Ballistic macroscopic fluctuation theory for Integrable systems

Takato Yoshimura (University of Oxford)

IGST2022, Budapest

Based on arXiv: 2206.14167 with B. Doyon, G. Perfetto, and T. Sasamoto

All Souls College
University of Oxford

Motivation

We are interested in evaluating dynamical correlation functions for a one-dimensional translation invariant Hamiltonian system in the infinite volume

$$\langle \hat{o}_1(x_1, t_1)\hat{o}_2(x_2, t_2)\cdots\hat{o}_n(x_n, t_n)\rangle_\ell^{\mathrm{c}}$$

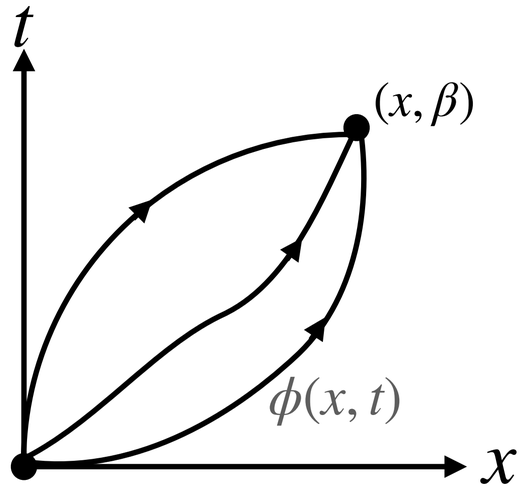
with respect to a local thermal equilibrium (or GGE)

$$\langle \bullet \rangle_\ell = \frac{1}{Z}\mathrm{Tr}\left[\exp\left[-\int_{\mathbb{R}} \mathrm{d}x\, \beta(x/\ell)\hat{h}(x,0)\right] \bullet\right]$$

when x_i, t_i are very large for $i = 1, \cdots, n$ but the ratio x_i/t_i is kept finite (**Euler** scaling)

In terms of the Feynman path integral

$$\langle \hat{o}_1(x_1,t_1)\hat{o}_2(x_2,t_2)\cdots\hat{o}_n(x_n,t_n)\rangle_{\text{thermal}}$$
$$= \int \mathscr{D}\phi\, e^{-\int_0^\beta \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x\, \mathscr{L}_{\mathrm{E}}[\phi(x,t)]}\hat{o}_1(x_1,t_1)\hat{o}_2(x_2,t_2)\cdots\hat{o}_n(x_n,t_n)$$



This is rather hard to evaluate! Could there be a more efficient way to compute it?

In fact, on the hydrodynamic (large) scale, we don't have to know all the possible trajectories of ϕ.

Relevant degrees of freedom on the hydro scale: those protected by **conservation laws**

We devise a theory, which we term the **ballistic macroscopic fluctuation theory** (BMFT), that provides a more efficient path-integral formula based on the reduction of DOFs.

Hydrodynamics and local relaxation

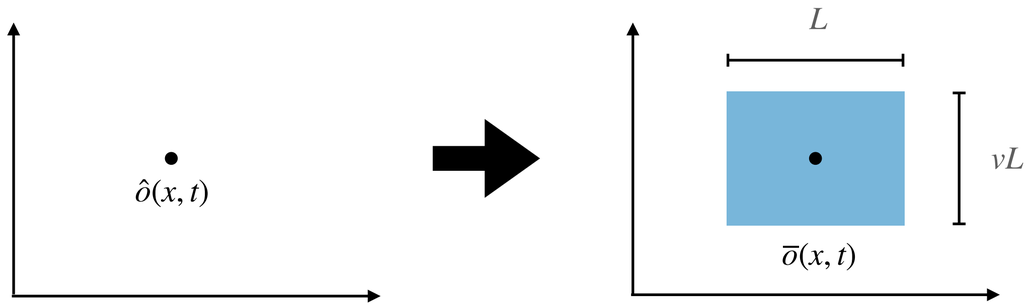
We consider a translation invariant many-body system with N conservation laws

Suppose the initial condition is given by some local (generalised) Gibbs ensemble with the statistical average

$$\langle \bullet \rangle_\ell = \frac{1}{Z}\mathrm{Tr}\left[\exp\left[-\sum_{i=0}^{N-1}\int_{\mathbb{R}} \mathrm{d}x\, \beta^i(x/\ell)\hat{q}_i(x,0)\right]\bullet\right]$$

We define the space-time averaged mesoscopic observables via **coarse-graining**

$$\overline{o}(x,t) = \frac{1}{vL^2}\int_{-L/2}^{L/2}\mathrm{d}y\int_{-vL/2}^{vL/2}\mathrm{d}s\,\hat{o}(x+y,t+s)$$



Separation of scales: $\ell_{\mathrm{micro}} \ll L \ll \ell$

The **key** observables in the BMFT are

$$o(x,t) = \overline{o}(\underbrace{\ell x, \ell t}_{\text{Euler scaling}})$$

We regard them as **fluctuating mesoscopic** variables. Their fluctuations are set by the initial condition

Hydrodynamics predicts

$$\mathrm{o}(x,t) := \lim_{\ell\to\infty} \langle \overline{o}(\ell x, \ell t)\rangle_\ell = \langle \hat{o}\rangle_{\underline{\beta}(x,t)}, \quad \langle \bullet \rangle_{\underline{\beta}} := \frac{1}{Z}\mathrm{Tr}\left[e^{-\sum_{i=0}^{N-1}\beta^i Q_i}\bullet\right]$$

On the Euler scale, the average of any local operator can be thought of as a functional of $\underline{\mathrm{q}}(x,t) = \langle \hat{q}\rangle_{\underline{\beta}(x,t)}$, i.e.

$$\mathrm{o}(x,t) = \langle \hat{o}\rangle[\underline{\mathrm{q}}(x,t)] = \mathrm{o}[\underline{\mathrm{q}}(x,t)]$$

Local relaxation of averages

Hydrodynamics describes the time-evolution of the **means** of fluctuating variables via the hydrodynamic equation $\partial_t \mathsf{q}_i(x,t) + \partial_x \mathsf{j}_i(x,t) = 0$, or equivalently,

$$\partial_t \mathsf{q}_i(x,t) + \mathsf{A}_i^{\ j}[\underline{\mathsf{q}}(x,t)]\partial_x \mathsf{q}_j(x,t) = 0, \quad \mathsf{A}_i^{\ j}[\underline{\mathsf{q}}] := \frac{\partial \mathsf{j}_j[\underline{\mathsf{q}}]}{\partial \mathsf{q}_i} = \frac{\partial \mathsf{j}_j}{\partial \beta^k}\mathsf{C}^{ki}$$

Here $\mathsf{C}^{ki} = (\mathsf{C}^{-1})_{ki}$ with the susceptibility matrix $\mathsf{C}_{ij} = -\frac{\partial \mathsf{q}_j}{\partial \beta^i} = \int_{\mathbb{R}} \mathrm{d}x \, \langle \hat{q}_i(x,0)\hat{q}_j(0,0)\rangle^{\mathrm{c}}_{\underline{\beta}}$

One can also write down a hydro equation for the Lagrange multipliers

$$\partial_t \beta^i(x,t) + \mathsf{A}_j^{\ i}[\underline{\beta}(x,t)]\partial_x \beta^j(x,t) = 0$$

Positivity of C implies a bijection $\underline{q} \leftrightarrow \underline{\beta}$

How would then the fluctuations propagate in space-time? Or more precisely, according to which **measure** would $o(x,t)$ fluctuate?

Initial fluctuations

To consider the propagation of fluctuations, we start with the fluctuations of the **initial condition**

We are interested in the fluctuations of the mesoscopic variables $q_i(x,0) = \overline{q}_i(\ell x,0)$. Its correlations can be obtained from a measure $\mathrm{d}\mathbb{P}_{\mathrm{ini}}[\underline{q}(\cdot,0)] = \mathrm{d}\mu[\underline{q}(\cdot,0)]\ \exp(-\ell\mathcal{F}[\underline{q}(\cdot,0)])/Z_\ell$ with

$$\mathcal{F}[\underline{q}(\cdot,0)] = \int_{\mathbb{R}} \mathrm{d}x \left(\beta^i_{\mathrm{ini}}(x)(q_i(x,0) - \mathrm{q}_{\mathrm{ini},i}(x)) + s[\underline{\mathrm{q}_{\mathrm{ini}}}(x)] - s[\underline{q}(x,0)]\right)$$

[Derrida, 2007]

Defining $\underline{\beta}(x,0)$ via $q_i(x,0) = \mathrm{q}_i[\underline{\beta}(x,0)]$, the **saddle point** of the path integral $\int_{(\mathbb{R})} \mathrm{d}\mathbb{P}_{\mathrm{ini}}[\underline{q}(\cdot,0)]$ gives the $\beta^i(x,0) = \beta^i_{\mathrm{ini}}(x)$

How do we generalise $\mathrm{d}\mathbb{P}_{\mathrm{ini}}[\underline{q}(\cdot,0)]$ to $\mathrm{d}\mathbb{P}[\underline{q}(\,\cdot\,,\,\cdot\,)]$?

Local relaxation of fluctuations (LRF)

We make the following assumption

Fluctuating variables $o(x,t) = \bar{o}(\ell x, \ell t)$ do not fluctuate **independently** but are **fixed** functionals of charge densities, i.e.

$$o(x,t) = o[\underline{q}(x,t)]$$

To fix $o[\,\bullet\,]$, we can determine it by taking $t = 0$ and invoke

$$\langle \hat{o} \rangle_{\underline{\beta}(x,0)} = \int_{(\mathbb{R})} \mathrm{d}\mathbb{P}_{\mathrm{ini}}[\underline{q}(\cdot,0)]\, o[\underline{q}(x,0)]$$ (because $o(x,0) = o[\underline{q}(x,0)]$ according to the ansatz)

The saddle point of it yields $o[\underline{q}(x,0)] = \mathrm{o}[\underline{q}(x,0)]$, i.e. $o(x,t) = \bar{o}(\ell x, \ell t) = \mathrm{o}[\underline{q}(x,t)]$

Local relaxation of averages	**Local relaxation of fluctuations**	
$\langle o(x,t) \rangle_{\ell} = \mathrm{o}[\underline{\mathrm{q}}(x,t)]$	$o(x,t) = \mathrm{o}[\underline{q}(x,t)]$	$\ell \to \infty$

LRF implies $j_i(x,t) = \bar{j}_i(\ell x, \ell t) = \mathtt{j}_i[\underline{q}(x,t)]$. The measure $\mathrm{d}\mathbb{P}[\underline{q}(\,\cdot\,,\cdot\,)]$ is given by

$$\mathrm{d}\mathbb{P}[\underline{q}(\,\cdot\,,\cdot\,)] = \underbrace{\mathrm{d}\mu[\underline{q}(\,\cdot\,,\cdot\,)]}_{\text{flat measure}}\, \underbrace{e^{-\ell\mathcal{F}[\underline{q}(\cdot,0)]}}_{\text{initial fluctuation}}\, \underbrace{\delta[\partial_t \underline{q} + \partial_x \underline{\mathtt{j}}[\underline{q}]]}_{\text{continuity equation}\ +\ \text{LRF}}$$

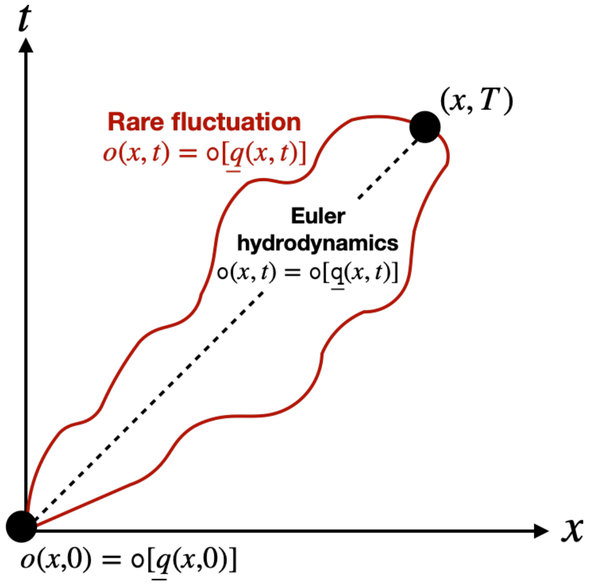
With this measure we write the BMFT average as the path-integral over $\mathbb{S} := \mathbb{R} \times [0,T]$

$$\langle\langle\, \bullet\, \rangle\rangle_\ell := \frac{1}{Z_\ell}\int_{(\mathbb{S})} \mathrm{d}\mu[\underline{q}(\,\cdot\,,\cdot\,)]\, e^{-\ell\mathcal{F}[\underline{q}(\cdot,0)]}\delta(\partial_t \underline{q} + \partial_x \underline{\mathtt{j}}[\underline{q}])\, \bullet$$

$$= \frac{1}{Z_\ell}\int_{(\mathbb{S})} \mathrm{d}\mu[\underline{q}(\,\cdot\,,\cdot\,)]\mathrm{d}\mu[\underline{H}(\,\cdot\,,\cdot\,)]\, e^{-\ell\mathcal{F}[\underline{q}(\cdot,0)]} e^{-\ell\int_{\mathbb{S}}\mathrm{d}x\mathrm{d}t\, H^i(\partial_t q_i + \partial_x \mathtt{j}_i[\underline{q}])}\, \bullet$$

The main claim of the BMFT is

$$\langle\, \bullet\, \rangle_\ell = \langle\langle\, \bullet\, \rangle\rangle_\ell$$



The power of the (B)MFT is that we do **not** have to evaluate the path-integrals, as they turn out to be dominated by their saddle points when $\ell \to \infty$! [Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim, 2015]

For instance, for an observable $O[\underline{q}]$ in space-time, we have

$$-\lim_{\ell\to\infty} \ell^{-1} \log\langle\langle \exp(\ell O[\underline{q}])\rangle\rangle_\ell = \mathscr{F}_O[\underline{q}^*]$$

Where $\mathscr{F}_O[\underline{q}(\,\cdot\,,\cdot\,)] = \mathscr{F}_O[\underline{q}(\cdot,0)] - O[\underline{q}(\,\cdot\,,\cdot\,)]$. Here, q^* is the **minimiser** of the BMFT action

$$S_O[\underline{q},\underline{H}] = \mathscr{F}_O[\underline{q}] + \int_{\mathbb{S}} \mathrm{d}x\mathrm{d}t\, H^i(\partial_t q_i + \partial_x \mathtt{j}_i[\underline{q}])$$

$\underline{H}$ serve as the Lagrange multipliers associated to the Euler equation, hence

$$\partial_t q_i^* + \partial_x \mathtt{j}_i[\underline{q}^*] = 0$$

As applications of the BMFT, we are interested in two objects:

$$S_{\hat{o}_1,\ldots,\hat{o}_n}(x_1,t_1;\cdots;x_n,t_n) := \lim_{\ell\to\infty} \ell^{n-1}\langle \overline{o_1}(\ell x_1,\ell t_1)\cdots\overline{o_n}(\ell x_n,\ell t_n)\rangle^{\mathrm{c}}_{\ell}$$

Euler scale dynamical correlation functions

$$F(\lambda,T) := \lim_{\ell\to\infty}\frac{1}{\ell T}\log\langle e^{\lambda \hat{J}_{i_*}(\ell T)}\rangle_{\ell}, \quad \hat{J}_i(\ell T) := \int_0^{\ell T} \mathrm{d}t\, \hat{j}_i(0,t)$$

Scaled cumulant generating function (SCGF)

Note $\hat{J}(\ell T) = \ell \int_0^T \mathrm{d}t\, j(0,t) =: \ell J(T)$

The BMFT allows us to evaluate these quantities by choosing $O_{\mathrm{corr}} = \sum_{a=1}^n \lambda_a \circ_a [\underline{q}(x_a,t_a)]$ and $O_{\mathrm{curr}} = \lambda J(T)$:

$$S_{\hat{o}_1,\ldots,\hat{o}_n}(x_1,t_1;\cdots;x_n,t_n) = -\frac{\mathrm{d}^n}{\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n}\mathscr{F}_{\mathrm{corr}}[\underline{q}^*]\Big|_{\lambda_1=\cdots=\lambda_n=0}$$

$$F(\lambda,T) = -\frac{1}{T}\mathscr{F}_{\mathrm{curr}}[\underline{q}^*]$$

with $\mathscr{F}_{\mathrm{corr}}[\underline{q}] := \mathscr{F}_{O_{\mathrm{corr}}}[\underline{q}]$ and $\mathscr{F}_{\mathrm{curr}}[\underline{q}] := \mathscr{F}_{O_{\mathrm{curr}}}[\underline{q}]$

Current fluctuations from the BMFT

The equation satisfied by $\underline{q}^*$, the BMFT equation, is obtained from $\delta S_{O_{\text{curr}}} = 0$

$$\boxed{\begin{aligned}
&\lambda \delta^i_{\ i_*} \Theta(x) - \beta^i(x,0) + \beta^i_{\text{ini}}(x) - H^i(x,0) = 0, \\
&\lambda \delta^i_{\ i_*} \Theta(x) - H^i(x,T) = 0, \\
&\partial_t \beta^i(x,t) + \mathsf{A}_j^{\ i}[\underline{\beta}(x,t)] \partial_x \beta^j(x,t) = 0, \\
&\partial_t H^i(x,t) + \mathsf{A}_j^{\ i}[\underline{\beta}(x,t)] \partial_x H^j(x,t) = 0.
\end{aligned}}$$

Since $S_{O_{\text{curr}}}$ is **stationary** under the change of $\underline{q}$ and $\underline{H}$, we readily see $\frac{\mathrm{d}}{\mathrm{d}\lambda} T F(\lambda, T) = J(T)$, i.e.

$$F(\lambda, T) = \frac{1}{T} \int_0^\lambda \mathrm{d}\lambda' \int_0^T \mathrm{d}t \ \mathtt{j}_{i_*}[\underline{q}^{(\lambda')}(0,t)]$$

Dynamical correlation functions from the BMFT

In a similar way one can also compute the Euler-scale dynamical correlation function

For simplicity we focus on the **two-point** function $S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t_1;x_2,t_2)$, which is given by

$$S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t_1;x_2,t_2) = -\frac{\mathrm{d}^2}{\mathrm{d}\lambda_1 \mathrm{d}\lambda_2}\mathscr{F}_{\mathrm{corr}}\Big|_{\lambda_1=\lambda_2=0}.$$

Since at the saddle point $q = q^*$ we have $\partial_{\lambda_2}\mathscr{F}_{\mathrm{corr}} = -\,q_{i_2}(x_2,t_2)$. Hence

$$S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t_1;x_2,t_2) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{q}_{i_2}(x_2,t_2)\Big|_{\lambda=0}, \quad \lambda := \lambda_1$$

The associated MFT equation is then

$$\boxed{\begin{aligned}
&\beta^i(x,0) - \beta^i_{\text{ini}}(x) + H^i(x,0) = 0,\\
&H^i(x,T) = 0,\\
&\partial_t \beta^i + \mathsf{A}_j{}^i \partial_x \beta^j = 0,\\
&\partial_t H^i + \mathsf{A}_j{}^i \partial_x H^j + \lambda \delta^i{}_{i_1} \delta(x - x_1)\delta(t - t_1) = 0.
\end{aligned}}$$

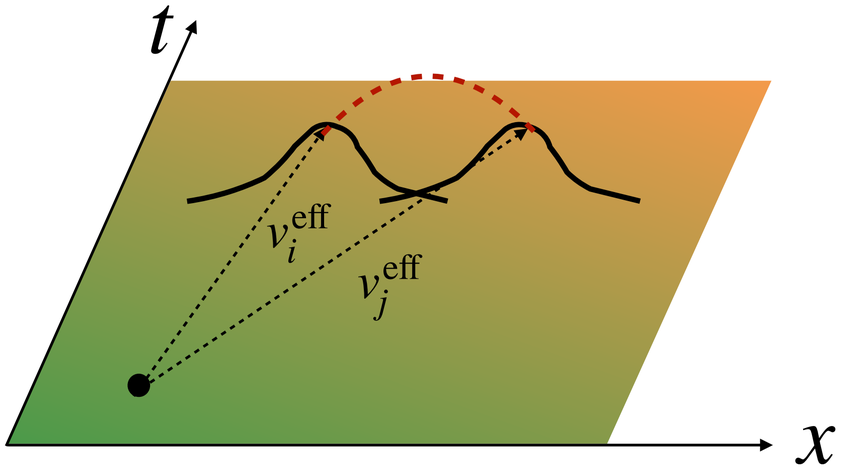
The solution of the MFT equation actually predicts the existence of the long-range correlations amongst the fluid cells on the same time slice $t_1 = t_2 = t$, i.e.

$$\lim_{\ell \to \infty} \ell^{-1} \langle \overline{q}_{i_1}(\ell x_1, \ell t) \overline{q}_{i_2}(\ell x_2, \ell t) \rangle_\ell = S_{\hat{q}_{i_1}, \hat{q}_{i_2}}(x_1, t; x_2, t) = \mathsf{C}_{i_1 i_2}(x_1, t)\delta(x_1 - x_2) + E_{i_1 i_2}(x_1, x_2; t)$$

Initially $E_{i_1 i_2}(x_1, x_2; 0) = 0$

Even if there is no long-range correlation initially, it could build up by the coupling between **normal modes** on an **inhomogeneous** background

Note that the long-range correlations observed here is of purely **hydrodynamic** nature



multiple conservation laws ✔
interaction ✔
initial inhomogeneity ✔

Having more than one conservation laws is crucial (TASEP shows no such correlations)

Long-range correlations could also exist on a different scale (diffusive) [Derrida, 2007; Ortiz de Zárate and Sengers, 2004]. Non-locality of the NESS density matrix also amounts to long-range correlations even without interactions [Doyon, Lucas, Schalm, and Bhaseen, 2015; De Nardis and Panfil, 2018].

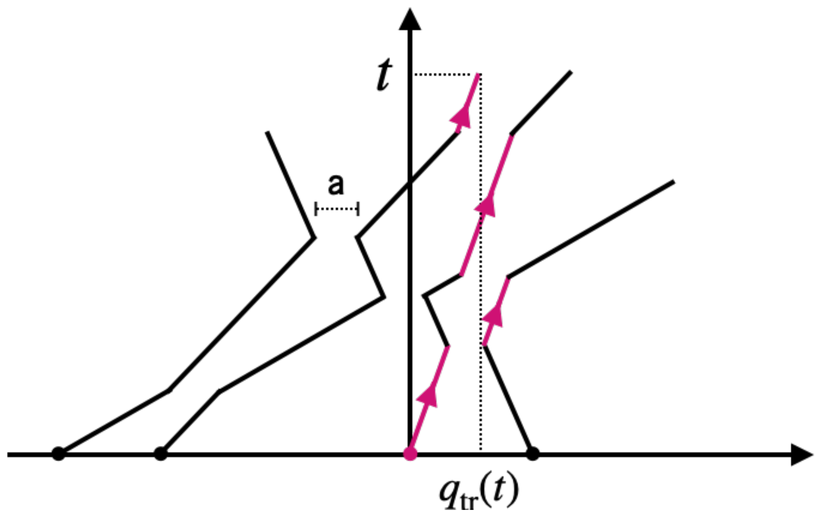
We are going to compute these quantities explicitly for **integrable systems**

Generalised hydrodynamics (GHD)

For simplicity, we consider a diagonally-scattering integrable model with single species (e.g. sinh-Gordon, Lieb-Liniger)

A kinetic intuition behind GHD is that on a hydrodynamic scale, quasi-particles in integrable systems behave pretty much like tracer particles of **hard-rods**.

[Boldrighini, Dobrushin, and Sukhov, 1983; Spohn, 1991; Doyon and Spohn, 2017; Doyon, TY, and Caux, 2018]



An exact expression of the current average (i.e. the matrix A) turns out to be instrumental in GHD.

[See reviews: Borsi, Pozsgay, and Pristyák, 2021; Cortés Cubero, TY, and Spohn, 2021]

On the Euler scale, quasi-particles in integrable systems are transported according to the GHD equation [Castro-Alvaredo, Doyon, and TY, 2016; Bertini, Collura, De Nardis, Fagotti, 2016]

$$\partial_t \underline{\mathtt{q}}(x,t) + \partial_x \underline{\mathtt{j}}(x,t) = 0 \quad \longleftrightarrow \quad \partial_t \rho_\theta(x,t) + \partial_x (v_\theta^{\mathrm{eff}}(x,t)\rho_\theta(x,t)) = 0$$

or equivalently $\partial_t \beta^\theta(x,t) + A_\phi^{\ \theta}[\rho_\cdot(x,t)]\partial_x \beta^\phi(x,t) = 0$ with a GGE $\varrho \sim e^{-\beta^\theta \hat{Q}_\theta}$. Here v_θ^{eff} is

$$v_\theta^{\mathrm{eff}} = \frac{E'_\theta}{p'_\theta} + \frac{1}{p'_\theta}\int_{\mathbb{R}} \mathrm{d}\phi\, \varphi_\theta^{\ \phi} \rho_\phi (v_\phi^{\mathrm{eff}} - v_\theta^{\mathrm{eff}})$$

To solve initial value problems we shall use the GHD equation in terms of the **normal mode**

$$\partial_t \epsilon_\theta(x,t) + v_\theta^{\mathrm{eff}}(x,t)\partial_x \epsilon_\theta(x,t) = 0$$

Note $(R^{-1})_\phi^{\ \theta}\partial_{t,x}\beta^\phi = \partial_{t,x}\epsilon^\theta$ where $R = 1 - n\varphi/2\pi$ diagonalises A: $RAR^{-1} = \mathrm{diag}\, v^{\mathrm{eff}}$

No **shock** appears in GHD!

Current fluctuations in integrable systems

We simply adopt the BMFT we formulated to (quantum) integrable systems. We want to compute $F(\lambda, T) = \lim_{\ell\to\infty} \log\langle e^{\lambda \hat{J}_{i_*}(\ell T)}\rangle_\ell/(\ell T)$. The MFT equation is

$$\boxed{\begin{aligned} \lambda h^\theta \delta^\theta_{\ \theta_*}\Theta(x) - \beta^\theta(x,0) + \beta^\theta_{\text{ini}}(x) - H^\theta(x,0) &= 0 \\ -\lambda h^\theta \delta^\theta_{\ \theta_*}\Theta(x) + H^\theta(x,T) &= 0 \\ \partial_t \beta^\theta(x,t) + \mathsf{A}^\theta_{\ \phi}(x,t)\partial_x \beta^\phi(x,t) &= 0 \\ \partial_t H^\theta(x,t) + \mathsf{A}^\theta_{\ \phi}(x,t)\partial_x H^\phi(x,t) &= 0 \end{aligned}}$$

$$\hat{Q}_i|\theta\rangle = h^\theta_i|\theta\rangle$$
$$h^\theta := h^\theta_{i_*}$$

We rewrite it in terms of normal modes. Recall that β^θ and ϵ_θ are related by $(R^{-1})^\theta_{\ \phi}\partial_{t,x}\beta^\phi = \partial_{t,x}\epsilon^\theta$. Motivated by this we define normal modes associated to H^θ:

$$(R^{-1})^\theta_{\ \phi}\partial_{t,x}H^\phi =: \partial_{t,x}G^\theta$$

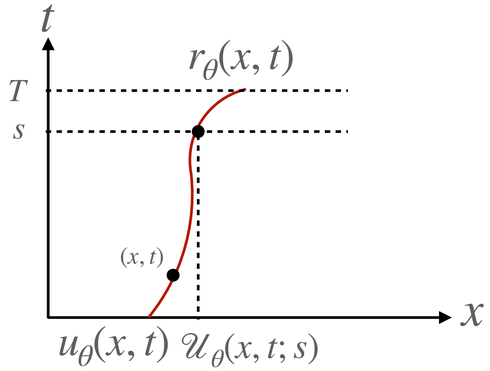
The property $\partial_t\partial_x G^\theta = \partial_x\partial_t G^\theta$ is instrumental

The MFT equation for the auxiliary field becomes

$$\begin{aligned} \lambda h^{\mathrm{dr};\theta}(0,T)\Theta(x) - G^{\theta}(x,T) &= 0 \\ \partial_t G^{\theta}(x,t) + v^{\mathrm{eff};\theta}(x,t)\partial_x G^{\theta}(x,t) &= 0 \end{aligned}, \quad (a^{\mathrm{dr},\theta} := (R^{-\mathrm{T}})^{\theta}{}_{\phi} a^{\phi})$$

The **method of characteristics** allows us to solve the equation via $G^{\theta}(x,t) = G^{\theta}(r^{\theta}(x,t),T) = h^{\mathrm{dr};\theta}(0,T)\Theta(r^{\theta}(x,t))$, where $r^{\theta}(x,t) = \mathscr{U}^{\theta}(x,t;T)$.

With this the MFT equation is now recast into the GHD equation with the λ-dependent initial condition



$$\boxed{\begin{aligned} &\beta^{\theta}(x,0) = \beta^{\theta}_{\mathrm{ini}}(x) + \lambda h^{\theta}\Theta(x) - \lambda R^{\theta}_{\phi}(0,T)\Theta(x - u^{\phi}(0,T))h^{\mathrm{dr};\phi}(0,T) \\ &\partial_t \beta^{\theta}(x,t) + \mathsf{A}^{\theta}_{\phi}(x,t)\partial_x \beta^{\phi}(x,t) = 0 \end{aligned}}$$

The evaluation of the current $j^{(\lambda)}_{\theta}(0,t)$ gives $F(\lambda,T)$

In the **homogeneous** case one can readily compute the cumulants and get

$$c_2^{\mathrm{hom}} = \chi_\theta \,|\, v_\theta^{\mathrm{eff}} \,|\, (h_{i_*}^{\mathrm{dr};\theta})^2, \quad \chi_\theta := \rho_\theta(1-n_\theta)$$

$$c_3^{\mathrm{hom}} = \chi_\phi \,|\, v_\phi^{\mathrm{eff}} \,|\, h_{i_*}^{\mathrm{dr};\phi} \left(s_\phi \tilde{f}_\phi (h_{i_*}^{\mathrm{dr};\phi})^2 + 3[sf(h_{i_*}^{\mathrm{dr}})^2]_\phi^{\mathrm{dr}} \right), \quad c_n := \left. \frac{\mathrm{d}^n F(\lambda,1)}{\mathrm{d}\lambda^n} \right|_{\lambda=0}$$

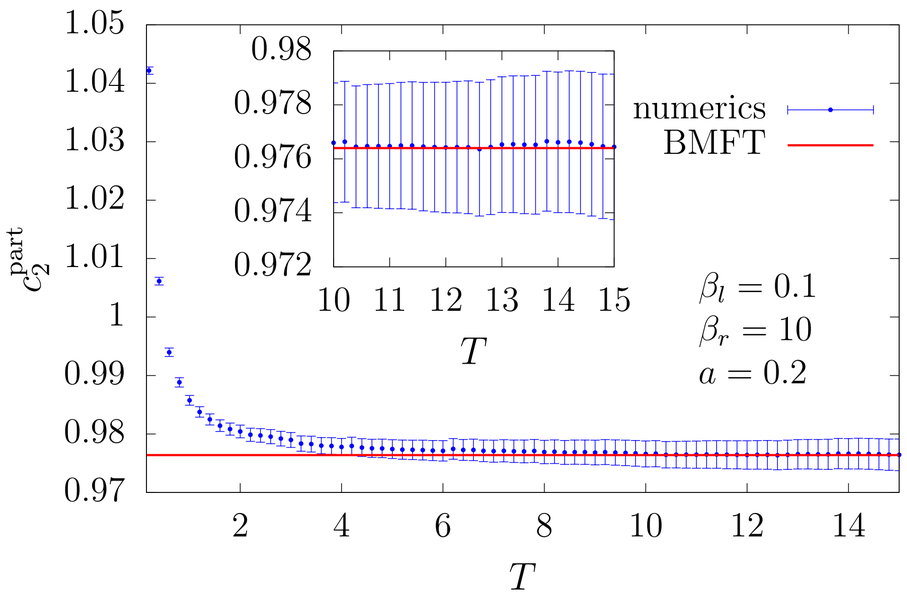
They coincide with the results obtained by the Ballistic fluctuation theory, which were also corroborated against Hard-rod simulations. [Doyon and Myers, 2020]

A virtue of the BMFT is that the extension to **inhomogeneous** cases is straightforward. For instance c_2 for the partitioning protocol (i.e. $\rho_{\mathrm{ini}} = \rho_L \otimes \rho_R$) and obtained

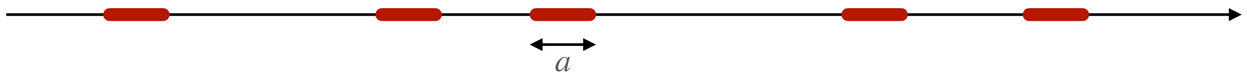
$$c_2^{\mathrm{part}} = \chi_\theta(0) \,|\, v_\theta^{\mathrm{eff}}(0) \,|\, (h_{i_*}^{\mathrm{dr};\theta}(0))^2$$

Fully fixed by the NESS at $\xi = x/t = 0$!

Comparison against hard-rod simulations



A gas of hard-rods consists of rigid rods that scatter elastically, and hence is an integrable system



$$H = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{j=1}^{N-1} V_{\text{HR}}(q_{j+1} - q_j), \quad V_{\text{HR}}(x) = \begin{cases} \infty & |x| < a \\ 0 & |x| \geq a \end{cases}$$

The onset to the stationary value is controlled by the diffusive corrections

Euler dynamical correlation functions in integrable systems

The MFT equation for the correlation function $S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t_1;x_2,t_2)$ is

$$\boxed{\begin{aligned}
&\beta^\theta(x,0) - \beta^\theta_{\text{ini}}(x) + H^\theta(x,0) = 0 \\
&H^\theta(x,T) = 0 \\
&\partial_t \beta^\theta + \mathsf{A}^\theta_\phi \partial_x \beta^\phi = 0 \qquad\qquad , \quad t_1, t_2 \neq 0, T \\
&\partial_t H^\theta + \mathsf{A}^\theta_\phi \partial_x H^\phi + \lambda h^\theta_{i_1} \delta(x - x_1)\delta(t - t_1) = 0
\end{aligned}}$$

In terms of the solution, the correlator is computed by

$$S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t_1;x_2,t_2) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathsf{q}_{i_2}(x_2,t_2)\Big|_{\lambda=0} = -\left[(h_{i_2})^{\mathrm{dr};\theta}\chi_\theta \partial_\lambda \epsilon^\theta\right](x_2,t_2)\Big|_{\lambda=0}.$$

As in the case of SCGF, the MFT equation is reduced to the following GHD equation

$$\beta^\theta(x,0) = \beta^\theta_{\rm ini}(x) - \lambda\partial\left((R^{\rm T})^\theta{}_\phi(x_1,t_1)h^{{\rm dr};\phi}(x_1,t_1)\Theta(x-u^\phi)\right)$$

$$\partial_t\beta^\theta(x,t) + \mathsf{A}^\theta_\phi(x,t)\partial_x\beta^\phi(x,t) = 0$$

Let us take $t_1 = t_2 = t$. The Euler dynamical correlator $S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t;x_2,t)$ turns out to be given by

$$S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t;x_2,t) = \mathsf{C}_{i_1i_2}(x_1,t)\delta(x_1-x_2) + E_{i_1i_2}(x_1,x_2;t), \quad \mathsf{C}_{i_1i_2}(x_1,t) := [(h_{i_1})^{{\rm dr};\theta}\chi_\theta(h_{i_2})^{{\rm dr};\theta}](x_1,t)$$

$$E_{i_1i_2}(x_1,x_2;t) := -[h^{{\rm dr};\theta}_{i_2}\chi_\theta\mathscr{E}^\theta](x_2,t)$$

$$\mathscr{E}^\theta(x,t) = \mathscr{E}^\theta_0(x,t) + w^\theta(x,t)\int_{-\infty}^{x}{\rm d}y[\chi\mathscr{E}]^{{\rm dr};\theta}(y,t)$$

$$w^\theta(x,t) := \frac{\partial\epsilon^\theta_{\rm ini}(u^\theta(x,t))}{\rho^{{\rm tot};\theta}(u^\theta(x,t),0)}$$

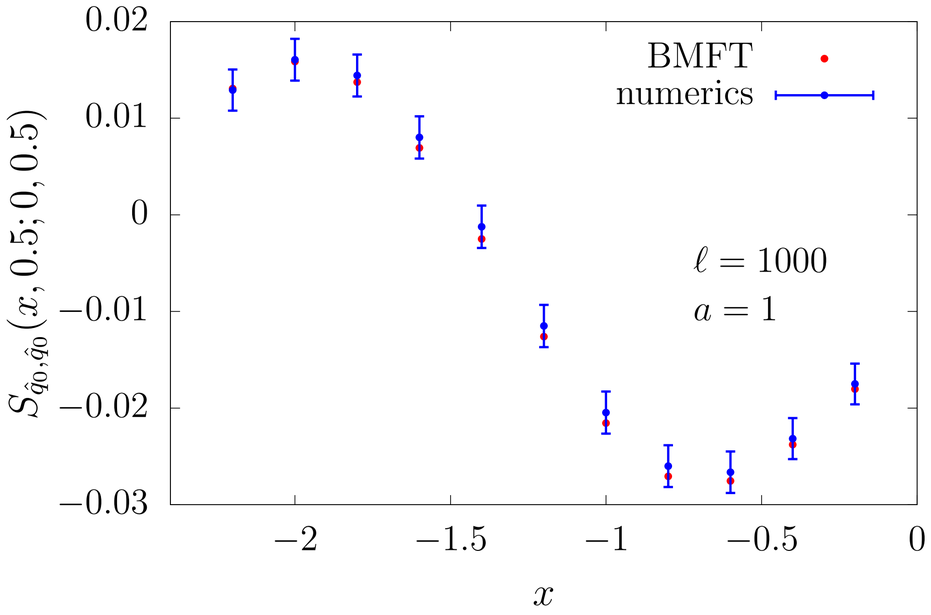
$$\mathscr{E}^\theta_0(x,t) = \mathscr{D}^\theta_1(x,t) + \mathscr{D}^\theta_2(x,t)$$

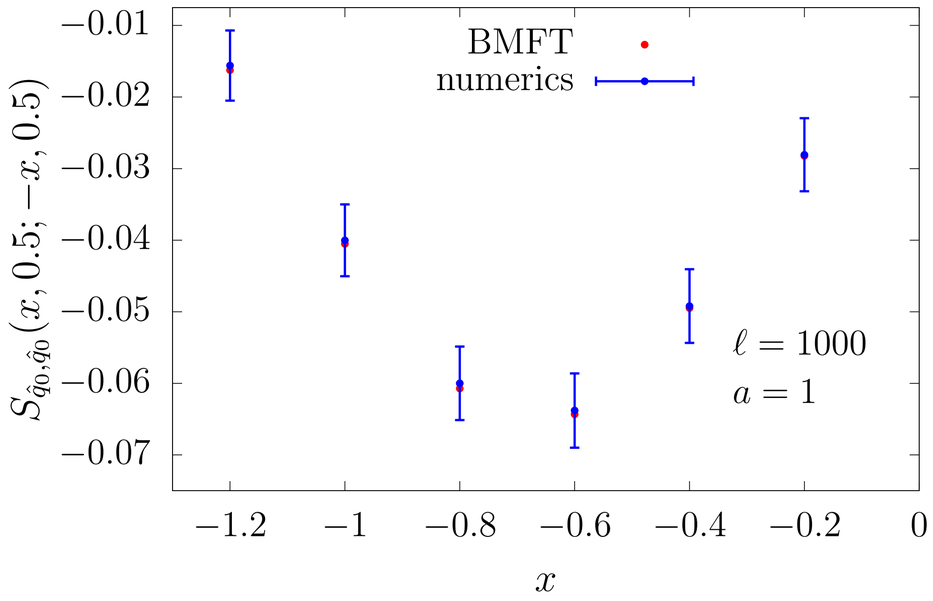
$$\mathscr{D}^\theta_1(x,t) := (R^{-{\rm T}})^\theta{}_\phi(u^\theta(x,t),0)\partial\left[(R^{\rm T})^\phi{}_\alpha h^{{\rm dr};\alpha}\right](x_1,t)\Theta(u^\theta(x,t)-u^\alpha(x_1,t)) - w^\theta(x,t)[\chi h^{\rm dr}]^{{\rm dr};\theta}(x_1,t)\Theta(x_2-x_1)$$

$$\mathscr{D}^\theta_2(x,t) := -w^\theta(x,t)\int_{-\infty}^{u^\theta(x,t)}{\rm d}y[\chi\mathscr{D}_3]^{{\rm dr};\theta}(y,0)$$

$$\mathscr{D}^\theta_3(x,0) := -h^{{\rm dr};\theta}(x_1,t)\frac{\delta(x-u^\theta(x_1,t))}{\partial\mathscr{U}^\theta(u^\theta(x_1,t),0;t)} + (R^{-{\rm T}})^\theta{}_\phi(x,0)\partial\left[(R^{\rm T})^\phi{}_\alpha h^{{\rm dr};\alpha}\right](x_1,t)\Theta(x-u^\alpha(x_1,t))$$

Comparison against hard-rod simulations (bump-release protocol)





Despite of the small numbers, the agreements are very satisfying!

By increasing ℓ, one can observe the convergence of the numbers

Conclusion and outlook

- The BMFT is a new theory to study the fluctuation-induced physics, such as current fluctuations and large scale dynamical correlation functions, in ballistic many-body systems
- The underlying idea of the BMFT is local relaxation of fluctuations
- It works particularly well for integrable systems. The results also agree with hard-rods simulations very well
- It is highly desirable to derive our predictions microscopically using a simple model such as the AHR model
- Obtaining the KPZ function from the BMFT+superdiffusive corrections?
- Quantum fluctuations?