Gaudin Integrability and Harmonic Analysis of Conformal Partial Waves

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based on works with
M. Isachenkov, S. Lacroix, J. A. Mann, L. Quintavalle, V. Schomerus and E. Sobko

Partial wave decompositions

Partial waves in quantum field theory

- Basic analytic tool in QFT, e.g. in $2 \to 2$ scattering

$$T(s,t) = -i\langle p_3, p_4|\ S - \hat{1}\ |p_1, p_2\rangle = \sum_{J=0}^{\infty} f_J(s) P_J^{(d)}(\cos\theta)$$

- partial wave = contribution to scattering of all spin J intermediate states
- $P_J^{(d)}(\cos\theta)$ = zonal spherical function on $SO(d-1)$

Partial waves in CFT

- Decompositions of four-point functions

$$\langle \phi(x_1)\dots\phi(x_4)\rangle = \Omega(x_i)\sum_{\Delta,l} C^2_{\phi\phi\mathcal{O}}\, g_{\Delta,l}(u,v)$$

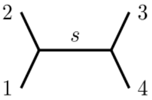
- partial wave = contribution of states in a single conformal representation

Conformal bootstrap

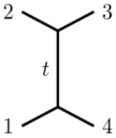
Bootstrap equations [Polyakov]

- Crossing symmetry = self-consistency

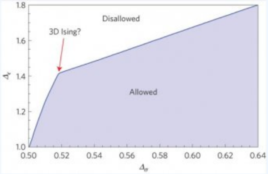
$$\sum_{s\in S} C_{12s}C_{s34}$$

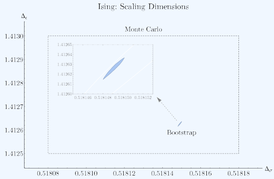


$$= \sum_{t\in S} C_{23t}C_{t41}$$



Applications [El-Showk, Kos, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi]





Partial waves for four-point functions

Casimir equations [Dolan, Osborn]

- Derived a 2^{nd} order differential equation for blocks
- Solved in two and four dimensions

$$g^{(4d)}_{\Delta,l}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\Big(k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - k_{\Delta-l-2}(z)k_{\Delta+l}(\bar{z})\Big)$$

Efficient methods for scalar and spinning four-point blocks

- Differential shifting operators [Costa, Penedones, Poland, Rychkov]
- Expansions in radial coordinates [Hogervorst, Rychkov; Costa, Hansen, Penedones, Trevisani]
- Recursion relations [Penedones, Trevisani, Yamazaki; Erramilli, Iliesiu, Kravchuk]
- From seed conformal blocks [Echeverri, Elkhidir, Karateev, Serone; Costa, Hansen, Penedones, Trevisani]
- Weight-shifting operators [Karateev, Kravchuk, Simmons-Duffin]

Relation to Calogero-Sutherland models [Isachenkov, Schomerus, Sobko]

- Mapped the Dolan-Osborn operator to the BC_2 CS Hamiltonian

Higher-point functions

Problem

- Much less known about about higher-point partial waves
- Solved in $d = 1, 2$ or for scalar exchanges [Rosenhaus; Fortin, Hoback, Ma, Parikh, Skiba]
- Weight-shifting for five-point blocks [Poland, Prilepina]

Potential uses

- Higher-point correlators contain a wealth of OPE data [Goncalves, Pereira, Zhou]
- "Abstract" bootstrap on the lightcone [Antunes, Costa, Goncalves, Vilas Boas]
- Away from the lightcone . . .

Aim for the talk

Questions

- In what sense are (higher-point) partial waves integrable?
- Elements of universality in Casimir equations?

Today

$$\begin{array}{ccc} \text{4-point functions} & \leftrightarrow & \text{(spinning) Calogero-Sutherland model} \\ \uparrow & & \\ \text{N-point functions} & \leftrightarrow & \text{Gaudin models} \\ \downarrow & & \\ \text{3-point functions} & \leftrightarrow & \text{elliptic Calogero-Moser model} \end{array}$$

Results

- Complete set of differential equations for the waves
- First structural consequences: 6-point blocks factorisation
- New relations between integrable models

Plan for the talk

Spherical functions

Why should Dolan-Osborn equations be integrable?

- $G_4(x_i) \sim$ two-point function of two-particle states. Two-particle little group

$$K = SO(1,1) \times SO(d)$$

- Ward identities: $G_4(x_i) \in C\left(G/K \times G/K\right)^G \cong C(K\backslash G/K)$

Spherical functions [Harish-Chandra, Gelfand, Godement]

- Lie group G and a subgroup K fixed by Cartan involution

$$G = SO(d+1,1), \quad K = SO(1,1) \times SO(d)$$

- Bi-covariant functions on G

$$\Gamma_{\rho,\sigma} = \{f : G \to \mathrm{Hom}(W_r, W_l) \mid f(k_l g k_r) = \rho(k_l) f(g) \sigma(k_r)\}$$

- $\Gamma_{1,1}$ commutative algebra under convolutions

Radial component map

Radial part of the Casimir [Harish-Chandra, Berezin]

- Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

$$\mathfrak{g} = \mathfrak{a}_p \oplus \mathfrak{m} \oplus \sum_{\lambda\in\Sigma} \mathfrak{g}^\lambda \quad \to \cdots \to \quad U(\mathfrak{g}) = U(\mathfrak{a}_p) \otimes U(\mathfrak{k}) \otimes_{U(\mathfrak{m})} U(\mathfrak{k})$$

- $\mathfrak{a}_p$ is the maximal abelian subspace of $\mathfrak{p}$, the split rank of (G, K)
- On the group level, decomposition takes the from

$$G = KA_pK$$

- Spherical functions depend effectively on $\dim(A_p)$ variables
- Casimir reduced to an operator on A_p using radial decomposition

$$C_2 = h^{ij} H_i H_j + \sum_{\alpha\in P_+} \coth(\alpha\cdot t) h_{\tilde{\alpha}} + \text{spin part}$$

- Harish-Chandra's map $\Pi : U(\mathfrak{g}) \to \mathrm{Fun}(A_p) \otimes U(\mathfrak{a}_p) \otimes U(\mathfrak{k}) \otimes U(\mathfrak{k})$

Quantum integrable systems [Feher, Pusztai; Reshetikhin, Stokman; Oblomkov; Etingof, Frenkel, Kirillov…]

Calogero-Sutherland Hamiltonian [Olshanetsky, Perelomov]

- Radial part of the Laplacian conjugate to the CS Hamiltonian

$$H = \partial_{t_1}^2 + \partial_{t_2}^2 + \frac{1-D_+^2}{2\sinh^2(t_1+t_2)} + \frac{1-D_-^2}{2\sinh^2(t_1-t_2)} - \frac{d^2-2d+2}{2}$$
$$+ \frac{M_{1a}M_{1a} - \frac{1}{4}(d-2)(d-4)}{\sinh^2 t_1} + \frac{M_{2a}M_{2a} - \frac{1}{4}(d-2)(d-4)}{\sinh^2 t_2} - \frac{1}{2}M^{ab}M_{ab}$$

- Higher Casimirs ensure integrability
- Equivalent to the Dolan-Osborn equation for scalars

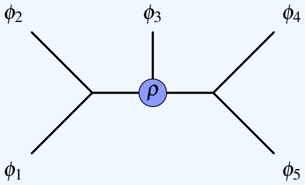
Related systems

- SUSY extension as a nilpotent perturbation [IB, Schomerus, Sobko]
- Systems involving defects [IB, Isachenkov, Liendo, Linke, Schomerus]
- Partial waves for ordinary QFT are rank 1 spherical functions

Insufficiency of Casimirs

Casimir operators [BLMQS, PRL 126 (2021)]

- For four-point functions, 2^{nd} and 4^{th} order Casimirs measure weight and spin of propagating field



- Four Casimir operators to measure weights and spins
- Five cross ratios $\rightarrow$ complete by the "vertex operator"

$$\mathcal{V}_4 = \kappa_4^{\alpha_1\ldots\alpha_4}(X^{(12)}_{\alpha_1} - X^{(3)}_{\alpha_1})\ldots(X^{(12)}_{\alpha_4} - X^{(3)}_{\alpha_4})$$

- Wish to generalise to higher-point functions

Phase space of N-point functions

Solutions to Ward identities

- Correlators = invariants in the N-fold tensor product of principal series

$$\mathcal{H}_{red} = \mathcal{H}^G = (\pi_1 \otimes \cdots \otimes \pi_N)^G$$

Gaudin model

- $\mathcal{H}$ carries the action of the algebra

$$[X_\alpha^{(i)}, X_\beta^{(j)}] = \delta_{ij} f_{\alpha\beta}^\gamma X_\gamma^{(i)}$$

- Gaudin Hamiltonians constructed with the help of Lax matrix

$$\mathcal{L}_\alpha(z) = \sum_{i=1}^N \frac{X_\alpha^{(i)}}{z - z_i}, \quad \mathcal{H}_p(z) = \kappa^{\alpha_1 \ldots \alpha_p} \mathcal{L}_{\alpha_1}(z) \ldots \mathcal{L}_{\alpha_p}(z) + \ldots$$

- Diagonal symmetry = Gaudin Hamiltonians preserve $\mathcal{H}_{red}$

$$[\mathcal{H}_p(z), \mathcal{H}_q(w)] = 0, \quad [\sum X_\alpha^{(i)}, \mathcal{H}_p(z)] = 0$$

Edge and vertex operators

Colliding punctures

- "Sites" $z_1, \ldots, z_N$ = parameters with no CFT counterpart
- Simple limit with $z_N \to \infty$ and each $z_{i<N} \to 0, 1$
- Scaling dependent on OPE topology

Edge and vertex operators [BLMQS, JHEP 10 (2021)]

- Each edge r breaks $\underline{N} = I_{r,1} \cup I_{r,2}$. Form Casimirs

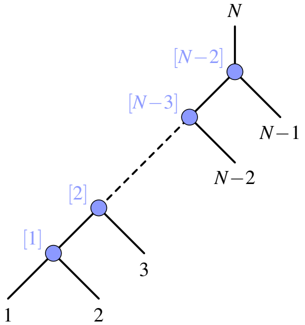
$$\mathcal{D}_r^p = \kappa_p^{\alpha_1 \ldots \alpha_p} X_{\alpha_1}^{(I_{r,1})} \ldots X_{\alpha_p}^{(I_{r,1})}$$

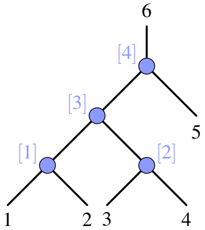
- Each vertex ρ breaks $\underline{N} = I_{\rho,1} \cup I_{\rho,2} \cup I_{\rho,3}$

$$\mathcal{D}_{\rho,12}^{p,\nu} = \kappa_p^{\alpha_1 \ldots \alpha_p} X_{\alpha_1}^{(I_{\rho,1})} \ldots X_{\alpha_\nu}^{(I_{\rho,1})} X_{\alpha_{\nu+1}}^{(I_{\rho,2})} \ldots X_{\alpha_p}^{(I_{\rho,2})}$$

- {Gaudin Hamiltonians} $\to$ {edge operators, vertex operators}
- Strong evidence for completeness

OPE limits of Gaudin models





$$z_1 = x^3, \qquad z_3 = x + x^3, \qquad z_5 = 1$$
$$z_2 = x^2 \qquad z_4 = x + x^2, \qquad z_6 = x^{-1}$$

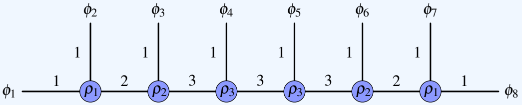
$$z_i = x^{N-1-i}$$

$$\mathcal{L}_\alpha(z) = \frac{X_\alpha^{(1)}}{z - x^3} + \frac{X_\alpha^{(2)}}{z - x^2} + \frac{X_\alpha^{(3)}}{z - x - x^3} + \frac{X_\alpha^{(4)}}{z - x - x^2} + \frac{X_\alpha^{(5)}}{z - 1} + \frac{X_\alpha^{(6)}}{z - x^{-1}}$$

$$\mathcal{L}_\alpha^{[i]}(z) = \frac{X_\alpha^{(1)} + \cdots + X_\alpha^{(i)}}{z} + \frac{X^{(i+1)}}{z - 1}$$

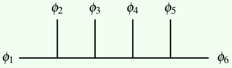
Counting dependencies

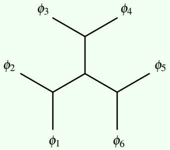
Counting of edge operators



- Increase by one in each step towards centre until rank(G) is reached

Completed by vertex operators

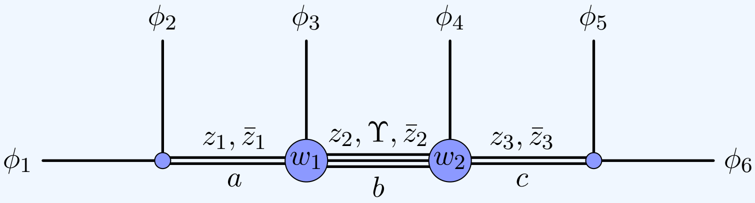




- Comb channel has single variable vertices, while the snowflake vertex has three variables

OPE-adapted coordinates

Comb channel six-point function [BLMQS, JHEP 06 (2022)]



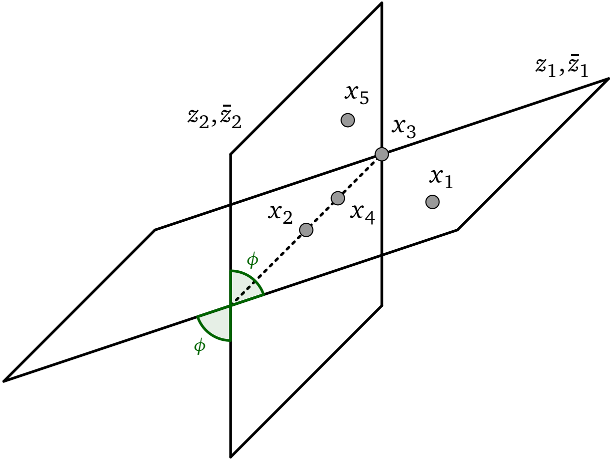
- coordinates $(z_i, \bar{z}_i)$ defined using four-point subdiagrams
- coordinates (w_1, w_2) defined using five-point subdiagrams
- six-point coordinate Υ

OPE limit

$$z_2, \bar{z}_2, \Upsilon \to 0$$

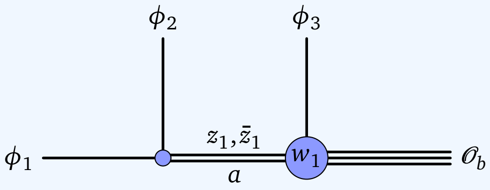
Conformal frame

$$x_2 = 0\,, \qquad x_4 = e_1\,, \qquad x_3 = \infty\,, \qquad w_1 = \sin^2\frac{\phi}{2}$$



Spinning Calogero-Sutherland Hamiltonian

Factorised partial wave



Properties

- 9 conformal invariants reduce to 6 in the limit

$$\psi(z_i, \bar{z}_i, w_j, \Upsilon) \overset{z_2, \bar{z}_2, \Upsilon \to 0}{\sim} z_2^{\frac{1}{2}(\Delta_b + l_b + \ell_b)} \bar{z}_2^{\frac{1}{2}(\Delta_b - l_b - \ell_b)} \Upsilon^{\ell_b} \psi(z_1, \bar{z}_1, w_1, z_3, \bar{z}_3, w_2)$$

- Hamiltonians separate into variables $(z_1, \bar{z}_1, w_1)$ and $(z_3, \bar{z}_3, w_2)$
- Quadratic Hamiltonians = spinning Calogero-Sutherland

Scalar-scalar-scalar-MST$_2$ Hamiltonian

The Hamiltonian

Substitutions for generators give the one-sided Hamiltonian

$$H^{(b)}_{l,\ell} = \partial^2_{t_1} + \partial^2_{t_2} + \frac{1-(2b+\ell-l+2X\partial_X)^2}{2\sinh^2(t_1+t_2)} + \frac{1-(2b-\ell+l-2X\partial_X)^2}{2\sinh^2(t_1-t_2)}$$

$$+ \frac{L_{l,\ell}(X)-\frac{1}{4}(d-2)(d-4)}{\sinh^2 t_1} + \frac{L_{l,\ell}(-X)-\frac{1}{4}(d-2)(d-4)}{\sinh^2 t_2} - \frac{d^2-2d+2}{2}$$

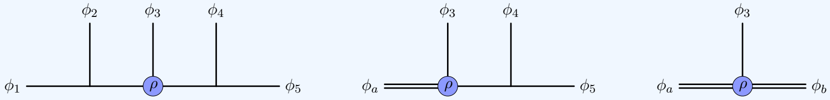
where $2b = \Delta_3 - \Delta_b$ and

$$L_{l,\ell}(X) = -X(1-X)^2\partial_X^2 - \left(\ell(1-X) - 2(1-l)X + \frac{d-2}{2}(1+X)\right)(1-X)\partial_X$$

$$+ \left(1 - l - \frac{d-2}{2}\right)(\ell(1-X)+lX) - \frac{l(d-2)}{2}$$

Reduction to a vertex

Vertex reduction [BLMQS, JHEP 11 (2021)]



Elliptic Calogero-Moser model

- Act with $\mathcal{V}_4$ on functions of the form

$$\psi(z_1, \bar{z}_1, z_2, \bar{z}_2, w) \overset{z_i, \bar{z}_i \to 0}{\sim} \bar{z}_1^{\frac{\Delta_a - l_a}{2}} z_1^{\frac{\Delta_a + l_a}{2}} z_2^{\frac{\Delta_b + l_b}{2}} \bar{z}_2^{\frac{\Delta_b - l_b}{2}} \psi(w)$$

- Produces 4^{th} order differential operator $H(w, \partial_w)$ with four singular points

$$z_0 = 0, \quad z_1 = \frac{1+i}{2}, \quad z_2 = \frac{i}{2}, \quad z_3 = \frac{1}{2}$$

- Coincides with elliptic CM Hamiltonian of Etingof, Felder, Ma, Veselov

Summary of results

1. A Gaudin model adapted to any CFT correlation function in any channel
2. Coordinates compatible with OPE limits
3. OPE factorisation of six-point conformal partial waves
4. Relations between Gaudin and CMS models

Future directions and open questions

1. Lightcone bootstrap for six-point functions

   and how to formulate numerical bootstrap?

2. General solution theory?
   $\rightarrow$ Bethe ansatz? Separation of variables? Weight-shifting operators?
   $\rightarrow$ Ideal: build partial waves by "gluing"edge and vertex wavefunctions

3. Universality and CFTs with defects:
   $\rightarrow$ thee-point blocks from the radial component map [with V. Schomerus]
   $\rightarrow$ cyclotomic Gaudin models [with S. Lacroix and V. Schomerus]

4. Extension of harmonic analysis methods to non-conformal theories
   $\rightarrow$ *S*-matrix partial waves [with F. Russo and A. Vichi]

Thank you!

Radial component map on $SL(2,\mathbb{R})$

Generators and bracket relations

$$[H,E_+]=E_+,\quad [H,E_-]=-E_-,\quad [E_+,E_-]=2H$$

Unprimed and primed generators of $\mathfrak{k}$

$$h=e^{tH},\quad Y=\frac{1}{2}(E_+-E_-),\quad Y'=h^{-1}Yh=\frac{1}{2}(e^{-t}E_+-e^{t}E_-)$$

Spherical functions and the Cartan decomposition

$$\Gamma_{m,n}=\{f:G\to\mathbb{C}\mid f(e^{\varphi Y}ge^{\psi Y})=e^{i(m\varphi-n\psi)}f(g)\},\quad g(\varphi,t,\psi)=e^{\varphi Y}e^{tH}e^{\psi Y}$$

Radial decomposition of the Casimir

$$C_2=H^2+\frac{1}{2}\{E_+,E_-\}=H^2+\coth t\ H+\frac{1}{\sinh^2 t}\left(Y'^2-2\cosh t\ Y'Y+Y^2\right)$$

Substitutions $\{H\to\partial_t,\ Y'\to im,\ Y\to -in\}$ lead to $\Delta|_{\Gamma_{m,n}}$

$$\Delta_{m,n}=\partial_t^2+\coth t\ \partial_t-\frac{1}{\sinh^2 t}\left(m^2+2mn\cosh t+n^2\right)$$

Decompositions of the conformal Lie algebra

Cartan decomposition

$$\mathfrak{g} = \mathfrak{a}_p \oplus h^{-1}\mathfrak{q}h \oplus \mathfrak{k}, \quad \mathfrak{k} = \mathfrak{m} \oplus \mathfrak{q}$$

Gauss decomposition $\mathfrak{g} = \mathfrak{a}_p \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}$

$$\text{rank } 1: \quad \mathfrak{g} = \mathfrak{so}(1,1) \oplus \mathfrak{so}(d) \oplus \mathbb{R}^d \oplus \mathbb{R}^d$$

$$\text{rank } 2: \quad \mathfrak{g} = \mathbb{R}^2 \oplus \mathfrak{so}(d-2) \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}$$

Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_p \oplus \mathfrak{n}$,

$$\text{rank } 1: \quad \mathfrak{g} = \mathfrak{so}(d+1) \oplus \mathfrak{so}(1,1) \oplus \mathbb{R}^d$$

$$\text{rank } 2: \quad \mathfrak{g} = (\mathfrak{so}(1,1) \oplus \mathfrak{so}(d)) \oplus \mathbb{R}^2 \oplus \mathfrak{n}$$

Three-point functions of spinning fields

Abstractly, the space of three-point functions is

$$(W_1 \otimes W_2 \otimes W_3)^{SO(d-1)}$$

If two fields are scalars, only one tensor structure

$$\# = \dim W_1^{SO(d-1)} = 1$$

STT-STT-scalar three-point function

$$\langle \mathcal{O}_1(x_1, z_1)\mathcal{O}_2(x_2, z_2)\varphi_3(x_3)\rangle = \Omega(x_i, z_j) t(X)$$

with

$$\Omega(x_i, z_j) = \frac{(X_{1;32}\cdot z_1)^{l_1}(X_{2;13}\cdot z_2)^{l_2}}{(X_{3;21}^2)^{-\frac{\Delta_3}{2}}(X_{2;13}^2)^{\frac{l_2-\Delta_2}{2}}(X_{1;32}^2)^{\frac{l_1-\Delta_1}{2}}}, \quad X = \frac{1}{x_{12}^2}\frac{z_{1\mu}J^{\mu\nu}(x_{12})z_{2\nu}}{(X_{1;32}\cdot z_1)(X_{2;13}\cdot z_2)}$$

Similarly MST_2-MST_2-scalar three-point function depends on two variables.

OPE limit of the Gaudin model

Combinatorially polynomials define "vertex Lax matrices"

$$\mathcal{L}^{\rho}_{\alpha}(z) \equiv \frac{X_{\alpha}^{(I_\rho,1)}}{z} + \frac{X_{\alpha}^{(I_\rho,2)}}{z-1} = \lim_{x\to 0} \, x^{n_\rho} \mathcal{L}_{\alpha}\left(x^{n_\rho} z + g_\rho(x), z_i\right) \, .$$

Associated Hamiltonians used to extract vertex operators

$$\mathcal{H}^{(p)}_{\rho}(z) = \kappa^{\alpha_1\dots\alpha_p} \mathcal{L}^{\rho}_{\alpha_1}(z)\dots\mathcal{L}^{\rho}_{\alpha_p}(z) + \dots = \sum_{\nu=0}^{p} \frac{\mathcal{D}^{p,\nu}_{\rho,12}}{z^{\nu}(z-1)^{p-\nu}} + \dots$$