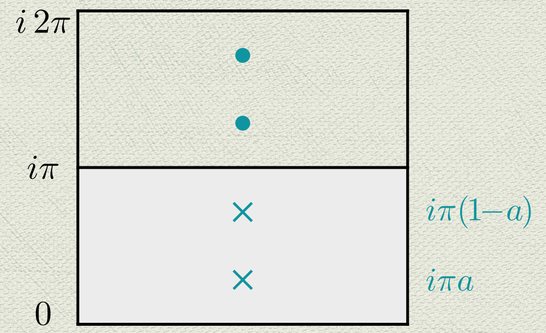
Results

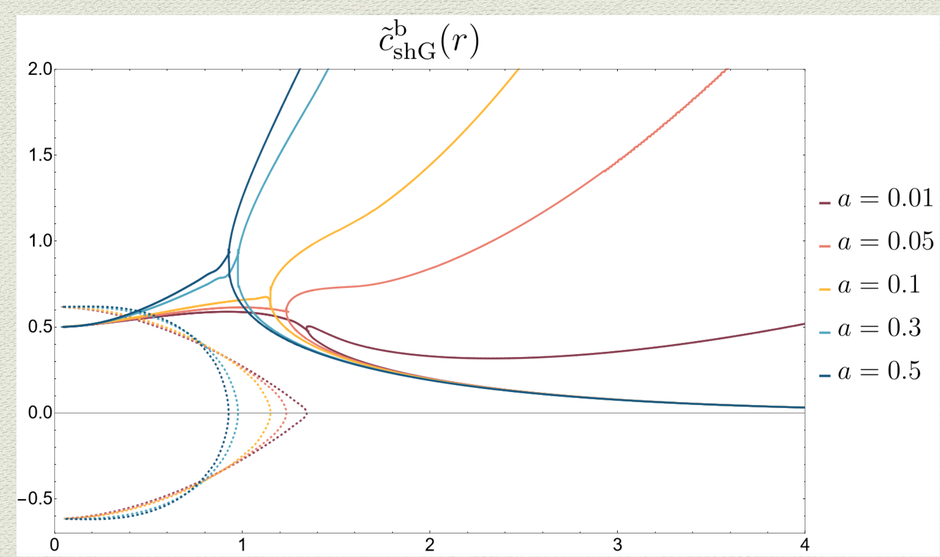
Warm up: Fermionic vs Bosonic sinh-Gordon

[Mussardo, Simon '99]

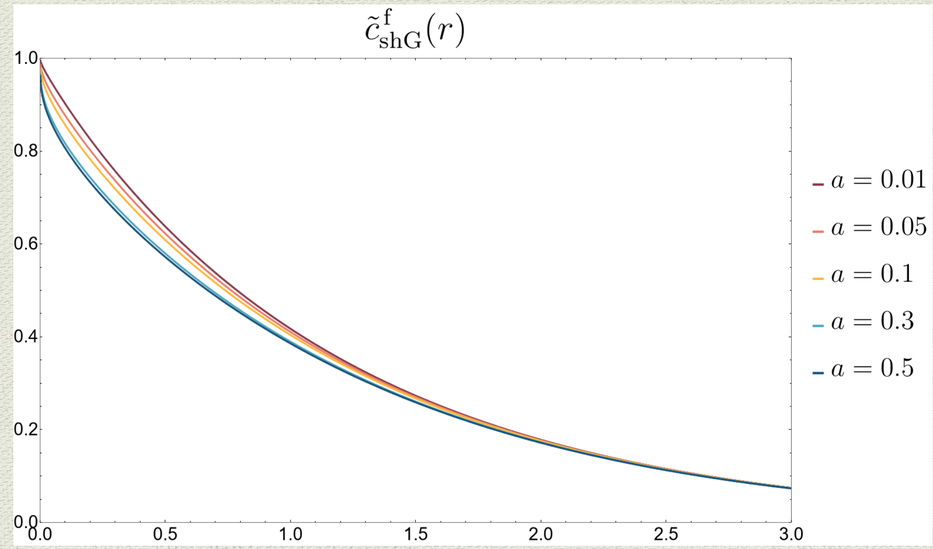
- Single resonance

$$S^{f,b}_{shG}(\theta) = \pm \frac{\sinh\theta - i\sin(\pi a)}{\sinh\theta + i\sin(\pi a)}$$





bosonic



fermionic

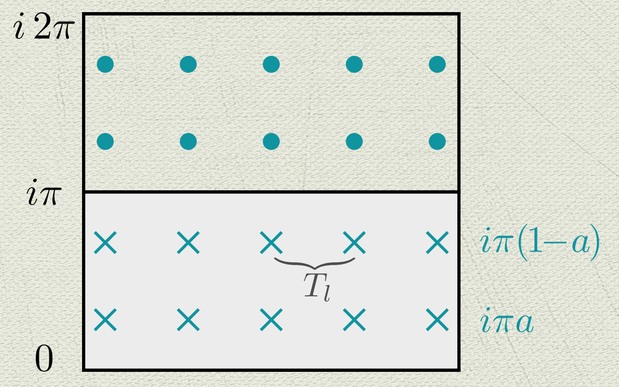
- $r_c(a)$
- $\tilde{c}(0) = \mathrm{L}(e^{-(\epsilon=\pm i\pi/3)}), \quad \epsilon = +\ln(1-e^{-\epsilon})$
- $(-1) \to$ extra resonance in fermionic

Elliptic sinh-Gordon

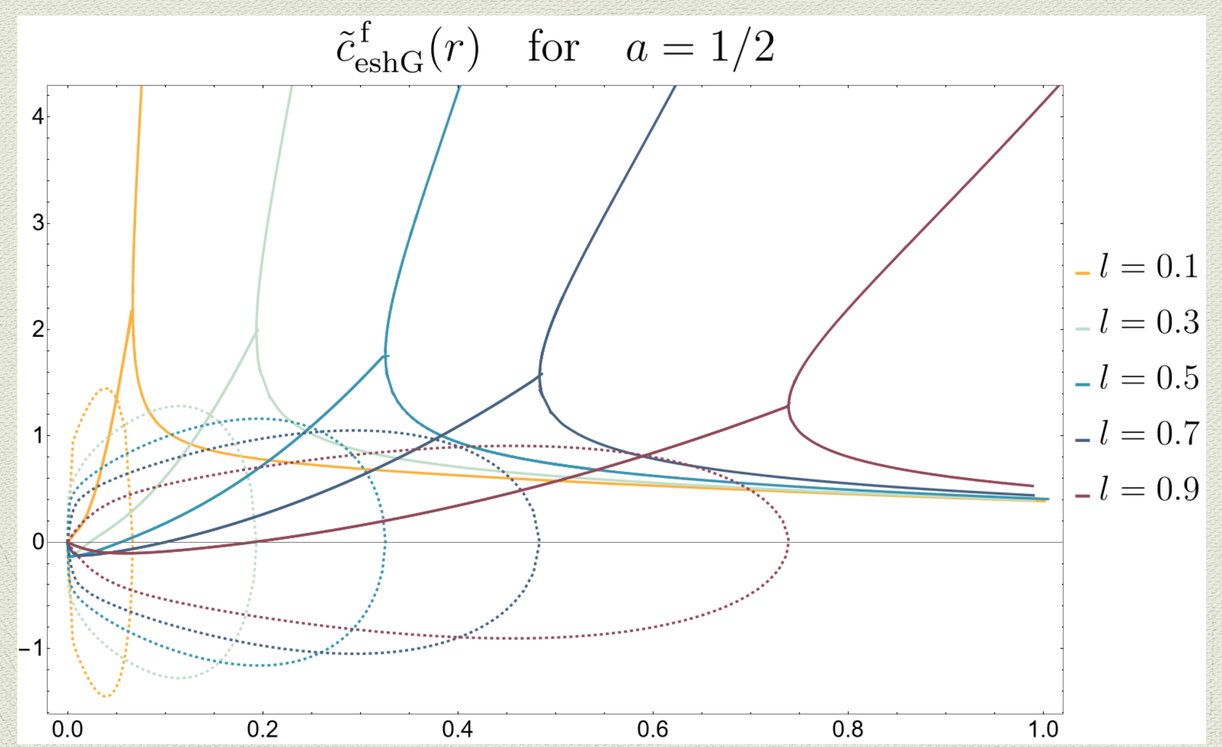
- ∞ resonances, periodic

[Mussardo, Penati '99]

$$S^{f,b}_{eshG}(\theta) = \pm \frac{\mathrm{sn}_l\left(2iK_l\theta/\pi\right) + \mathrm{sn}_l(2K_l a)}{\mathrm{sn}_l\left(2iK_l\theta/\pi\right) - \mathrm{sn}_l(2K_l a)}$$

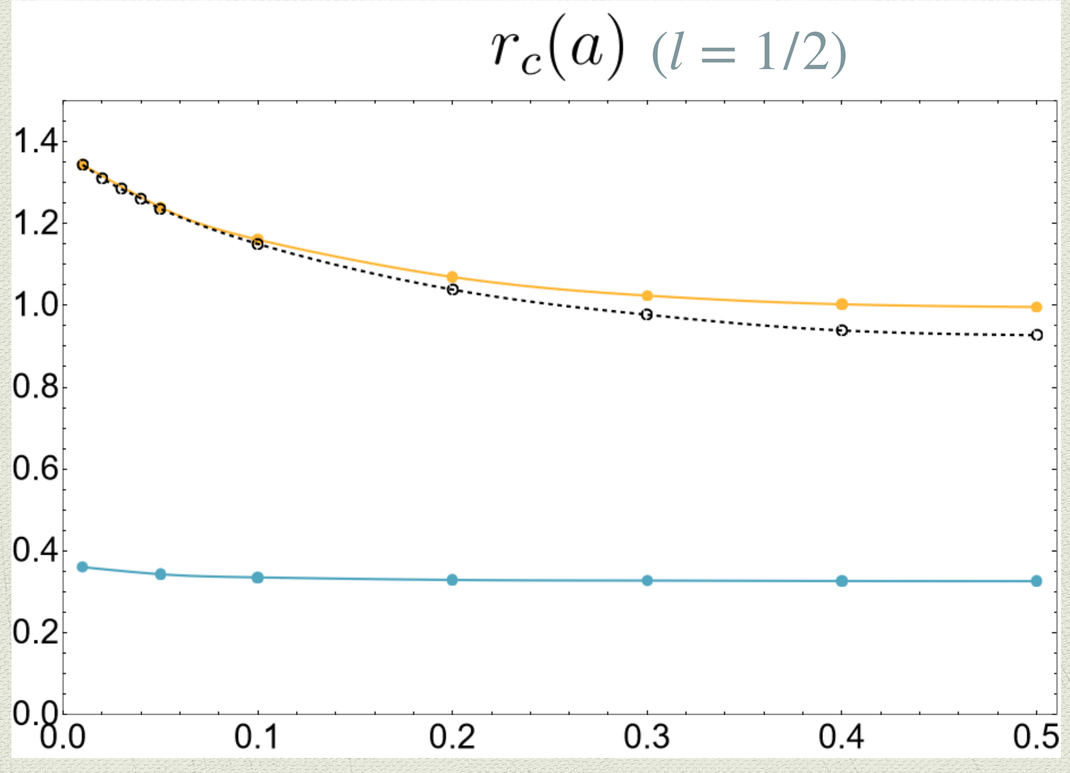


- free $a \to 0$
- sinh-Gordon $T_l \to \infty$ $(l \to 0)$



- $r_c(l, a)$
- $\tilde{c}(0) \to 0$ (like $T\bar{T}$)

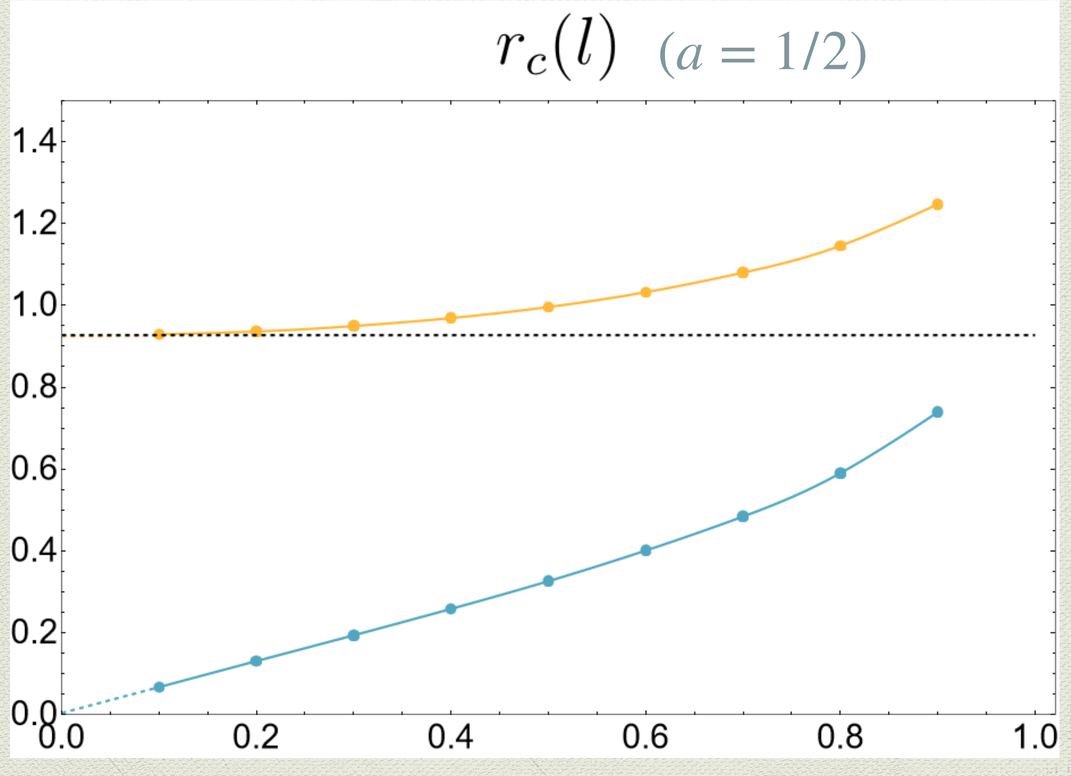
Elliptic sinh-Gordon



bos

bshG

ferm



k resonances

- Fermionic ↔ bosonic TBA

$$\epsilon^-(\theta) = \ln\left[e^{\epsilon^+(\theta)} - 1\right],$$

$$\varphi^-(\theta) = \varphi^+(\theta) + 2\pi\,\delta(\theta) \longrightarrow \lim_{u\to 0^+} \frac{\sinh\theta - i\sin u}{\sinh\theta + i\sin u}$$

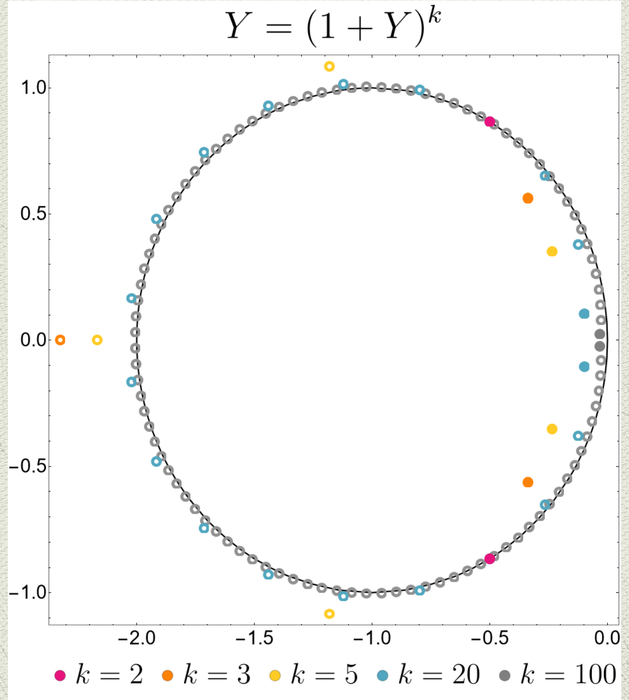
k+1 fermionic CDDs ↔ k bosonic CDDs

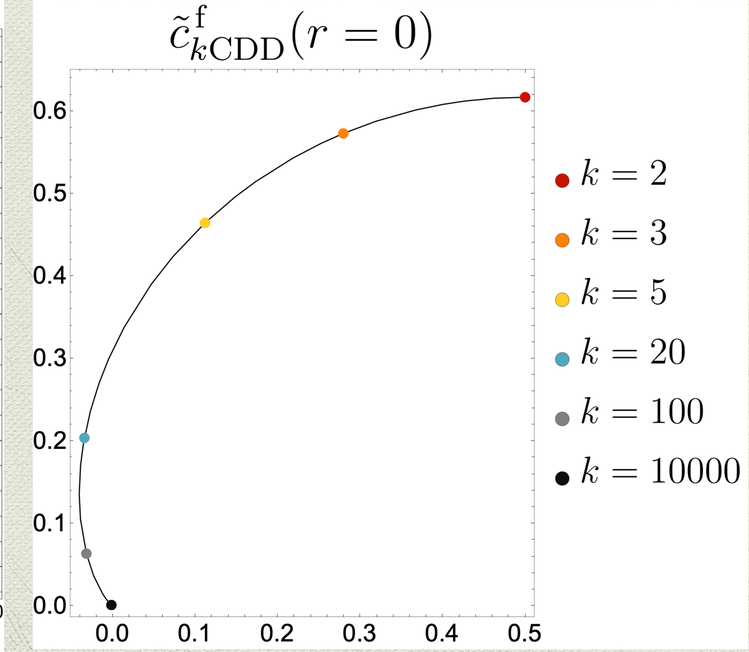
- UV ($r \to 0$) plateau solutions

$$Y = (1+Y)^k, \quad Y = e^{-\epsilon}$$

k possible solutions
2 picked by TBA s.t. $|\tilde{c}(0)|$ minimized

(Confirmed numerically for $k \leq 7$)





"Bosonic min models"

- Bound states?

non-unitary min models $\mathscr{M}_{2,2n+3} + \Phi_{1,3}$

[Smirnov '89]

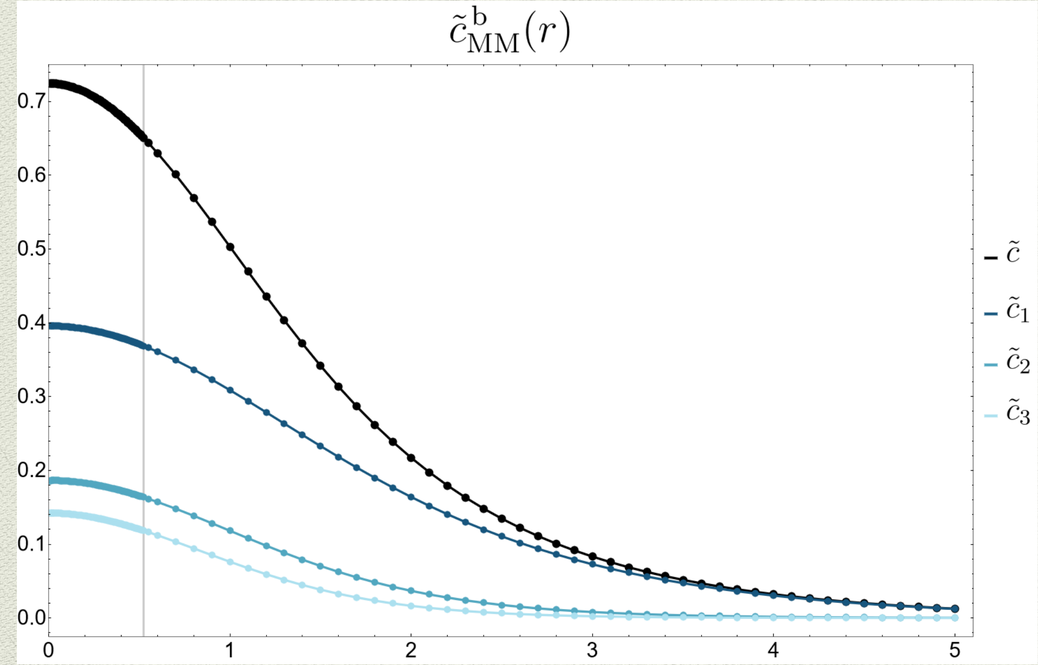
$$S_{11}(\theta) = \mathrm{th}_{\frac{2}{2n+1}}(\theta)\,,$$

$$S_{ab}(\theta) = \mathrm{th}_{\frac{|a-b|}{2n+1}}(\theta)\,\mathrm{th}_{\frac{a+b}{2n+1}}(\theta) \prod_{k=1}^{min(a,b)-1} \left[\mathrm{th}_{\frac{|a-b|+2k}{2n+1}}(\theta)\right]^2$$

$$\mathrm{th}_x(\theta) = \frac{\sinh(\theta) + i\sin(\pi x)}{\sinh(\theta) - i\sin(\pi x)}$$

$$\tilde{c}^{(n)}(0) = 2n/(2n+3)$$

$$S_{ab}(\theta) \to -S_{ab}(\theta)$$



n=3

- UV complete. Physical model?

$$\tilde{c}(r=0) \approx 0.5, 0.641304, 0.724253, 0.778979, 0.817083 \quad \text{for} \quad n = 1,2,3,4,5$$

Final Remarks

Summary/ future directions

- Analyzed TBA for models with one stable particle and k resonances (b/f).
- In all cases (except $k=1$ (f)) there is a turning point at r_c.
- We were able to follow complex solutions past turning point all the way to $r \to 0$.
- Always two branches. In UV solutions such that $|\tilde{c}|$ is minimized.
- More stable particles: New (?) family of UV complete theories.

  $\to$ Classification for N bound states, k resonances [massless flows: Ahn, LeClair '22]

- Evidence for: conventional UV fixed point $\leftrightarrow$ $|\varphi|_1 = \#\text{resonances - bound states} \leq 1$.
- Most consistent S-matrices incompatible with standard UV fixed point.
- Complex CFTs in UV? Excited TBA, conformal perturbation theory, …

Thank you!