BOOTSTRAP THE LATTICE YANG-MILLS THEORY

work with Vladimir Kazakov

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Bootstrap

Basically bootstrap method is solving problems in theoretical physics by optimization theory.

Motivation

- Non-perturbative formulation.
- A competitive numerical method.

Small pieces of Optimization theory

- Quadratic programming:

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & y = x^2 + 3x + 1 \end{aligned} \tag{3}$$

- Linear programming:

$$\begin{aligned} \max \quad & 300x + 100y \\ \text{s.t.} \quad & 6x + 3y \leq 40 \\ & x - 3y \leq 0 \\ & x + \tfrac{1}{4}y \leq 4 \end{aligned} \tag{4}$$

Semi-definite Programming

- Semi-definite Programming:

$$\begin{aligned} \min \quad & 2x + 3y \\ \text{s.t.} \quad & \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0 \end{aligned} \tag{5}$$

- Linear programming and Quadratic programming are special situations of Semi-definite Programming(SDP).
- They all fall into the class of Convex Optimization.
- Generally we cannot solve large-scale non-convex optimization problem (NP hard).

A Simplest bootstrap model

Consider the single-variable integral:

$$Z = \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2} - g\frac{x^4}{4})\mathrm{d}x, \;\; g > 0, \tag{6}$$

We want to compute its k-moment for a given g:

$$\mathcal{W}_k = \frac{1}{Z}\int_{-\infty}^{\infty} x^k \exp(-\frac{x^2}{2} - g\frac{x^4}{4})\mathrm{d}x \tag{7}$$

We have a lot of choices to do the integration!

Loop equations and global symmetries

Loop equations are Dyson-Schwinger equations. They can be derived by make the variable translation $x \to x + \epsilon$ or in our model by integration by part:

$$(k+1)\mathcal{W}_k = \mathcal{W}_{k+2} + g\mathcal{W}_{k+4} \tag{8}$$

Global symmetry:

$$\mathcal{W}_k = 0, \text{ for odd } k \tag{9}$$

The conclusion is all the k-moments are linear functions of $\mathcal{W}_2$, so correlation matrix is a linear function of $\mathcal{W}_2$.

Correlation matrix

The bootstrap method is that considering the expectations of square of polynomials are always positive semi-definite:

$$\frac{1}{Z}\int_{-\infty}^{\infty}(\sum \alpha_i x^i)^2 \exp(-\frac{x^2}{2} - g\frac{x^4}{4}) \geq 0,\ \forall \alpha \tag{10}$$

This is a quadratic form in α, its positivity is equivalent to:

$$\mathbb{W} = \begin{pmatrix} \mathcal{W}_0 & \mathcal{W}_1 & \mathcal{W}_2 & \dots \\ \mathcal{W}_1 & \mathcal{W}_2 & \mathcal{W}_3 & \dots \\ \mathcal{W}_2 & \mathcal{W}_3 & \mathcal{W}_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \succeq 0 \tag{11}$$

This condition will be referred as the positivity of **correlation matrix**.

Bootstrap

We can solve the Semi-Definite Programming(SDP) maximizing or minimizing $\mathcal{W}_2$ constrained by a truncation of the positivity of correlation matrix:

$$\textsf{min or max } \mathcal{W}_2 \tag{12}$$

$$\mathbb{W}_\Lambda \succeq 0 \tag{13}$$

Here $\mathbb{W}_\Lambda$ is the top $(\Lambda + 1) \times (\Lambda + 1)$ sub-matrix of $\mathbb{W}$.

Result

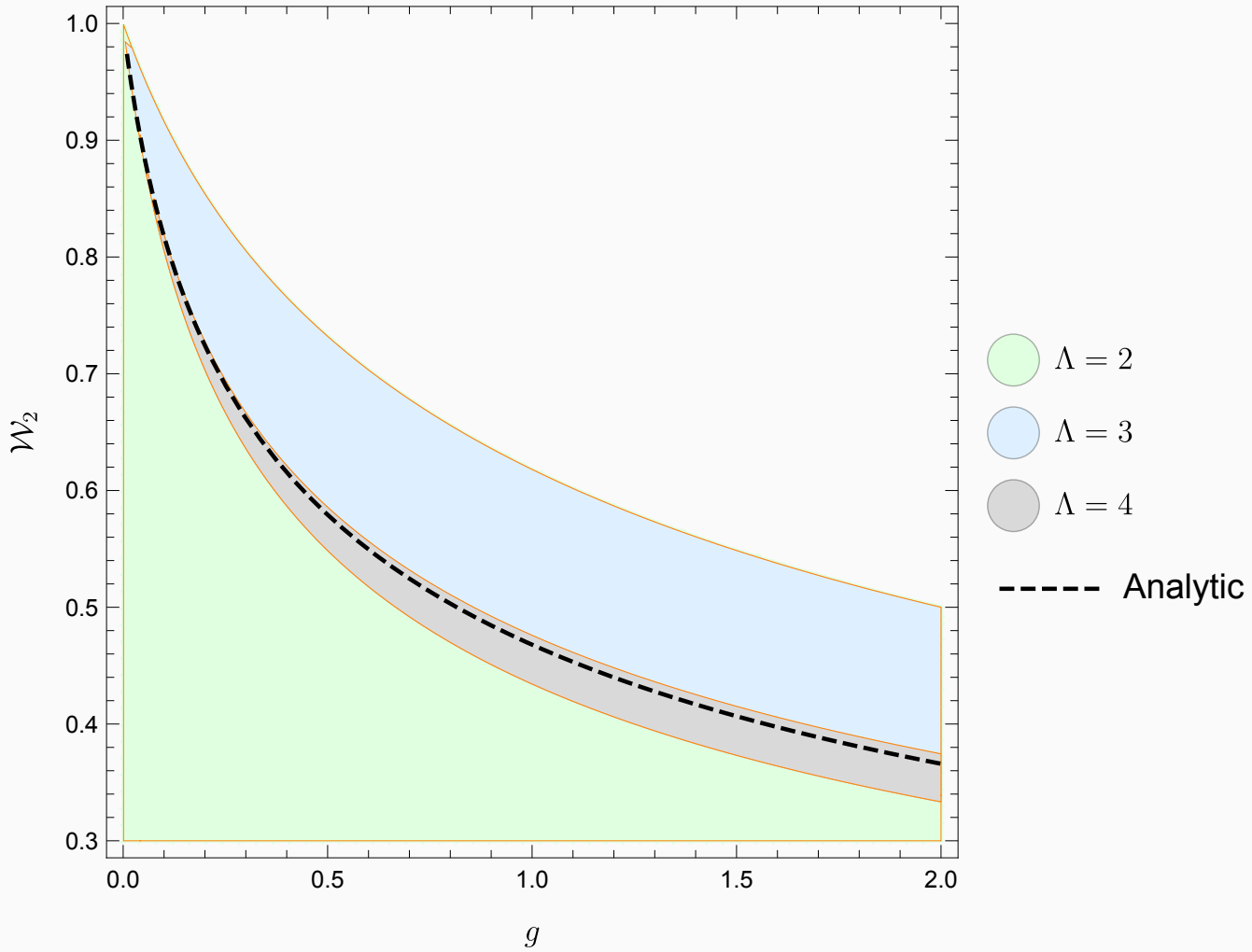
Analytic result:

$$\mathcal{W}_2 = \frac{\pi\left(-I_{-\frac{1}{4}}\left(\frac{1}{8g}\right) + (4g+1)I_{\frac{1}{4}}\left(\frac{1}{8g}\right) - I_{\frac{3}{4}}\left(\frac{1}{8g}\right) + I_{\frac{5}{4}}\left(\frac{1}{8g}\right)\right)}{2\sqrt{2}gK_{\frac{1}{4}}\left(\frac{1}{8g}\right)} \tag{14}$$

For $g = 1$, $\Lambda = 10$, we can get the numerical bootstrap result:

$$0.4679137 \leq \mathcal{W}_2 = 0.4679199170 \leq 0.4679214 \tag{15}$$

Result


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
ν_2
0.0
0.5
1.0
1.5
2.0
g
$\Lambda = 2$
$\Lambda = 3$
$\Lambda = 4$
Analytic

Remarks

- Weak coupling perturbation totally fails at $g = 1$.
- Justification/Convergence
- Uniqueness/Generalization
- Symmetry: Not necessary but efficient If $\mathcal{W}_1 \neq 0$,

$$-2.9968 \times 10^{-6} \leq \mathcal{W}_1 \leq 2.9968 \times 10^{-6} \tag{16}$$

Remarks

- Symmetry: block-diagonal through re-arrangement of the vector:

$$
\begin{array}{c}
\\
1 \\
x^2 \\
x^4 \\
\vdots \\
x \\
x^3 \\
\vdots
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & x^2 & x^4 & \dots & x & x^3 & \dots \end{array} \\
\begin{pmatrix}
1 & \mathcal{W}_2 & \mathcal{W}_4 & & & & \\
\mathcal{W}_2 & \mathcal{W}_4 & \mathcal{W}_6 & \dots & & 0 & \\
\mathcal{W}_4 & \mathcal{W}_6 & \mathcal{W}_8 & & & & \\
 & \vdots & & \ddots & & & \\
 & & & & \mathcal{W}_2 & \mathcal{W}_4 & \dots \\
 & 0 & & & \mathcal{W}_4 & \mathcal{W}_6 & \\
 & & & & \vdots & & \ddots
\end{pmatrix}
\end{array}
\succeq 0.
$$

Multi-matrix bootstrap: an example

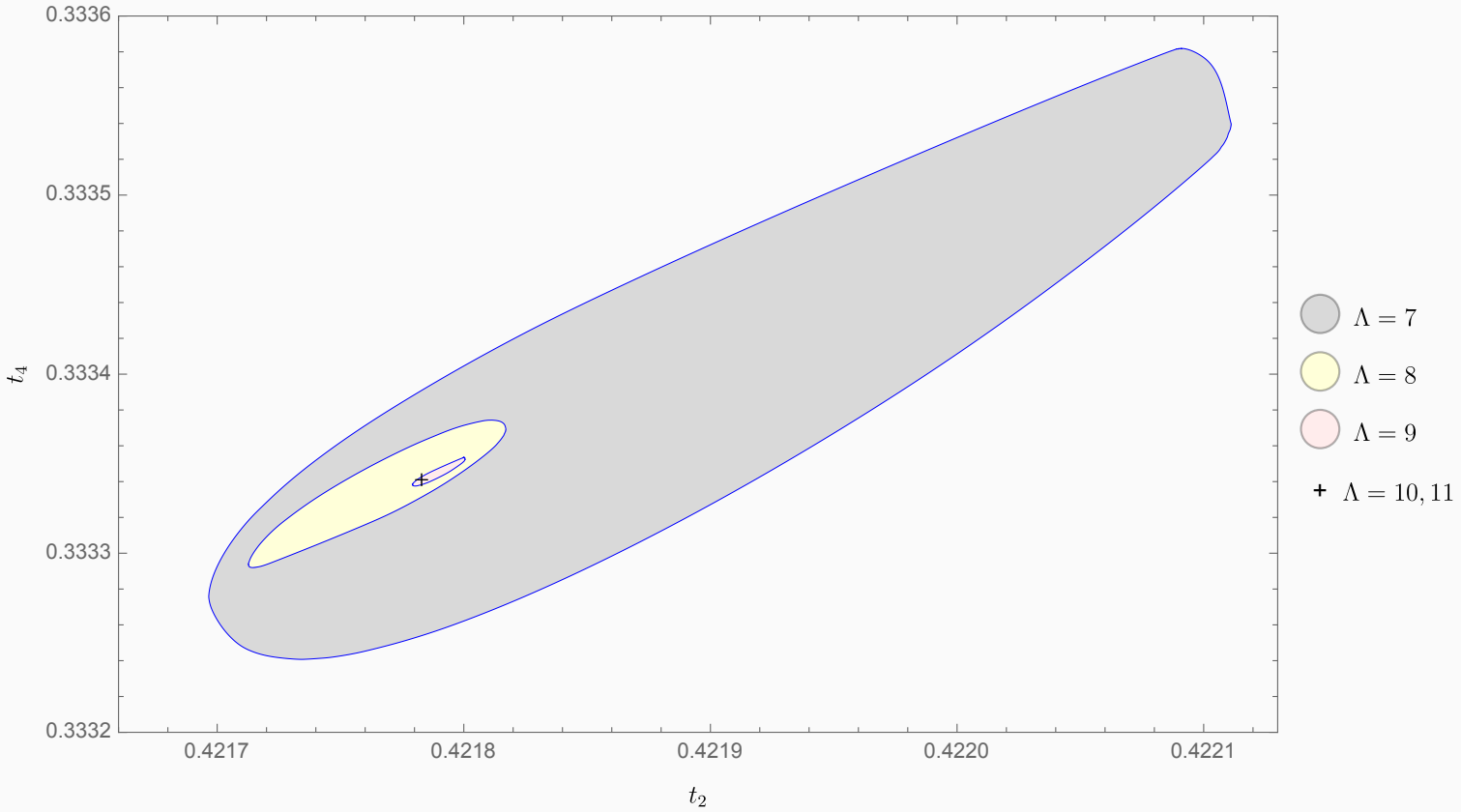
Here we propose to study the following two-matrix model:

$$Z = \lim_{N\to\infty} \int dA^{N^2}\, dB^{N^2}\, \mathrm{e}^{-N\mathrm{tr}\left(-h[A,B]^2/2+A^2/2+gA^4/4+B^2/2+gB^4/4\right)} \tag{17}$$

The integration is over Hermitian matrix. To the best of our knowledge, this model with general g and h value, is not solvable!

$$\mathrm{Tr}A^2,\ \mathrm{Tr}A^4,\ \mathrm{Tr}A^2B^2,\ \mathrm{Tr}ABAB,\ \mathrm{Tr}A^6,\ \mathrm{Tr}A^4B^2,\ \mathrm{Tr}A^3BAB,\ \mathrm{Tr}A^2BA^2B,\ \mathrm{Tr}A^8,$$
$$\mathrm{Tr}A^6B^2,\ \mathrm{Tr}A^5BAB,\ \mathrm{Tr}A^4BA^2B,\ \mathrm{Tr}A^4B^4,\ \mathrm{Tr}A^3BA^3B,\ \mathrm{Tr}A^3BAB^3,\ \mathrm{Tr}A^3B^2AB^2,$$
$$\mathrm{Tr}A^2BABAB^2,\ \mathrm{Tr}A^2BAB^2AB,\ \mathrm{Tr}A^2B^2A^2B^2,\ \mathrm{Tr}ABABABAB\ \ldots \tag{18}$$

Result



$$\Lambda = 11,\ g = h = 1 : \begin{cases} 0.421783612 \leq \langle \mathrm{Tr} A^2 \rangle \leq 0.421784687 \\ 0.333341358 \leq \langle \mathrm{Tr} A^4 \rangle \leq 0.333342131 \end{cases} \tag{19}$$

Compare with MC

Compared to the MC study of the same model[Jha, 2021], we are convinced that for this model bootstrap is at least two order of magnitude more efficient than MC.

- MC: 80-85 hours for N=800 simulation to get 4.5 digits.
- Bootstrap: $\sim$ 40 hours to get 6 digits. (These are old results and can be greatly improved by at least one order of magnitude.)

Lattice Gauge theory

We are going to bootstrap the large N_c limit of the following theory:

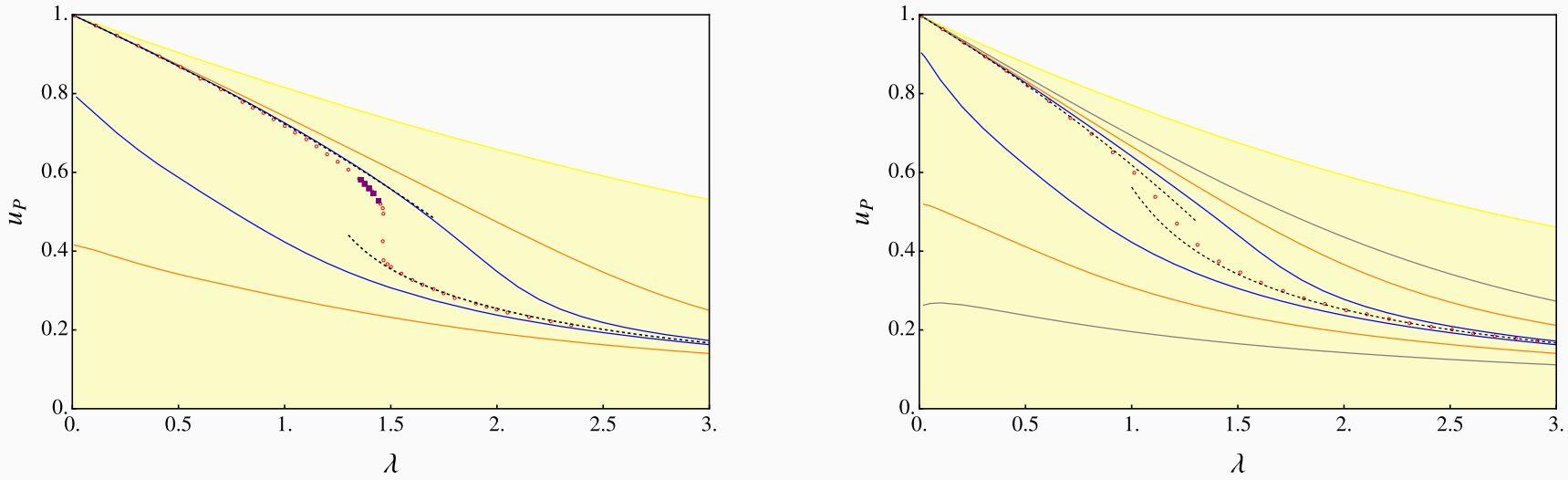
$$Z = \int \prod_{x,\,\mu} \mathrm{d}U_\mu(x)\, \exp(-S) \tag{20}$$

$$S = -\frac{N_c}{2\lambda} \sum_P \mathrm{Re}\,\mathrm{tr} U_P \tag{21}$$

where U_P is the product of four unitary link variables around the plaquette P and we sum up over all plaquettes P, including both orientations. In our last work we bootstrap the one plaquette average:

$$u_P = \frac{1}{N_c}\langle \mathrm{tr} U_P \rangle \tag{22}$$

Result



(a) 4D

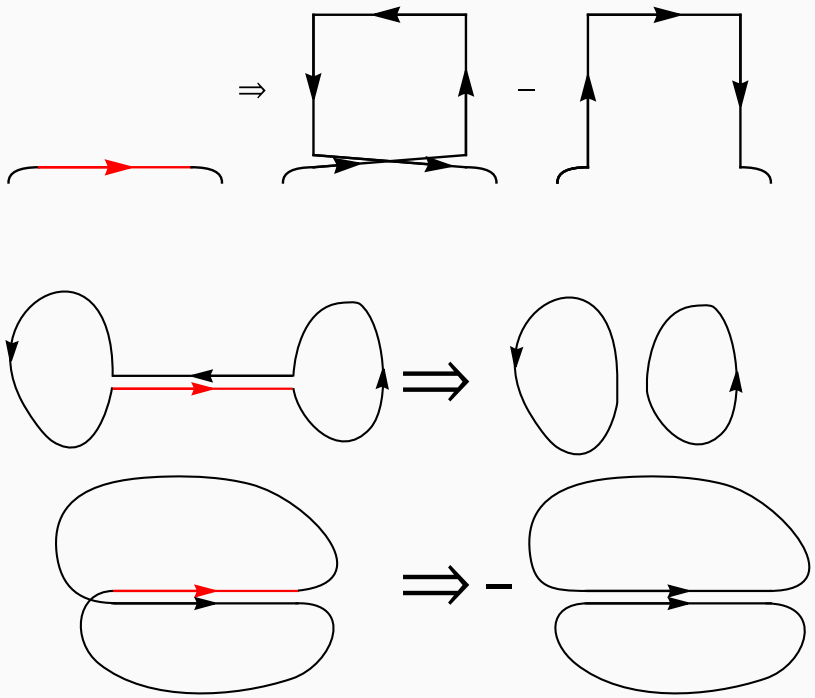
(b) 3D

Figure: Our bootstrap results for plaquette average in 4*D* and 3*D* LGT: upper bounds at $L_{\max} = 8$ (yellow domain) at $L_{\max} = 12$ (orange curves) and $L_{\max} = 16$ (blue curves). The red circles represent the MC data for $SU(10)$ LGT (with 5 purple squares for $SU(12)$). Dashed upper and lower lines represent the 3-loop PT and strong coupling expansion, respectively.

Makeenko-Migdal loop equations

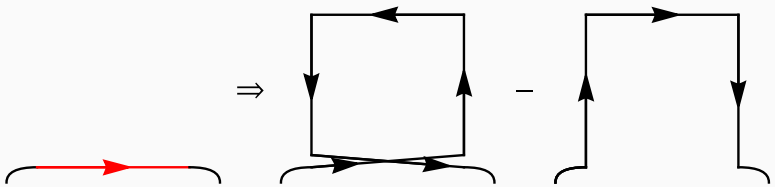
Doing the following infinitesimal transformation $U_\mu(x) \to U_\mu(x)(1 + i\epsilon)$ to the Wilson loop $\mathcal{W}[C]$, we can get the following loop equations schematically:

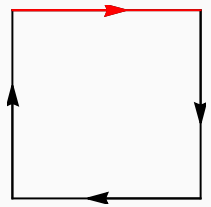
$$(\text{linear}) + 2\lambda\mathcal{W}[C] = 2\lambda(\text{nonlinear}) \tag{23}$$

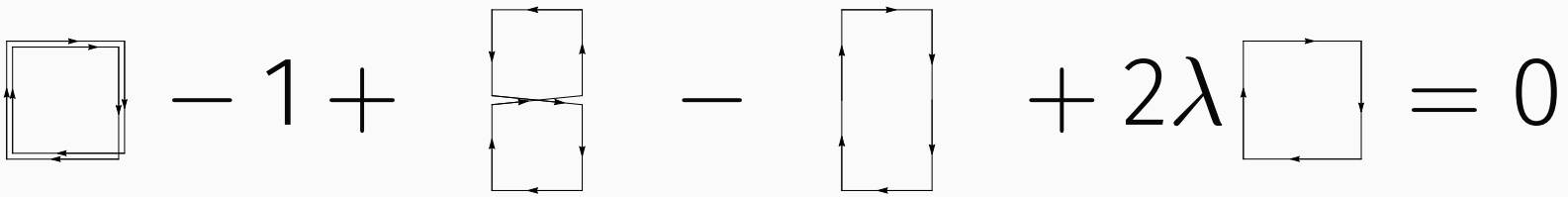


Makeenko-Migdal loop equations

$$(\text{linear}) + 2\lambda \mathcal{W}[C] = 2\lambda(\text{nonlinear}) \tag{24}$$

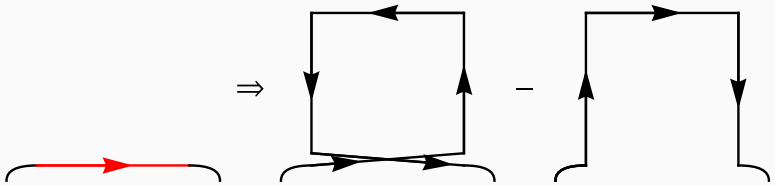


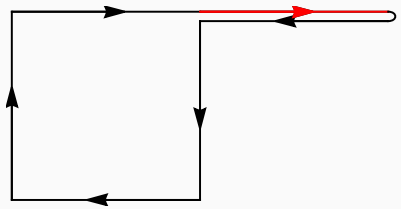


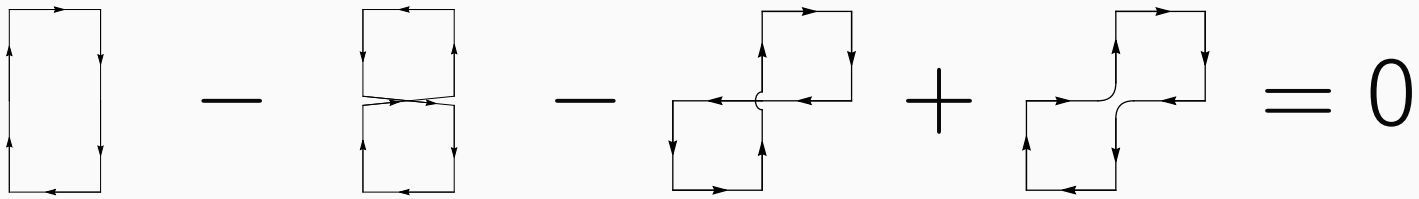


Makeenko-Migdal loop equations

$$(\text{linear}) + 2\lambda \mathcal{W}[C] = 2\lambda(\text{nonlinear}) \tag{25}$$

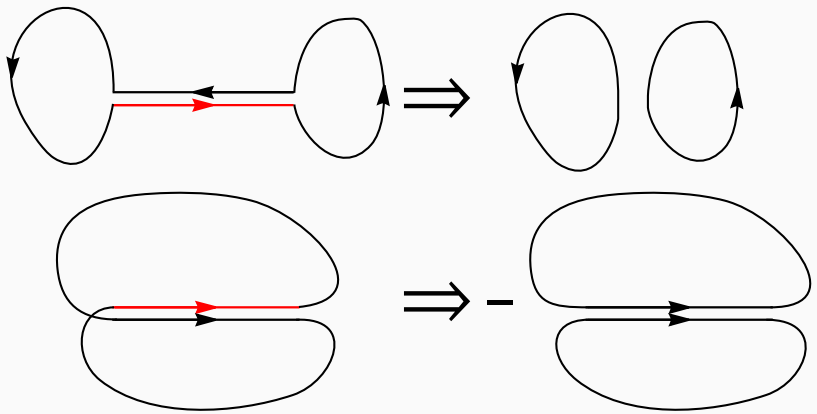


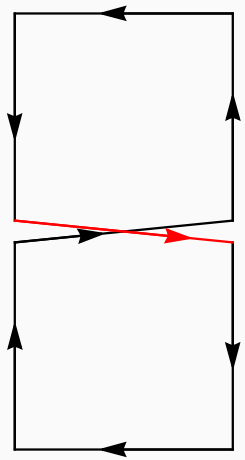




Makeenko-Migdal loop equations

$$(\text{linear}) + 2\lambda \mathcal{W}[C] = 2\lambda(\text{nonlinear}) \tag{26}$$





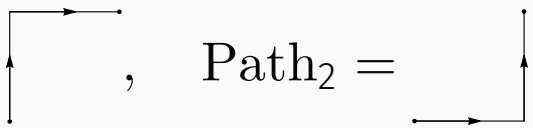
Positivity by Hermitian Conjugation

In parallel to the bootstrap for Hermitian matrix model, we have:

$$\mathrm{Path}^{*\mathrm{T}} = \mathrm{Reverse} \circ \mathrm{Path} \tag{27}$$

For a simplest example:

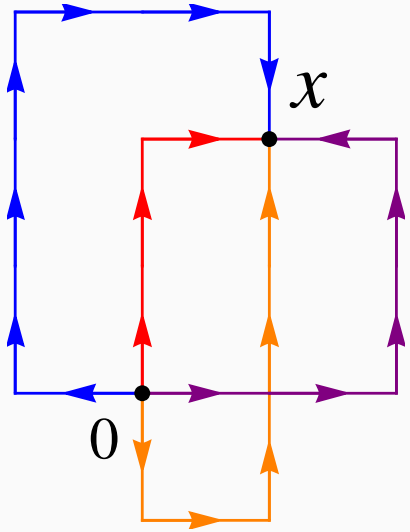
$$\mathrm{Path}_1 = \quad , \quad \mathrm{Path}_2 = \tag{28}$$

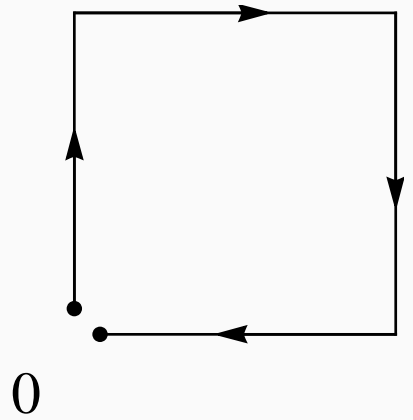


$$\begin{array}{c} \\ \mathrm{Path}_1^\dagger \\ \mathrm{Path}_2^\dagger \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{Path}_1 & \mathrm{Path}_2 \end{array} \\ \begin{pmatrix} 1 & u_P \\ u_P & 1 \end{pmatrix} \end{array} \succeq 0. \tag{29}$$

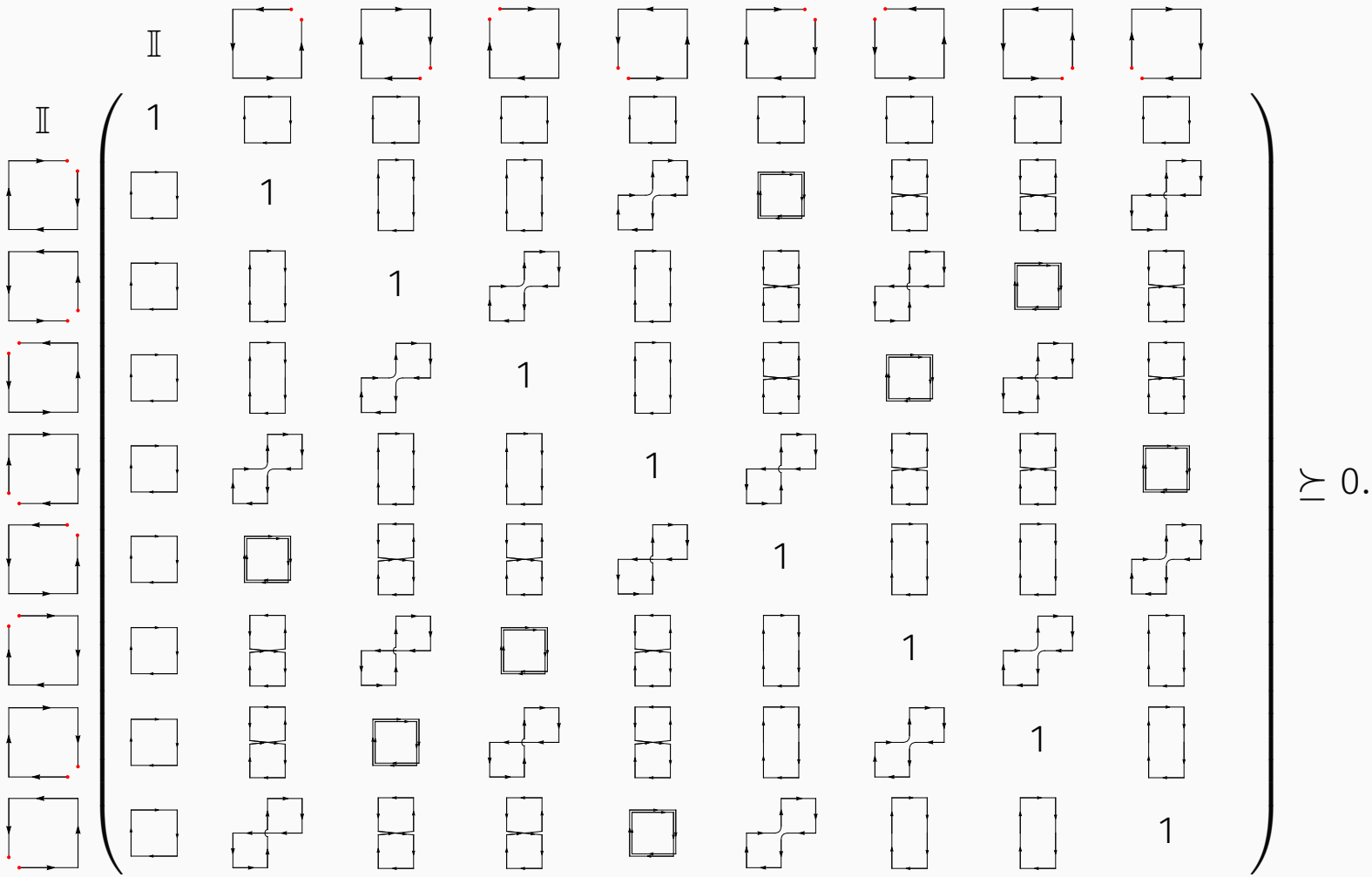
Positivity by Hermitian Conjugation

Of course the matrix can be arbitrarily big when we consider multiple Wilson paths:



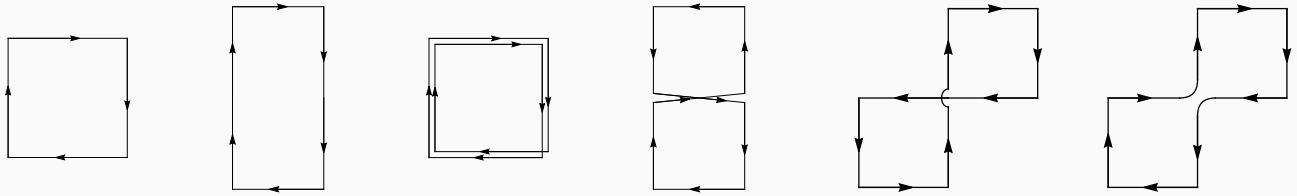


Positivity by Hermitian Conjugation



Bootstrap

There are actually 6 Wilson loops in the matrix:



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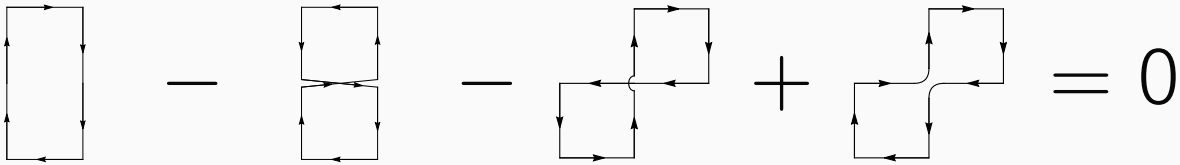
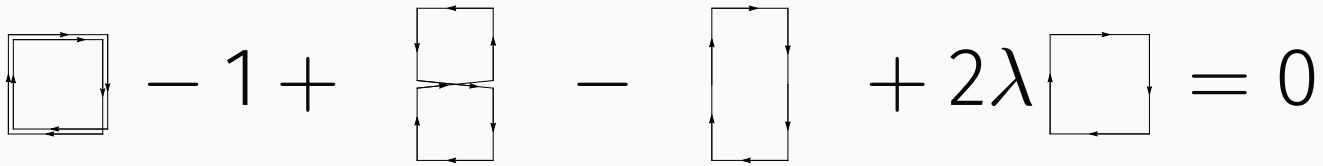




After the optimization, we get ($\lambda = 1$):

$$0 \leq \square \leq 0.69300$$

Positivity by inner product

Generalization: Any inner products defined on the vector space of operators or its subspace could leads to positivity condition:

$$\langle \mathcal{O} | \mathcal{O} \rangle = \langle \mathcal{O}^\dagger \mathcal{O} \rangle = \alpha^{*\mathrm{T}} \mathcal{M} \alpha \geq 0 \Leftrightarrow \mathcal{M} \succeq 0. \tag{31}$$

In the above case of single-variable integration and Hermitian matrix integration, we were taking adjoint to be Hermitian conjugation:

$$\mathcal{O}^\dagger = \mathcal{O}^{*\mathrm{T}} \tag{32}$$

Reflection Positivity

We can also define the inner product by reflection positivity:

$$\mathcal{O}^\dagger = \Theta\mathcal{O} \tag{33}$$

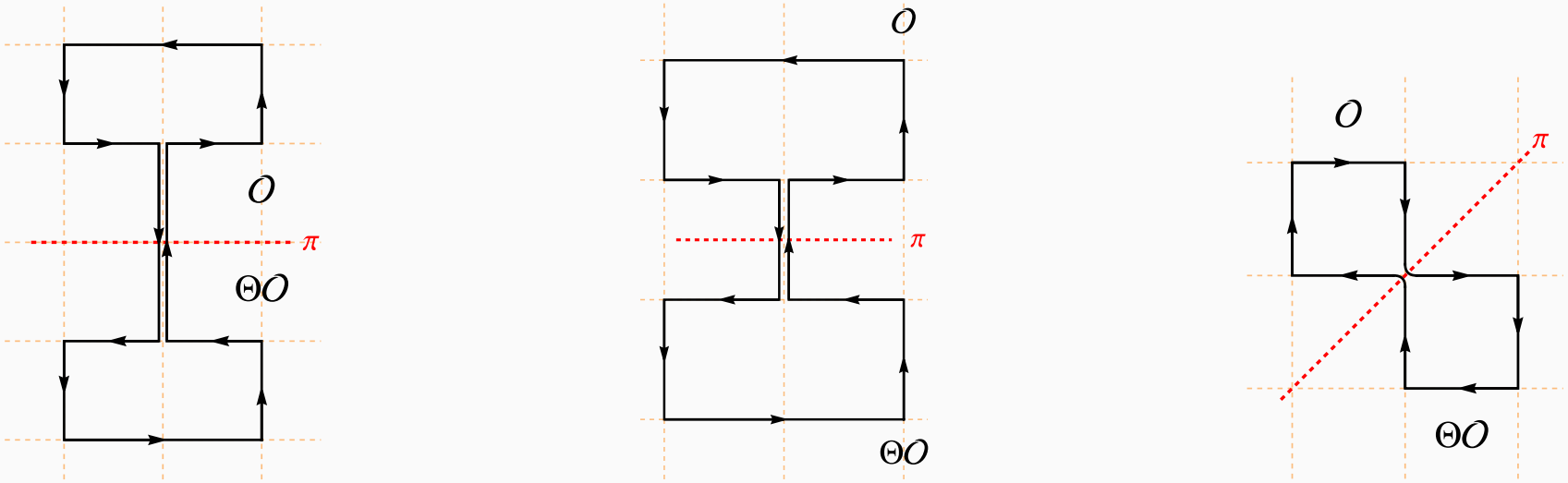
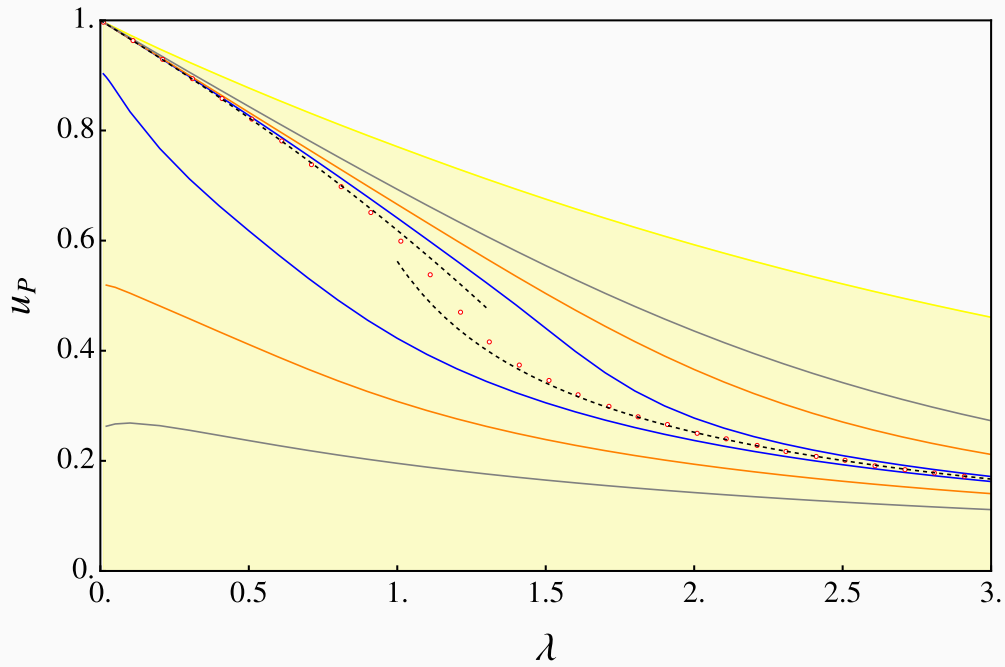


Figure: Three reflection symmetries on the lattice allowing new positivity conditions on Wilson loops combining the original and reflected Wilson lines.

Reflection Positivity

Reflection Positivity is a new independent positivity condition (Gray curve and Orange curve).



Symmetry reduction

Coming from the matrix of inner products, our positivity conditions is formally similar to S-matrix. It is a well-known fact that we can decompose the S-matrix w.r.t the spin channel.

We notice that our inner product defined above is invariant under some symmetry group.

$$\langle (g \circ \mathcal{O}_1) | (g \circ \mathcal{O}_2) \rangle = \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle, \, \forall g \in G \tag{34}$$

We can decompose the positivity conditions w.r.t the irreducible representation of the symmetry group. (This is mathematically guaranteed)

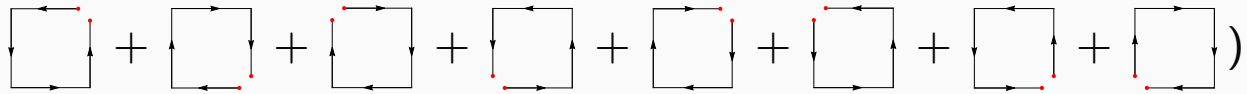
Symmetry Group

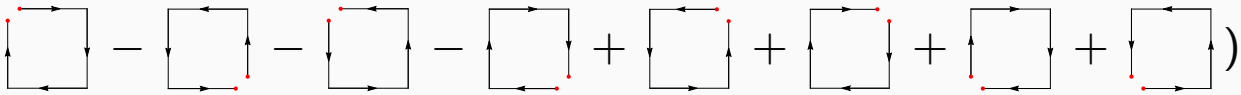
For the correlation matrix with $0 \to 0$, the invariant group G is $B_d \times \mathbb{Z}_2$. Here B_d is the Hyperoctahedral group in D spacetime dimensions. It acts on the Wilson path by doing the corresponding spacetime rotation and reflection on the lattice. $\mathbb{Z}_2$ is the group reversing the path.

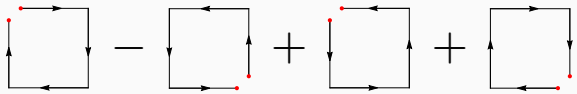
Dimension	Hermitian Conjugation	site&link reflection	diagonal reflection
2	$B_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	$B_3 \times \mathbb{Z}_2$	$B_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$
4	$B_4 \times \mathbb{Z}_2$	$B_3 \times \mathbb{Z}_2$	$B_2 \times \mathbb{Z}_2^2$

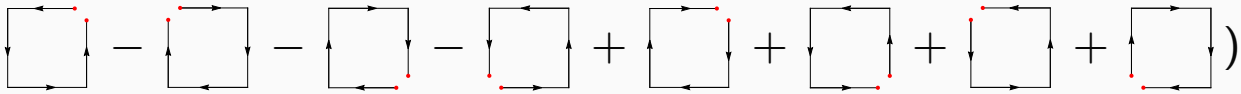
Table: Invariant groups of correlation and reflection matrices $0 \to 0$

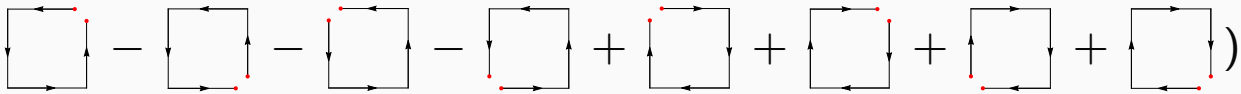
Symmetry Group

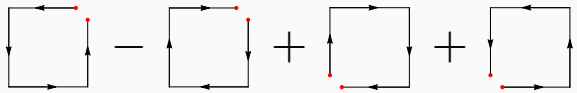
$(A_1, +1) : \mathbb{I},$ 

$(B_2, +1) :$ 

$(E, +1) :$ 

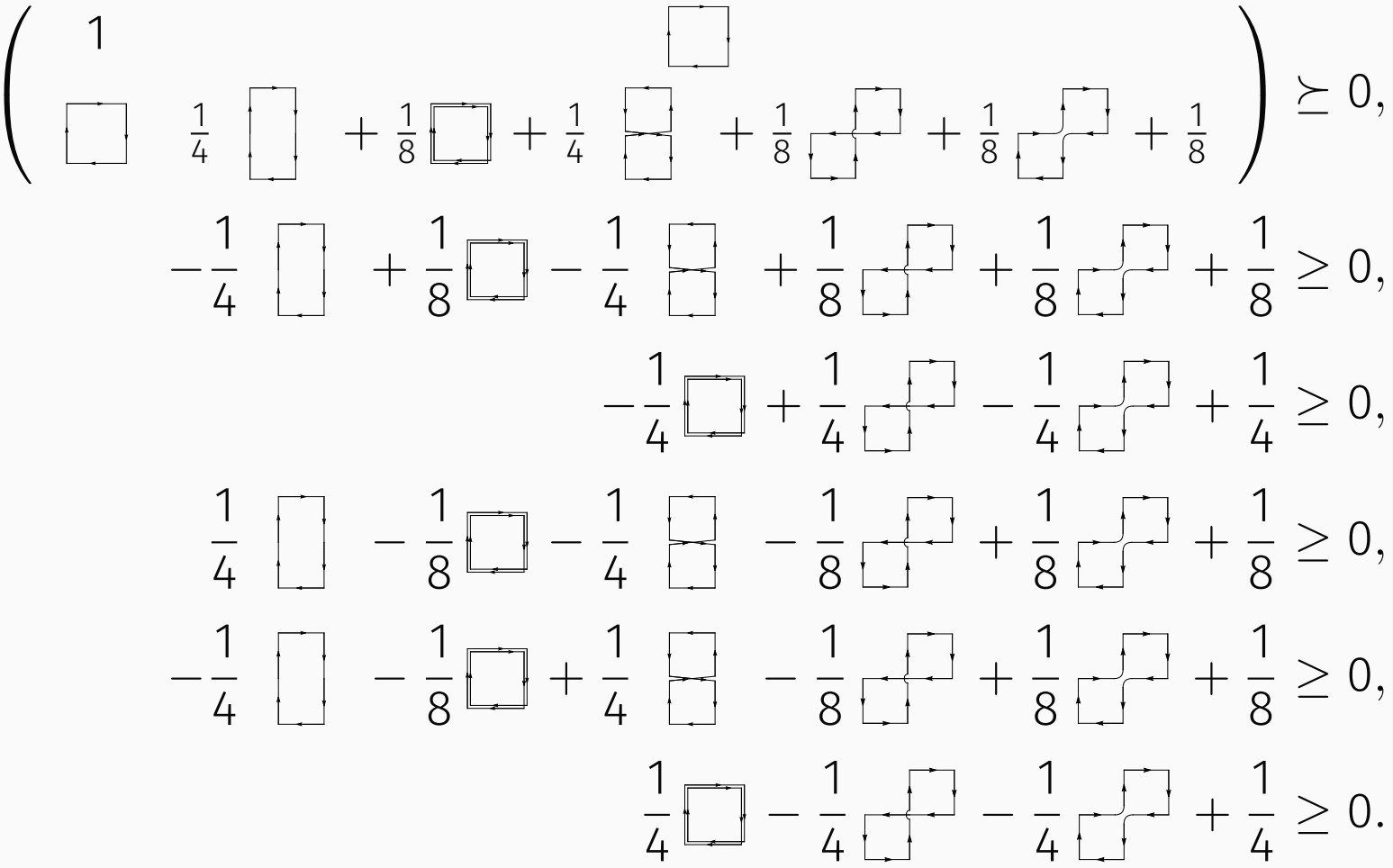
$(B_1, -1) :$ 

$(A_2, -1) :$ 

$(E, -1) :$ 

(35)

Symmetry Group



(36)

Selection of multiplets of Wilson path

We discard positivity constraints from less important Wilson Paths.

To better illustrate the efficiency of these reduction and selection techniques, take an example of the correlation matrix for the paths $0 \to 0$, at $3D$ and $L_{\max} = 16$: it has a huge size 6505×6505. After the symmetry reduction and truncation of the multiplets, the positivity of the correlation matrix becomes the positivity conditions of 20 smaller matrices [1], each with size:

$$\begin{aligned} &38, 15, 25, 18, 62, 33, 68, 75, 56, 78, \\ &22, 18, 34, 15, 56, 33, 57, 76, 69, 73 \end{aligned} \tag{37}$$

So the SDP gets greatly simplified.

[1] The invariant group $B_3 \times \mathbb{Z}_2$ has 20 irreducible representations.

Relaxation

Most naively we treat all the quadratic terms as independent variables, and eliminate them from the loop equations. Actually we have better choice.

Relaxation

Suppose we have only three quadratic "loop equations":

$$\begin{cases} x^2 = T_1 \\ y^2 = T_2 \\ xy = T_3 \end{cases} \tag{38}$$

Here $T_i (i = 1, 2, 3)$ is the new variable we introduce to replace the quadratic terms in the loop equations.

We can relax them to make them convex by replacing $x^2 = T_1$ with $x^2 \leq T_1$ or, in the positive semi-definite matrix form,

$$\begin{pmatrix} 1 & x \\ x & T_1 \end{pmatrix} \succeq 0 \,. \tag{39}$$

Relaxation

But the same operation cannot be reproduced for equation $xy = T_3$, since neither $xy \leq T_3$ nor $xy \geq T_3$ is convex [2]. It is tempting to consider the positive semi-definite combinations:

$$(x + \alpha y)^2 \leq T_1 + \alpha^2 T_2 + 2\alpha T_3, \forall \alpha \in \mathbb{R}\,. \tag{40}$$

In its turn, it is equivalent to:

$$\mathrm{Det} \begin{pmatrix} 1 & x & y \\ x & T_1 & T_3 \\ y & T_3 & T_2 \end{pmatrix} \geq 0. \tag{41}$$

[2] Because the bilinear form xy is not positive semi-definite.

Relaxation

We come to the conclusion that:

$$\begin{pmatrix} 1 & x & y \\ x & T_1 & T_3 \\ y & T_3 & T_2 \end{pmatrix} \succeq 0. \tag{42}$$

Our general strategy: we treat the quadratic terms in the loop equations as independent variable, and replace the algebraic equality by the convex inequality:

$$X = xx^{\mathrm{T}} \tag{43}$$

to:

$$\mathcal{R} = \begin{pmatrix} 1 & x^{\mathrm{T}} \\ x & X \end{pmatrix} \succeq 0. \tag{44}$$

Relaxation

In our example, at $L_{\max} = 12$, the relaxation matrix is:

$$\begin{pmatrix} 1 & \square \\ \square & q \end{pmatrix} \succeq 0. \tag{45}$$

Here q is the variable in place of square of $\square$.

Final scheme

$$
\begin{aligned}
\text{min / max} \quad & \square, \\
\text{subject to} \quad & \text{MM loop equations} \\
& \text{CM}^{\text{irrep}} \succeq 0, \\
& \text{RefM}^{\text{irrep}} \succeq 0, \times 3 \\
& \mathcal{R} \succeq 0
\end{aligned}
\tag{46}
$$

Result

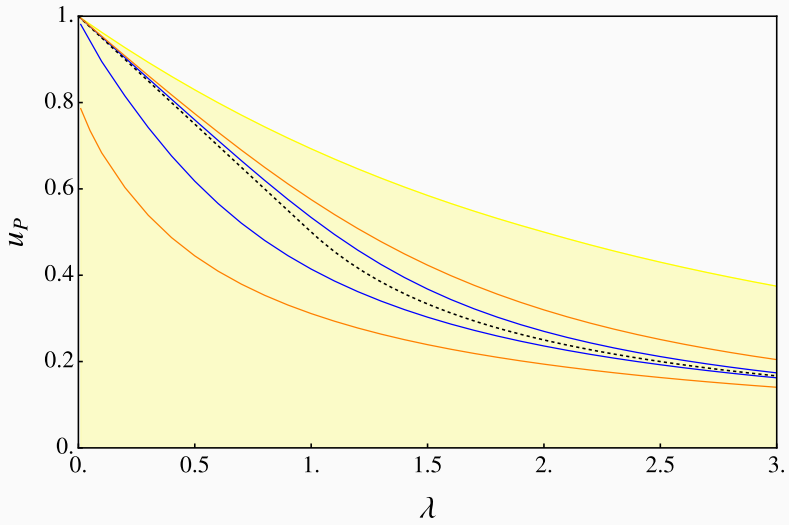
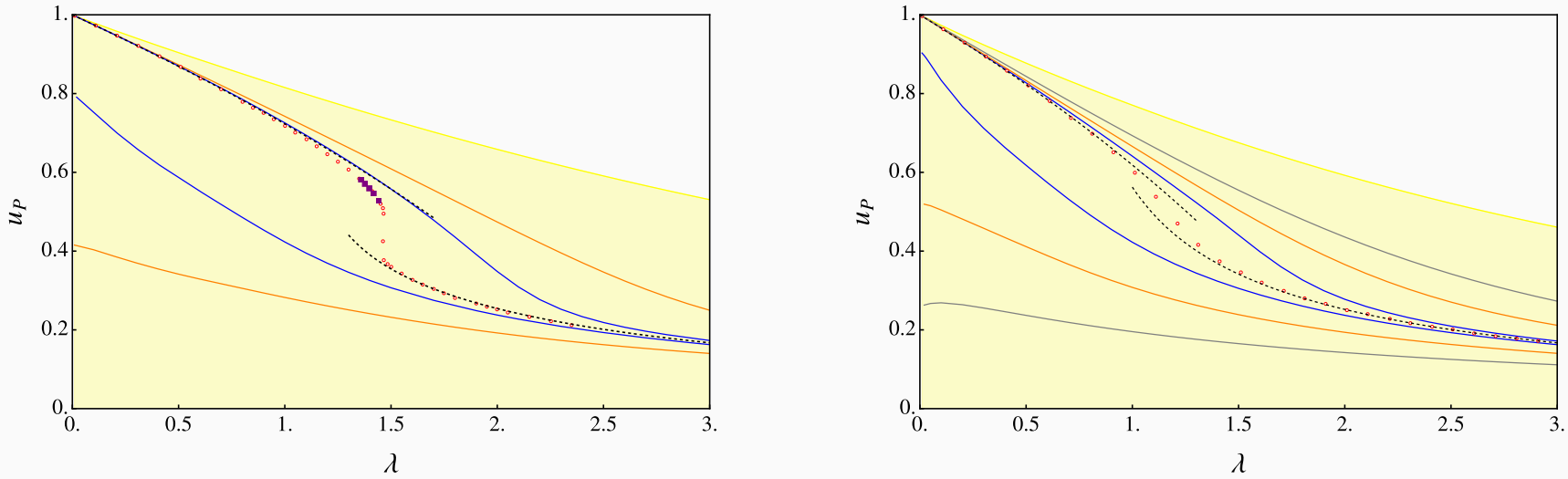


Figure: 2D: the upper and lower bounds from our bootstrap at $L_{\max} = 8$ (yellow region), $L_{\max} = 12$ (orange curves) and $L_{\max} = 16$ (blue curves). The dashed line is the exact solution.

$$u_P = \begin{cases} 1 - \frac{\lambda}{2}, & \text{for} \lambda \leq 1 \\ \frac{1}{2\lambda}, & \text{for} \lambda \geq 1 \end{cases} \tag{47}$$

Result



(a) 4D

(b) 3D

Figure: Our bootstrap results for plaquette average in 4*D* and 3*D* LGT: upper bounds at $L_{\max} = 8$ (yellow domain) at $L_{\max} = 12$ (orange curves) and $L_{\max} = 16$ (blue curves). The red circles represent the MC data for $SU(10)$ LGT (with 5 purple squares for $SU(12)$). Dashed upper and lower lines represent the 3-loop PT and strong coupling expansion, respectively.

Some remarks

- MM loop equation is over-determined, i.e. contains linear redundancy!
- Number of independent MM loop equations is smaller than the number of Wilson loops.
- Wilson loop average is real. This is from the charge conjugation symmetry. (Path reversing symmetry)

Some remarks

- Better than MC in weak coupling limit, which is physical.
- Is perfectible. We expect much better result at $L = 24$.
- Not limited to large N, i.e. could be straightforwardly generalized to $N = 3$.
- We can get estimation for all higher Wilson loops up to bootstrap cutoff.
- Solver: MOSEK. 20 hours CPU time for a single point.
- We are not far from real physics!

Questions?

Reference

Jha, R. G. (2021).
Introduction to Monte Carlo for Matrix Models.