

The Gaudin model from 3d mixed BF theory

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based on 2201.07300 w/ J. Winstone

Brief summary

4d CS theory

[Costello '13,
Costello - Wilten]

- Yamazaki '17, '18,

$$\{L_1, L_2\} = [r_{12}, L_1, L_2] \quad (R_{12} L_1 L_2 = L_2 L_1 R_{12})$$

Line defects
surface defects

- Integrable lattice models

- 2d integrable field theories

$$\{L_1, L_2\} = [r_{12}, L_1 + L_2] \delta + [s_{12}, L_1 - L_2] \delta + 2s_{12} \delta'$$

Brief summary

4d CS theory

[Costello '13,
Costello - Wilten]

- Yamazaki '17, '18,

$$[Costello - Yamazaki '19] \quad \{L_1, L_2\} = [r_{12}, L_1, L_2] \delta + [s_{12}, L_1, -L_2] \delta + 2s_{12} \delta'$$

3d BF theory

Line
defects

- Integrable lattice models

$$\{L_1, L_2\} = [r_{12}, L_1, L_2] \quad (R_{12} L_1 L_2 = L_2 L_1 R_{12})$$

Line
defects

Surface
defects

- 2d integrable field theories

- Finite-dim. integrable systems

$$\{L_1, L_2\} = [r_{12}, L_1 + L_2] \quad ([L_1, L_2] = [r_{12}, L_1 + L_2])$$

4d CS theory/Affine Gaudin & 2d IFTs

4d Chern-Simons

on $\mathbb{R} \times S^1 \times \mathbb{C}\mathbb{P}^1$

Hamiltonian
formalism

Affine Gaudin
model

+ gauge [BV'19]
fixing

Gauge fixing

+ surface defects

[Costello - Yamazaki '19]

[Delduc - Lacroix

- Magro - BV '19]

[Fukushima - Sakamoto
- Yoshida '20] ...

Integrable Field
theory on $\mathbb{R} \times S^1$

realisation

[BV'17]

[Delduc - Lacroix
- Magro - BV '19]

Finite-dimensional analogue? [BV-Winstone '22]

3d mixed BF

on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$

Gauge fixing
+ line defects
in Hamiltonian
formalism

Gaudin
model

realisation

Integrable
System

Plan of the talk

III 3d mixed BF

on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$

IV Gauge fixing

V + line defects

VI in Hamiltonian
formalism

I Gaudin
model

II realisation

Integrable
System

Plan of the talk

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on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$

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The Gaudin model - dynamical variables

\mathfrak{g} - semisimple complex Lie algebra.

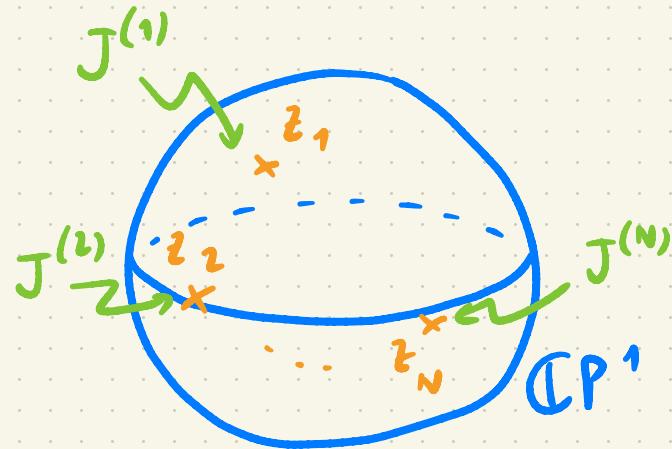
$\{I_a\}, \{I^a\}$ - dual bases w.r.t. $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$.

Fix $z_1, \dots, z_N \in \mathbb{C}$.

Dynamical variables:

$I^{a(1)}, \dots, I^{a(N)} \in \mathfrak{g}^{\oplus N}$

$J^{(i)} := I_a \otimes I^{a(i)} \in \mathfrak{g} \otimes \mathfrak{g}^{\oplus N}$



The Gaudin model - Lax matrix

Lie bracket $[I^a(i), I^b(j)] = [I^a, I^b]^{(i)} \delta_{ij}$ of $\mathfrak{g}^{\oplus N}$

encoded in Kostant-Kirillov Poisson bracket:

$$\{J_1^{(i)}, J_2^{(j)}\} = -[C_{12}, J_2^{(i)}] \delta_{ij} \quad i, j = 1, \dots, N.$$

$\overset{I_a \otimes I^a}{\nearrow}$

Lax matrix $L(z) := \sum_{i=1}^N \frac{J^{(i)}}{z - z_i}$ satisfies

$$\{L_1(z), L_2(w)\} = \left[\frac{C_{12}}{w-z}, L_1(z) + L_2(w) \right].$$

The Gaudin model - Hamiltonians

forall adjoint-invariant polynomials $P, Q : \mathfrak{g} \rightarrow \mathbb{C}$

$$P(gXg^{-1}) = P(X) \quad \left\{ P(L(w)), Q(L(z)) \right\} = 0.$$

involution property

Fix $w \in \mathbb{C}$ & $P : \mathfrak{g} \rightarrow \mathbb{C}$, then define

Hamiltonian $H_w^P := P(L(w)) \in S(\mathfrak{g}^{\otimes N}) = S(\mathfrak{g})^{\otimes N}$.

The Gaudin model - Lax equation

Time flow $\partial_t \theta := \{P(L(w)), \theta\}, \forall \theta \in S(\mathfrak{g}^{\otimes n})$

Dynamics of Lax matrix:

Lax equation $\partial_t L(z) = [M(z), L(z)]$

$$M(z) := \frac{P'(L(w))}{z - w}$$

e.g. with $P(x) = \text{tr}(x^n)$

Derivative of $P: \mathfrak{g} \rightarrow \mathbb{C}$ at $X \in \mathfrak{g}$
is Linear map $\langle P'(X), \cdot \rangle: \mathfrak{g} \rightarrow \mathbb{C}$

$$\text{have } M(z) = \frac{n L(w)^{n-1}}{z - w}$$

Plan of the talk

3d mixed BF

on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$

Gauge fixing
+ line defects
in Hamiltonian
formalism

Integrable
System

Gaudin
model

II realisation

The Gaudin model - A realisation

Fix $u_1, \dots, u_N \in \mathfrak{g}$. Product of adjoint orbits

$$\mathcal{O}_{u_i} := \left\{ \hat{u}_i := h_i u_i h_i^{-1} \mid h_i \in G \right\} \subset \mathfrak{g} \cong \mathfrak{g}^*$$

is a symplectic manifold with

$$\{\hat{u}_{i1}, \hat{u}_{j2}\} = -[C_{12}, \hat{u}_{i2}] \delta_{ij} \quad i, j = 1, \dots, N.$$

Realisation $\pi : S(\mathfrak{g})^{\otimes N} \rightarrow \text{Fun} \left(\prod_{i=1}^N \mathcal{O}_{u_i} \right)$.

$$(J^{(i)} = I_a \otimes I^{a(i)} \mapsto \hat{u}_i = I_a \hat{u}_i^a)$$

The Gaudin model - Summary

Lax matrix $L(z) = \sum_{i=1}^N \frac{\widehat{u}_i}{z - z_i} = \sum_{i=1}^N \frac{h_i u_i h_i^{-1}}{z - z_i}$

satisfies Lax algebra

$$\{L_1(z), L_2(z')\} = \left[\frac{C_{12}}{z' - z}, L_1(z) + L_2(z') \right]$$

and Lax equation

$$\partial_t L(z) = \left[\frac{P'(L(w))}{z - w}, L(z) \right].$$

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3d mixed BF theory - fields

G - semisimple complex Lie group.

$$\mathfrak{g} = \text{Lie}(G), \quad \langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}.$$

$$A = A_t(t, z, \bar{z}) dt + A_z(t, z, \bar{z}) dz + A_{\bar{z}}(t, z, \bar{z}) d\bar{z}$$

\mathfrak{g} -valued smooth 1-form on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$.

$$B = B_z(t, z, \bar{z}) dz$$

\mathfrak{g} -valued smooth $(1,0)$ -form on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$.

3d mixed BF theory - action

$$\rightarrow F(A) = dA + \frac{1}{2}[A, A]$$

$$S_{\text{3d-BF}}[A, B] = \frac{1}{2\pi i} \int_{\mathbb{R} \times \mathbb{C}\mathbb{P}^1} \langle B, F(A) \rangle$$

[Gwilliam-Williams'19]
[Gwilliam-Rabinovich
-Williams '21]

Obvious gauge invariance: $A \mapsto A + \chi(t, z, \bar{z}) dz$

Fix by letting $A_z = 0$.

$$A = A_t(t, z, \bar{z}) dt + A_{\bar{z}}(t, z, \bar{z}) d\bar{z}, \quad B = B_z(t, z, \bar{z}) dz$$

3d mixed BF theory - gauge invariance

$g(t, z, \bar{z})$ - smooth G -valued field on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$

$$\left\{ \begin{array}{l} A_t \mapsto -\partial_t g g^{-1} + g A_t g^{-1} \\ A_{\bar{z}} \mapsto -\partial_{\bar{z}} g g^{-1} + g A_{\bar{z}} g^{-1} \\ B_z \mapsto g B_z g^{-1} \end{array} \right. \quad F_{t\bar{z}} \mapsto g F_{t\bar{z}} g^{-1}$$

$\langle B_z, F_{t\bar{z}} \rangle$ invariant

3d mixed BF theory - equations of motion

$$\text{Eom for } B: \partial_{\bar{z}} A_t = \partial_t A_{\bar{z}} + [A_t, A_{\bar{z}}]$$

$$\text{Eom for } A: \begin{cases} \partial_{\bar{z}} B_z = [B_z, A_{\bar{z}}] \\ \partial_t B_z = [-A_t, B_z] \end{cases}$$

Looks like Lax equation! $L \equiv B_z, M \equiv -A_t$

Issue: L & M depend smoothly on (z, \bar{z}) .

Plan of the talk

3d mixed BF

on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$

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3d mixed BF theory - equations of motion

$$\text{Eom for } B: \partial_{\bar{z}} A_t = \partial_t A_{\bar{z}} + [A_t, A_{\bar{z}}]$$

$$\text{Eom for } A: \left\{ \begin{array}{l} \partial_{\bar{z}} B_z = [B_z, A_{\bar{z}}] \\ \partial_t B_z = [-A_t, B_z] \end{array} \right.$$

$$F_{t\bar{z}}(A) = 0$$

Looks like Lax equation! $L \equiv B_z, M \equiv -A_t$

Issue: L & M depend smoothly on (z, \bar{z}) .

3d mixed BF theory - gauge fixing

Use gauge fixing condition : $A_{\bar{z}} = 0$.

$$\left. \begin{array}{l} \partial_{\bar{z}} A_t = 0 \\ \partial_{\bar{z}} B_z = 0 \\ \partial_t B_z = [-A_t, B_z] \end{array} \right\} \Rightarrow L = B_z, M = -A_t \text{ are holomorphic!}$$

- Issues with L & M :
- 1) no poles in \mathbb{C} .
 - 2) unrelated.

Plan of the talk

3d mixed BF
on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$

Gauge fixing
① + Line defects
in Hamiltonian
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3d BF w/ defects – action

$$S[A, B] = \frac{1}{2\pi i} \int_{\mathbb{R} \times \mathbb{CP}^1} \langle B, F(A) \rangle - \sum_{j=1}^N \int_{\mathbb{R} \times \{z_j\}} \langle u_j, h_j^{-1}(\partial_t + A_t) h_j \rangle dt - \int_{\mathbb{R} \times \{\omega\}} P(B_\omega) dt$$

type A
defects

type B
defect

3d BF w/ defects – action

$$S[A, B] = \frac{1}{2\pi i} \int_{\mathbb{R} \times \mathbb{CP}^1} \langle B, F(A) \rangle - \sum_{j=1}^N \int_{\mathbb{R} \times \{z_j\}} \langle u_j, h_j^{-1}(\partial_t + A_t) h_j \rangle dt - \int_{\mathbb{R} \times \{\omega\}} P(B_\omega) dt$$

$h_j \in C^\infty(\mathbb{R}, G)$

R

\mathbb{CP}^1

z_1, z_2, \dots, z_N, w

$u_j \in g$

polynomial

$P: g \rightarrow \mathbb{C}$

3d BF w/ defects – gauge invariance

$$\left\{ \begin{array}{l} A_t \mapsto -\partial_t g g^{-1} + g A_t g^{-1} \\ A_{\bar{z}} \mapsto -\partial_{\bar{z}} g g^{-1} + g A_{\bar{z}} g^{-1} \\ B_z \mapsto g B_z g^{-1} \end{array} \right. \quad \begin{array}{l} g \in C^\infty(\mathbb{R} \times \mathbb{C}\mathbb{P}^1, G) \\ \text{choose } P: g \rightarrow \mathbb{C} \\ \text{adjoint-invariant.} \end{array}$$

$$h_j \mapsto g|_{z_j} h_j \Rightarrow \left(h_j^{-1} (\partial_t + A_t|_{z_j}) h_j \right) \text{ invariant.}$$

3d BF w/ defects - equations of motion

Eom for B :

$$\partial_{\bar{z}} A_t = \partial_t A_{\bar{z}} + [A_t, A_{\bar{z}}] + 2\pi i P'(B_z(w)) \delta_{zw}$$

Eom for A :

$$\begin{cases} \partial_{\bar{z}} B_z = [B_z, A_{\bar{z}}] - 2\pi i \sum_{j=1}^N \hat{u}_j \delta_{zz_j} \\ \partial_t B_z = [-A_t, B_z] \end{cases}$$

3d BF w/ defects - gauge fixing

Use gauge fixing condition : $A_{\bar{z}} = 0$.

Equations of motion \Leftrightarrow

Lax equation $\partial_t L(z) = [M(z), L(z)]$

with $\left(\partial_{\bar{z}} \frac{1}{z-w} = -2\pi i \delta_{zw} \right)$

$$L(z) = B_z(z) = \sum_{j=1}^N \frac{\hat{u}_j}{z-z_j}, \quad M(z) = -A_k(z) = \frac{P'(L(w))}{z-w}.$$

Plan of the talk

3d mixed BF

on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$

Gauge fixing
+ line defects

VI in Hamiltonian
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3d BF w/ defects – Hamiltonian formalism

Conjugate momenta

$$(\Pi_t, \Pi_{\bar{z}}, P_z, X_i) \text{ for } (A_t, A_{\bar{z}}, B_z, h_i)$$

satisfy primary constraints

$$\Pi_t \approx 0$$

$$\mathcal{C}_z := B_z + 2\pi i \Pi_{\bar{z}} \approx 0$$

$$P_z \approx 0$$

second class pair

$$\mathcal{C}_i := X_i + u_i \approx 0$$
$$i = 1, \dots, N$$

$$\{P_{z_1}(z), \mathcal{C}_{z_2}(z')\} = C_{12} \delta_{zz'}$$

3d BF w/ defects - Dirac bracket for $\mathcal{L}_{z_1}, P_{z_2}$

Can set $P_z = 0$ and $\mathcal{L}_z = 0$ strongly

by replacing $\{\cdot, \cdot\}$ with Dirac bracket

$$\{F(z), G(z')\}^* := \{F(z), G(z')\}$$

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{(i,z)} := \int_{CP^1} \langle \cdot, \cdot \rangle_i dz \wedge d\bar{z}$$

$$-\langle\!\langle \{F(z), P_{z_1}(z'')\}, \langle\!\langle C_{12} \delta_{z'' z'''}, \{\mathcal{L}_{z_2}(z'''), G(z')\} \rangle\!\rangle_{(2,z''')} \rangle\!\rangle_{(1,z'')}$$

$$-\langle\!\langle \{F(z), \mathcal{L}_{z_1}(z'')\}, \langle\!\langle C_{12} \delta_{z'' z'''}, \{P_{z_2}(z'''), G(z')\} \rangle\!\rangle_{(2,z''')} \rangle\!\rangle_{(1,z'')}$$

3d BF w/ defects - Dirac bracket for \mathcal{E}_i :

$$\mathcal{E}_i := X_i + u_i \approx 0 \quad \rightarrow \text{first class part}$$
$$i = 1, \dots, N$$
$$\langle v_p^i, \mathcal{E}_i \rangle \approx 0$$

$\{v_p^i\}_{p=1}^{\dim q^{u_i}}$ basis of centraliser $q^{u_i} = \ker(\text{ad } u_i)$.

Can fix $\mathcal{E}_i = 0$ by introducing a Dirac bracket

$$\{\hat{u}_{i_1}, \hat{u}_{j_2}\}^* = -[C_{12}, \hat{u}_{i_2}] \delta_{ij} \quad i, j = 1, \dots, N.$$

↙ Kostant-Kirillov bracket on $\prod_{i=1}^N O_{u_i}$!

3d BF w/ defects - Hamiltonian

Remaining primary constraint

$$\Pi_t \approx 0.$$

Hamiltonian $H = \int_{\mathbb{CP}^1} \langle \hat{\mu}, A_t \rangle dz \wedge d\bar{z} + P(B_z(w))$

$$\hat{\mu} := \frac{1}{2\pi i} \left(\partial_{\bar{z}} B_z + [A_{\bar{z}}, B_z] \right) + \sum_{j=1}^N \hat{u}_j \delta_{zz_j}$$

Secondary constraint:

$$\partial_t \Pi_t = \{H, \Pi_t\} = \hat{\mu} \approx 0$$

3d BF w/ defects - gauge fixing

$\hat{\mu} \approx 0$ is first class (generates gauge symmetry)

$$\left\{ \hat{\mu}_1(z), \hat{\mu}_2(z') \right\} = - [C_{12}, \hat{\mu}_2(z)] \delta_{zz'} \approx 0.$$

Impose gauge fixing condition $A_{\bar{z}} \approx 0$.

Have $\left\{ \hat{\mu}_1(z), A_{\bar{z}2}(z') \right\} = - \partial_{\bar{z}} (C_{12} \delta_{zz'})$,

& $\left\langle \left\langle -\partial_{\bar{z}} (C_{12} \delta_{zz'}), \frac{1}{2\pi i} \frac{C_{23}}{z' - z''} \right\rangle \right\rangle_{(2,z')} = C_{13} \delta_{zz''}$.

3d BF w/ defects - Dirac bracket for $\hat{\mu}, A_{\bar{z}}$

Dirac bracket

$$\left\{ U_1(z), V_2(z') \right\}^{\star} := \left\{ U_1(z), V_2(z') \right\}$$

$$\langle\langle \cdot, \cdot \rangle\rangle_{(i,z)} := \int \langle \cdot, \cdot \rangle_i dz \wedge dz$$

$\xrightarrow{(\mathbb{CP}^1)}$

$$-\langle\langle \left\{ U_1(z), \hat{\mu}_3(z'') \right\}, \langle\langle \frac{1}{2\pi i} \frac{C_{34}}{z''-z'''}, \left\{ A_{\bar{z}4}(z'''), V_2(z') \right\} \rangle\rangle_{(4,z''')} \rangle\rangle_{(3,z'')}$$
$$-\langle\langle \left\{ U_1(z), A_{\bar{z}3}(z'') \right\}, \langle\langle \frac{1}{2\pi i} \frac{C_{34}}{z''-z'''}, \left\{ \hat{\mu}_4(z'''), V_2(z') \right\} \rangle\rangle_{(4,z''')} \rangle\rangle_{(3,z'')}$$

3d BF w/ defects - Lax algebra

Dirac bracket

$$\{B_{z_1}(z), B_{z_2}(z')\}^{\star} = \left[\frac{C_{12}}{z' - z}, B_{z_1}(z) + B_{z_2}(z') \right].$$

cf. $P_{A,B} := \bar{\partial}_A \Phi$

Now $\hat{\mu} = 0$ & $A_{\bar{z}} = 0$ ($\hat{\mu} := \underbrace{\frac{1}{2\pi i} (\partial_{\bar{z}} B_z + [A_{\bar{z}}, B_z])}_{\text{cf. } P_{A,B}} + \sum_{j=1}^N \hat{u}_j \delta_{zz_j}$):

- **Hamiltonian** $H = P(B_z(w))$

- $B_z(z) = \sum_{j=1}^N \frac{\hat{u}_j}{z - z_j}$

3d BF w/ defects – involution property

For polynomials $P, Q : \mathfrak{g} \rightarrow \mathbb{C}$, clearly have

$$\{P(B_z(z)), Q(B_z(z'))\} = 0$$

For adjoint-invariant P, Q have

$$\{\hat{r}(z), P(B_z(z'))\} = 0 = \{\hat{r}(z), Q(B_z(z'))\}$$

\Rightarrow **involution** $\{P(B_z(z)), Q(B_z(z'))\}^* = 0$.

Conclusion & outlook

- Establish dictionary

$$\left\{ \begin{array}{l} \text{"Type A" Line defects} \\ \text{in 3d BF theory} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Realisations of} \\ \text{Gaudin model} \end{array} \right\}$$

e.g. in this talk:

$$\int_{\mathbb{R} \times \{z_j\}} \langle u_j, h_j^{-1} (\partial_t + A_t) h_j \rangle dt \leftrightarrow$$

$$\mathcal{O}_{u_j} \text{ realisation of} \\ \{ J_1^{(i)}, J_2^{(i)} \} = - [C_{12}, J_2^{(i)}]$$

$$(J^{(i)} \mapsto \hat{u}_i)$$

Conclusion & outlook

- Generalised Gaudin models :

- Irregular singularities

$$L(z) = \sum_{i=1}^N \sum_{r=0}^{n_i} \frac{J_{[r]}^{(i)}}{(z - z_i)^{r+1}}$$

↔ "thickened" type A
Line defects ?

[Benini - Schenkel - BV'20]

[Lacroix - BV'20]

- Γ -equivariance

$$L(z) = \sum_{i=1}^N \sum_{g \in \Gamma} \frac{g \cdot J^{(i)}}{z - g \cdot z_i}$$

↔ 3d BF on orbifold
 $\mathbb{R} \times \mathbb{C}\mathbb{P}^1/\Gamma$.

[Bittleston - Skinner '19]
[Schmidtt '21]

- ...

Conclusion & outlook

- Quantum Gaudin model very well understood
[Beilinson-Drinfeld, Feigin-Frenkel-Reshetikhin,
Mukhin-Tarasov-Varchenko, Rybnikov, ...]

Based on representation theory of $\tilde{\mathfrak{g}}$ at $k = -h^*$.

Classically,

$$S_{\text{3dCS}}[A] = \frac{k}{4\pi i} \int_{\mathbb{R} \times \mathbb{C}\mathbb{P}^1} CS(A) \xrightarrow[k \rightarrow 0]{(A_z = \frac{1}{k} B_z)} S_{\text{3dBf}}[A, B]$$

Conclusion & outlook

Expect [Gaiotto-Witten'21, Zeng'21]

Quantum
 $k = -h^\vee$ 3d CS

on $\mathbb{R} \times \mathbb{C}\mathbb{P}^1$

gauge fixing
+ line defects

Quantum
Gaudin model

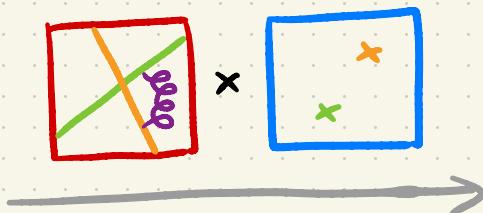
quantum
realisation

Quantum
Integrable System

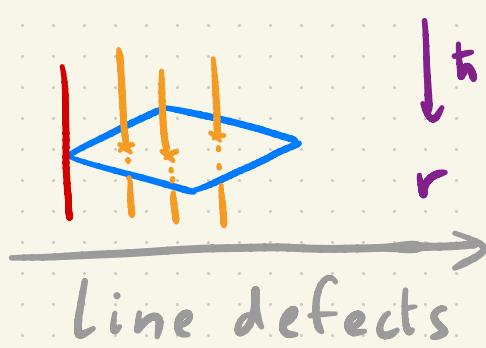
Conclusion & outlook

- Originally, [Costello '13, Costello-Witten - Yamazaki '17 '18]

4d CS
on $\mathbb{R}^2 \times \mathbb{C}\mathbb{P}^1$
?



Integrable
Lattice models
 $R = 1 + \hbar r + \dots$ e.g. Heisenberg
spin chain



Quantum
Gaudin model

Thank you!