Segmented strings in AdS_3

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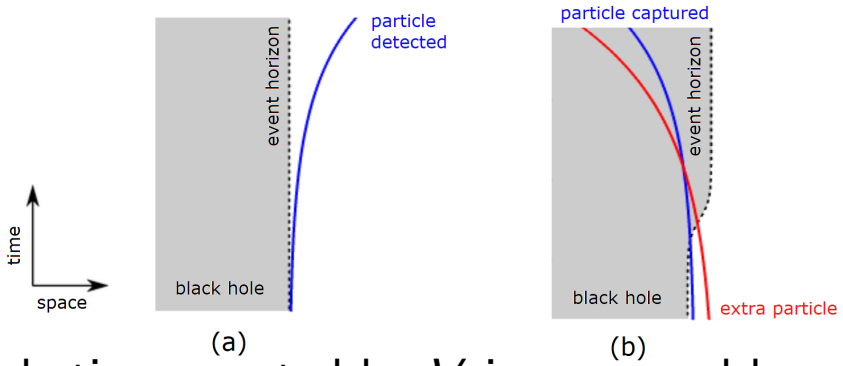
Motivation for studying strings in AdS

- String worldsheet as a **toy model for quantum gravity**
  Dubovsky-Flauger-Gorbenko 2012



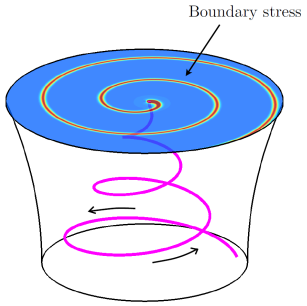
Shenker-Stanford 2013
sensitivity of W to perturbations created by V is measured by out-of-time order correlation functions

$$C(t,\vec{x}) = -\langle [W(t,\vec{x}), V(0,0)]^2 \rangle_T \sim e^{\lambda_L (t - |\vec{x}|/v_B)}$$

λ_L: Lyapunov exponent $\qquad$ v_B: butterfly velocity

universal bound: $\lambda_L \leq 2\pi T$ Maldacena-Shenker-Stanford 2015

- Lyapunov exponent for the string saturates the universal bound: $\lambda_L = 2\pi T$
  de Boer-Llabres-Pedraza-DV 2017



- **AdS/CFT in the planar limit**
- Applications of AdS/CFT
- Wilson lines, scattering amplitudes

Segmented strings in AdS_3

Segmented strings in AdS_3

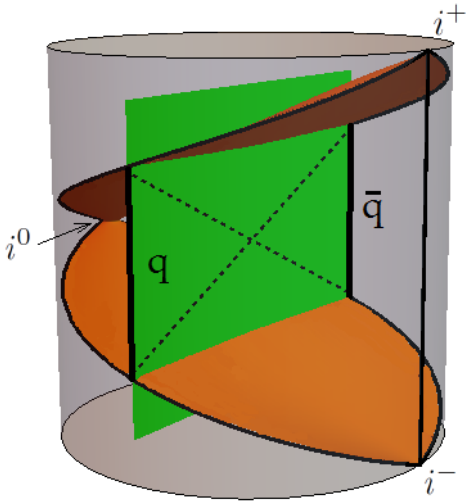
- $AdS_3 \subset \mathbb{R}^{2,2}$
- normal vector of the string:

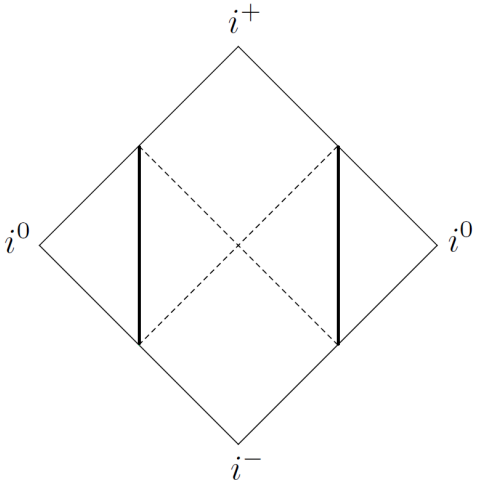
$$N_a \propto \epsilon_{abcd} Y^b \partial Y^c \bar{\partial} Y^d \qquad Y \in \mathbb{R}^{2,2}$$

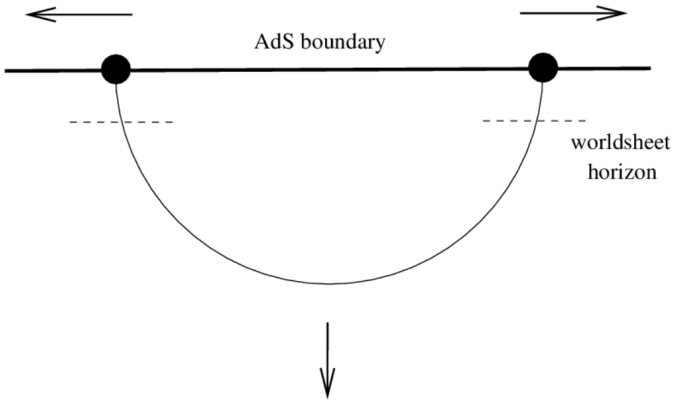
- **Idea: glue segments with constant normal vectors**
  DV 2015 Callebaut-Gubser-Samberg-Toldo 2015

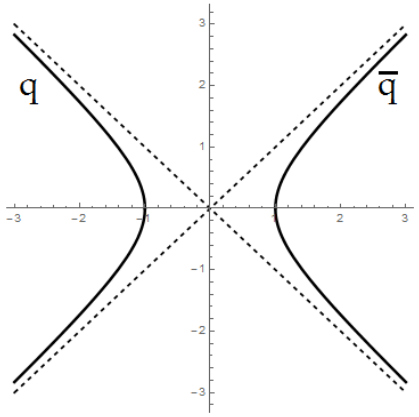
- This generalizes piecewise linear strings in flat space

Elementary segment



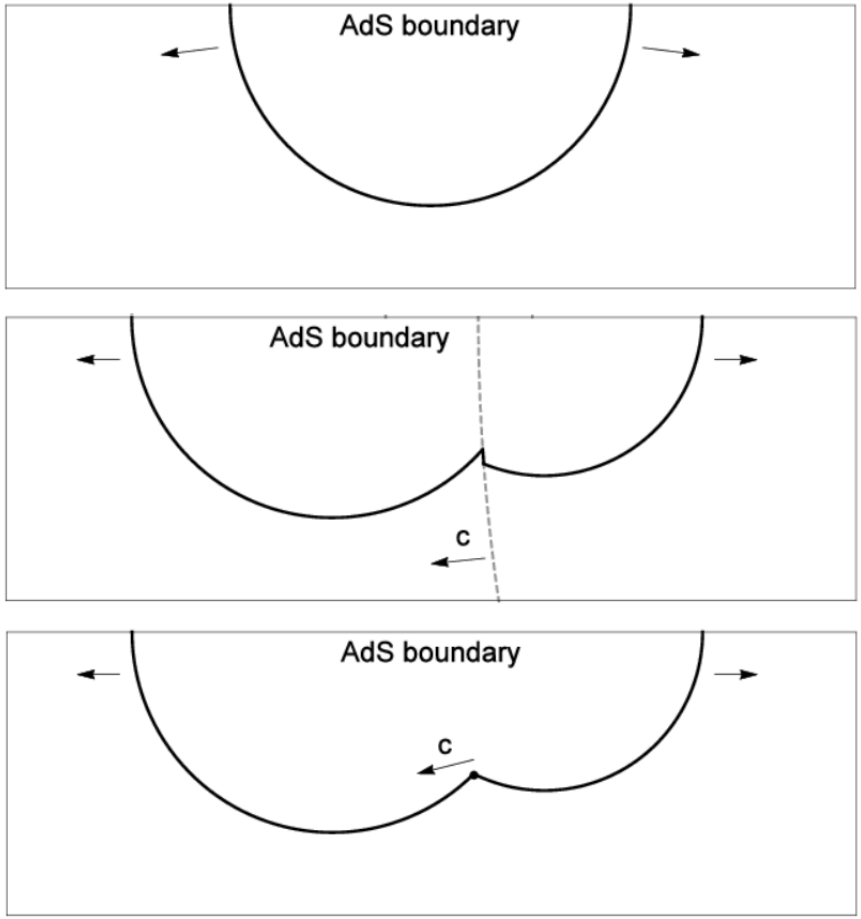






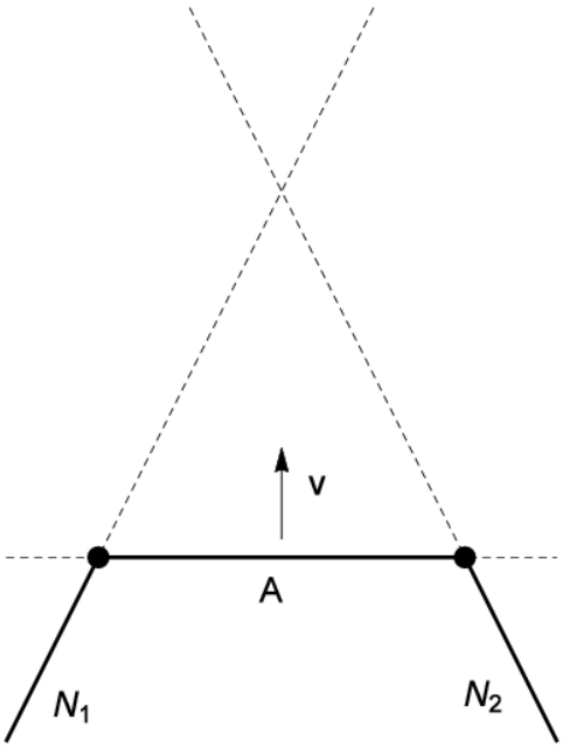
Connecting segments

- 3-parameter family of string embeddings



- normal vectors must satisfy: $\vec{N}_L \cdot \vec{N}_R = 1$.

Colliding cusps



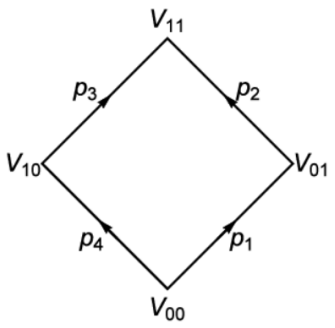
- normal vector after the collision:

$$\boxed{\vec{A}' = -\vec{A} + (\vec{A} \cdot \vec{N})\vec{N} \qquad \text{with} \quad \vec{N} = \sqrt{2}\,\frac{\vec{N}_1 + \vec{N}_2}{|\vec{N}_1 + \vec{N}_2|}}$$

Examples

Segmented strings and celestial variables

Compute the string action



- Mandelstam variables: $s = (p_1 + p_2)^2$ and $u = (p_1 - p_4)^2$

$$S = Area = \log\left[u/s\right]^2$$

- Since $p^2 = \det(p_{a\dot{a}}) = 0$, we can write $\sigma^{\mu}_{a\dot{a}} p_\mu = \lambda_a \tilde{\lambda}_{\dot{a}}$, where $\sigma^\mu = (1, -i\sigma_2, \sigma_1, \sigma_3)$

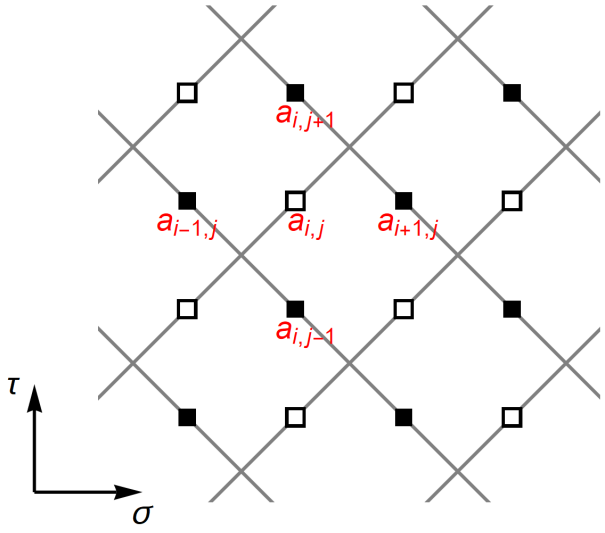
$$S = 2\log\left|\frac{\langle\lambda_1, \lambda_4\rangle\langle\lambda_2, \lambda_3\rangle}{\langle\lambda_1, \lambda_2\rangle\langle\lambda_3, \lambda_4\rangle}\right|$$

Now define the **celestial variable** $a := \frac{\lambda_1}{\lambda_2} = \frac{p_{-1}+p_2}{p_0+p_1}$.

$$Area = 2\log\left|\frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_2)(a_3 - a_4)}\right|$$

- summary: **vertices** $\rightarrow$ **difference vectors** $\rightarrow$ **spinors** $\rightarrow$ **celestial variables**

Discrete integrable system

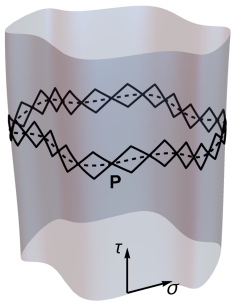


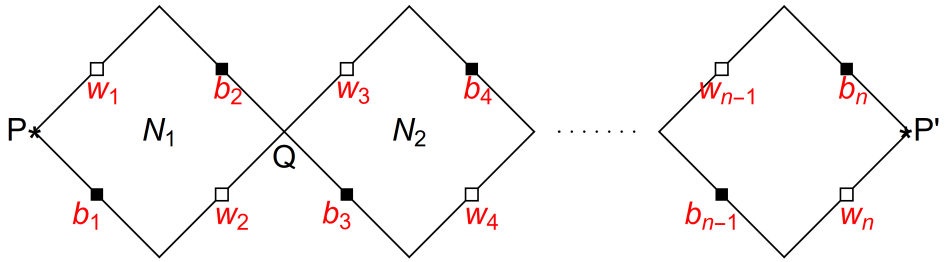
In terms of the celestial variables, the string EOM is simply

$$\frac{1}{a_{ij} - a_{i,j+1}} + \frac{1}{a_{ij} - a_{i,j-1}} = \frac{1}{a_{ij} - a_{i+1,j}} + \frac{1}{a_{ij} - a_{i-1,j}}$$

DV 2016

Reconstructing the string embedding





- Define the 'reflection matrix' DV 2019

$$\mathcal{R}_{b,w} = \frac{1}{b-w}\begin{pmatrix} 0 & bw+1 & bw-1 & -b-w \\ -1-bw & 0 & b+w & bw-1 \\ -1+bw & b+w & 0 & -1-bw \\ -b-w & bw-1 & bw+1 & 0 \end{pmatrix}$$

- Normal vectors and vertices can be computed from the celestial variables

$$N_1 = \mathcal{R}_{b_1,w_1} P$$

$$Q = \mathcal{R}_{b_2,w_2} N_1$$

$$etc.$$

The spectral curve

Lax matrices

- Pohlmeyer reduction

$$e^{2\alpha(z,\bar{z})} = \frac{1}{2}\partial Y \cdot \bar{\partial} Y \qquad u(z) = \frac{1}{2}\partial N \cdot \partial Y \qquad v(\bar{z}) = -\frac{1}{2}\bar{\partial} N \cdot \bar{\partial} Y$$

- α satisfies the generalized sinh-Gordon equation

$$\partial\bar{\partial}\alpha - e^{2\alpha} + u(z)v(\bar{z})e^{-2\alpha} = 0$$

- Lax matrices with spectral parameter ζ

$$B_z = \begin{pmatrix} \frac{1}{2}\partial\alpha & -\frac{1}{\zeta}e^{\alpha} \\ -\frac{u}{\zeta}e^{-\alpha} & -\frac{1}{2}\partial\alpha \end{pmatrix}, \quad B_{\bar{z}} = \begin{pmatrix} -\frac{1}{2}\bar{\partial}\alpha & -\zeta v e^{-\alpha} \\ -\zeta e^{\alpha} & \frac{1}{2}\bar{\partial}\alpha \end{pmatrix}.$$

- Solutions ($\psi_{\alpha,a}$, where $a = 1, 2$) to the linear problem

$$\partial\psi_\alpha + (B_z)_\alpha^{\ \beta}\psi_\beta = 0\,, \qquad \bar{\partial}\psi_\alpha + (B_{\bar{z}})_\alpha^{\ \beta}\psi_\beta = 0$$

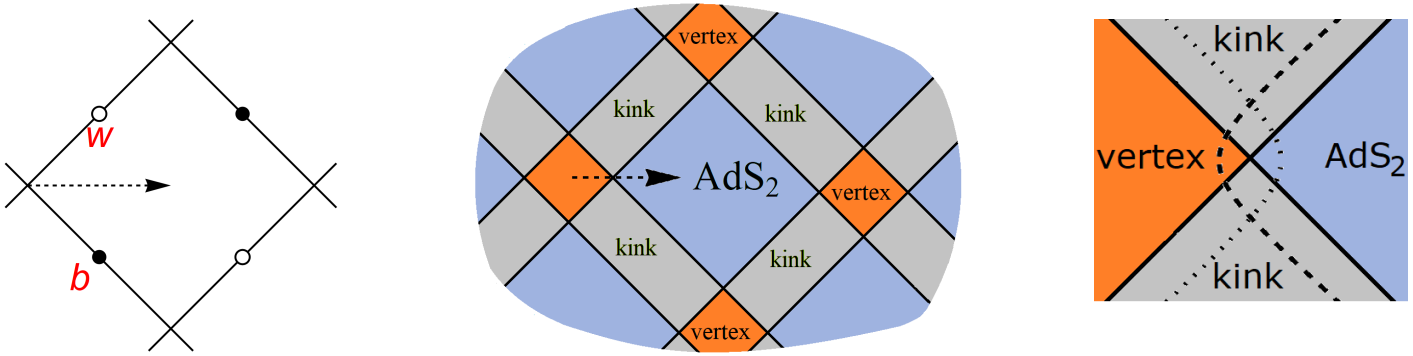
  suffer an $\Omega \in SL(2)$ monodromy around the loop of a closed string.

- The string embedding is obtained from

$$Y_{a\dot{a}} = \psi^L_{\alpha,a} M^{\alpha\dot{\beta}} \psi^R_{\dot{\beta},\dot{a}}\,, \qquad M^{\alpha\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.1}$$

  Alday-Maldacena 2009

Matching spinor solutions

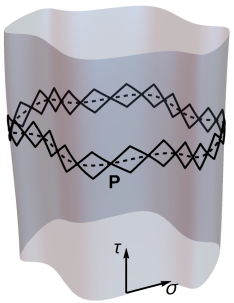


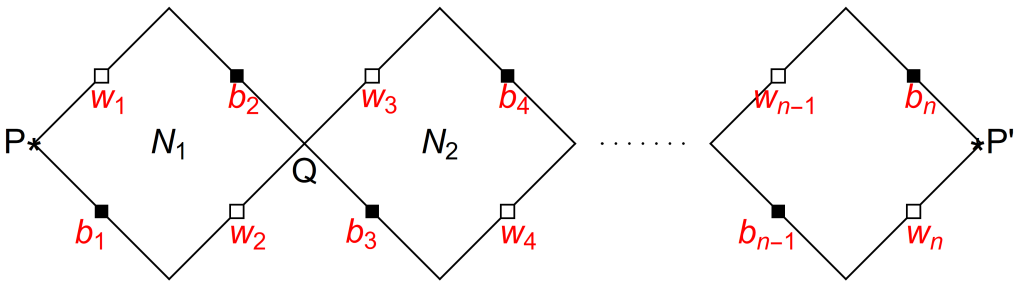
- Match solutions in the overlapping region (matched asymptotic expansion)
- Lax matrix DV 2021

$$\Omega_{b,w}(\zeta) = \frac{1}{(b-w)\zeta}\begin{pmatrix} b\zeta^2 - w & bw(1-\zeta^2) \\ \zeta^2 - 1 & b - w\zeta^2 \end{pmatrix}$$

$$\det \Omega_{b,w} = 1\,, \qquad \Omega_{b,w}(\zeta = 1) = \mathbb{1}$$

Monodromy





- Suppose there are N segments in a closed string. The monodromy is given by

$$\Omega(\zeta) = \Omega^{-1}_{b_N, w_N} \Omega_{b_{N-1}, w_{N-1}} \cdots \Omega^{-1}_{b_2, w_2} \Omega_{b_1, w_1}$$

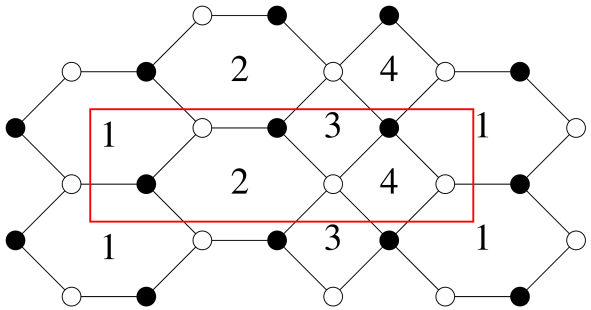
- For generic values of b_i, w_j, the string will not close in AdS_3.
  Closure is equivalent to demanding

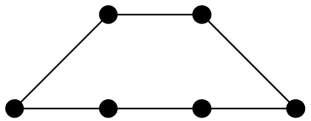
$$\Omega(\zeta = i) = \mathbb{1}$$

  This gives 3 constraints on the celestial variables.

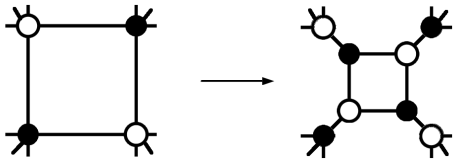
The spectral curve and brane tilings

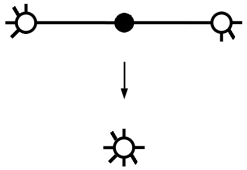
Brane tilings



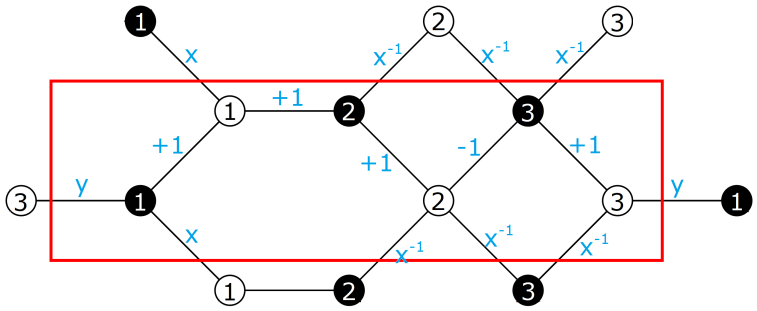


- **brane tiling:** doubly-periodic bipartite graph
- its dual graph is the quiver
  (in the context of 4d $\mathcal{N} = 1$ theories, the tiling also encodes the superpotential Franco-Hanany-Kennaway-DV-Wecht 2005)
- related to on-shell diagrams
  Arkani-Hamed-Bourjaily-Cachazo-Goncharov-Postnikov-Trnka 2012
- Newton polygon of a Laurent polynomial (toric diagram of CY threefold)
- invariant under the transformations





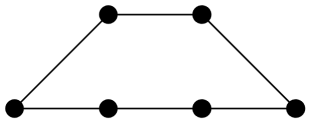
Kasteleyn matrix



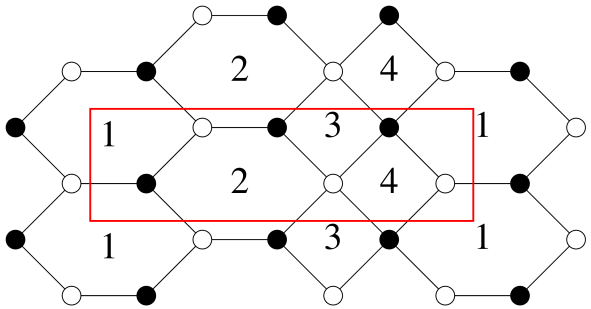
- brane tiling dressed with edge weights
- white and black vertices label the rows and columns of the Kasteleyn matrix

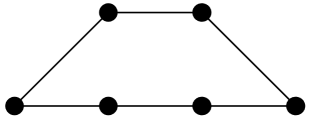
$$K = \begin{pmatrix} 1+x & 1 & 0 \\ 0 & 1+x^{-1} & -1+x^{-1} \\ y & 0 & 1+x^{-1} \end{pmatrix} \tag{2.2}$$

$$\det K = 3 + x^{-2} + 3x^{-1} + x - y + x^{-1}y \tag{2.3}$$



Associated cluster integrable system



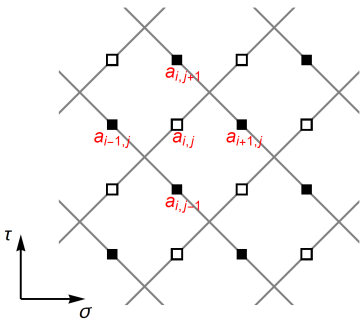


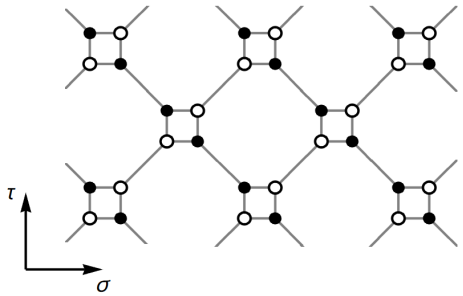
- A (quantum) integrable model is associated to the brane tiling Goncharov-Kenyon 2011
- Dynamical variables = oriented edge loops in the tiling
- a basis: f_i faces + two non-trivial loops f_x, f_y
- Poisson brackets

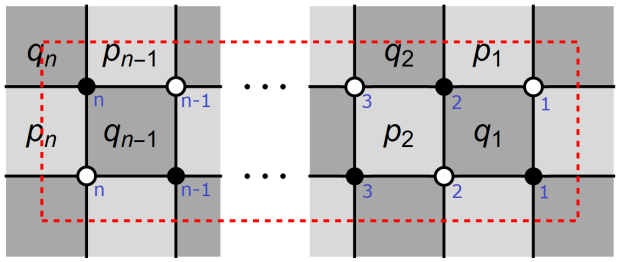
$$
\begin{aligned}
\{f_i, f_j\} &= \epsilon_{f_i,f_j}\, f_i f_j \\
\{f_x, f_y\} &= \left(\langle f_x, f_y\rangle + \epsilon_{f_x,f_y}\right) f_x f_y \\
\{f_x, f_i\} &= \epsilon_{f_x,f_i}\, f_x f_i \\
\{f_y, f_i\} &= \epsilon_{f_y,f_i}\, f_y f_i\,.
\end{aligned}
$$

here $\epsilon_{x,y}$ is the number of edges appearing in both loops x and y (counted with orientation) and $\langle f_x, f_y\rangle$ is the intersection number of the two loops

$Y^{n,0}$ tilings







- cross-ratio

$$(a,\ b;\ c,\ d) \equiv \frac{(a-b)(c-d)}{(a-d)(b-c)}$$

$$p_{i,j} = (a_{i+1,j},\ a_{i+1,j+1};\ a_{i+2,j},\ a_{i+2,j+1})$$

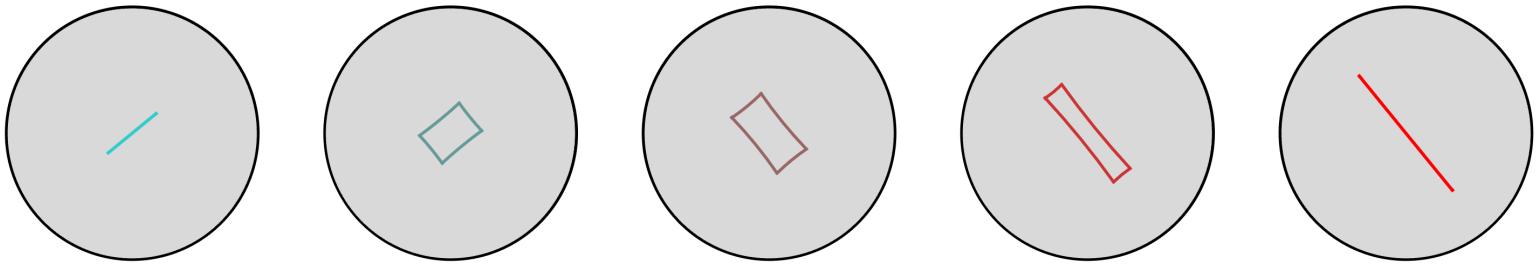
- performing cluster transformations on all faces gives DV 2021

$$q_{i,j+1} = \frac{1}{p_{i,j}}\ , \qquad p_{i,j+1} = q_{i,j}\, p_{i,j}^2 \frac{(1+p_{i-1,j})(1+p_{i+1,j})}{(1+p_{i,j})^2}$$

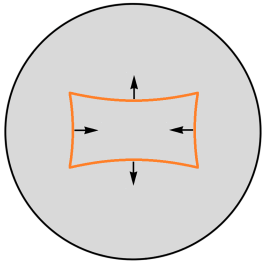
compatible with the EOM: $\dfrac{1}{a_{ij}-a_{i,j+1}} + \dfrac{1}{a_{ij}-a_{i,j-1}} = \dfrac{1}{a_{ij}-a_{i+1,j}} + \dfrac{1}{a_{ij}-a_{i-1,j}}$

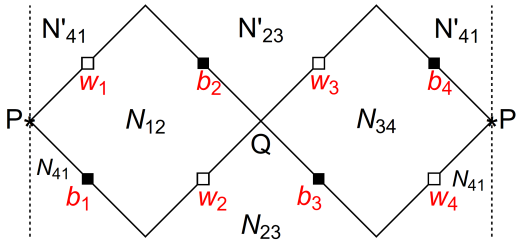
- $\det \tilde{K} = 0$ gives the spectral curve, where $\tilde{K}(x,y)$ is the dressed Kasteleyn matrix

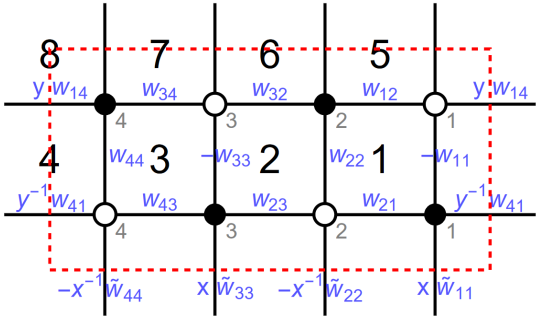
An example: string with four segments



Example: string with four segments







- The spectral curve can be computed from (i) the monodromy $\det(y\mathbb{1} + \Omega(\zeta)) = 0$

$$\Omega(\zeta) = \Omega^{-1}_{b_4,w_4}\Omega_{b_3,w_3}\Omega^{-1}_{b_2,w_2}\Omega_{b_1,w_1}$$

  or (ii) from $\det \tilde{K} = 0$. In terms of the energy Δ we get

$$y + y^{-1} - \frac{\Delta^2}{16}(x^2 + x^{-2}) + 2 + \frac{\Delta^2}{8} = 0$$

- Define new variables

$$x = \frac{\check{x} - 1}{\check{x} + 1} \qquad u = (\check{x} + \check{x}^{-1})g \qquad y = e^p$$

$$e^p + e^{-p} + 2 - \frac{\Delta^2 g^2}{u^2 - 4g^2} = 0$$

  p and u are canonically conjugate variables Dorey-Vicedo 2006

- Ruijsenaars-Schneider-type Hamiltonian

$$H = \Delta = 2g^{-1}\cosh\left(\frac{p}{2}\right)\sqrt{u^2 - 4g^2}$$

An observation

$$e^p + e^{-p} + 2 - \frac{\Delta^2 g^2}{(u+2g)(u-2g)} = 0$$

- Spectral curve does not depend on the squashing parameter
- **Squash the string** (folded string) & go to **flat space limit**
- In this limit, a quantization is given by the 't Hooft equation 't Hooft 1974

$$2\pi^2\Delta^2\varphi(x) = \left[\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}\right]\varphi(x) - \fint_0^1 dy\,\frac{\varphi(y)}{(y-x)^2} \qquad (x \equiv p_1/P)$$

- Switch to rapidity coordinates $\theta = \frac{1}{2}\log\frac{x}{1-x}$ and Fourier transform $\varphi(\theta) \to \varphi(\nu)$
  Fateev-Lukyanov-Zamolodchikov 2009 Brower-Spence-Weis 1979

$$Q(\nu) = \nu\cosh\left(\frac{\pi\nu}{2}\right)\varphi(\nu)$$

$$Q(\nu+2i) + Q(\nu-2i) - 2Q(\nu) = -4\pi\Delta^2\frac{\tanh(\frac{\pi\nu}{2})}{\nu}Q(\nu)$$

- Compare this equation with the classical one where we **cut out the** $-2g < u < 2g$ **region**:

$$e^p + e^{-p} + 2 - \frac{\Delta^2 g}{4|\tilde{u}|} = 0 \qquad \tilde{u} = \begin{cases} u - 2g & u > 2g \\ u + 2g & u < -2g \end{cases}$$

Summary

- We studied a bosonic string in AdS_3 spacetime
- Exact discretization $\rightarrow$ finite number of DoF
- We computed the spectral curve using brane tiling technology

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- Quantized segmented strings?
- Extension to $\mathrm{AdS}_5 \times S^5$?
- Holographic dual?