On generally covariant mathematical formulation of Feynman integral in Lorentz signature

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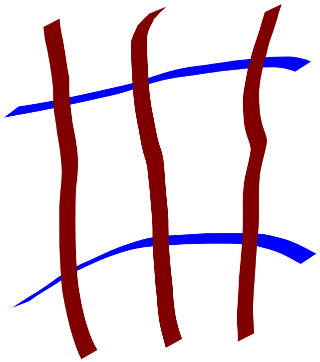
Outline

- Structure of model building in fundamental physics.
- Model building attempts in QFT.
- Heuristic Feynman integral formulation.
- Classical field theory.
- Rigorous form of master Dyson-Schwinger (MDS) equation.
- Existence condition for regularized MDS solutions.
- Outlook: tentative rigorous definition for the Wilsonian renormalization.
- Summary.

Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Space of solutions turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space.



Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Space of finite charge weak solutions turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space.

Most important ingredient: a one-liner, the equation of motion or field equation.
$\longrightarrow$ Then, one is working on the solution space.

Can one find a one-liner equation to summarize Quantum Field Theory?

Model building attempts in QFT

Common QFT formalisms in physics:

- Often non-manifestly covariant formalism.
  (Hamiltonian, reminescents of non-relativistic QM as seen by an inertial observer.)
- In momentum space.
- Splitting Lagrangian to free + interacting terms.
- Often perturbative handling.
- Need for regularization and renormalization. (What this is precisely?)
- Not easy to see what is legitimate and what is not.
- In some cases the “right” thing is done, even without the adequate formalism.

Common formalisms in mathematical QFT:

- Loop quantum gravity. (Spacetime is emergent, but far from finalized.)
- Algebraic QFT: easy to understand math/physics concept, but no known 3+1d example.
- Perturbative AQFT formalized over fixed spacetime, and known examples.
  (Still cannot put down a one-liner.)
- Feynman integral in Wick rotated signature one can. But still free + interaction splitting.

Our guidelines:

- Do not assume spacetime manifold to be affine space. (Spacetime not flat.)
- Do not refer to a fixed spacetime metric. (Not even to a fixed causal structure.)
- Only refer to underlying spacetime manifold and generic fields over it.
  (Metric is not distinguished field – but spacetime manifold is usual 4d continuum.)
- Do not refer to a known splitting of Lagrangian to free + interaction parts.
  (No canonical way to split e.g. Yang-Mills or Einstein-Hilbert Lagrangian.)

Some consequences:

- Cannot go to momentum space, have to stay in spacetime description.
- Cannot refer to any affine property of Minkowski spacetime, e.g. asymptotics.
  (No notion of space $\mathcal{S}$ of rapidly decreasing functions.)
- Cannot use Wick rotation to Euclidean signature metric.
- Even if Wick rotated, no free + interaction splitting, so no Gaussian Feynman measure.

We revisit Feynman integral formulation in Lorentz signature.
[We conclude that its differential reformulation can be well defined.]

Heuristic Feynman integral formulation

Fix some $\psi_0 \in F$ reference field for transforming the problem $F \to \mathbb{F}$.
Let $J_1, \ldots, J_n \in \mathbb{F}^*$ be test functionals.

Then, Feynman type quantum expectation value of polynomial observable $(J_1| \cdot - \psi_0) \cdot \ldots \cdot (J_n| \cdot - \psi_0) : \; F \to \mathbb{R}$ in vacuum state ρ postulated as:

$$\int\limits_{\psi \in F} (J_1|\psi - \psi_0) \cdot \ldots \cdot (J_n|\psi - \psi_0) \;\; \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\psi)} \, [\mathrm{d}\psi]_\rho \;\; \Big/ \;\; \int\limits_{\psi \in F} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\psi)} \, [\mathrm{d}\psi]_\rho$$

Partition function is often invoked to book-keep all this (Fourier transform of $\mathrm{e}^{\mathrm{i} S(\psi)} \, [\mathrm{d}\psi]_\rho$):

$$Z_{\psi_0,\rho} : \quad \mathbb{F}^* \longrightarrow \mathbb{C}, \quad J \longmapsto Z_{\psi_0,\rho}(J) := \int\limits_{\psi \in F} \mathrm{e}^{\mathrm{i}\,(J|\psi - \psi_0)} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\psi)} \, [\mathrm{d}\psi]_\rho,$$

and from this one can define

$$G^{(n)}_{\psi_0,\rho} \quad := \quad \left((-\mathrm{i})^n \, \frac{1}{Z_{\psi_0,\rho}(J)} \, D^{(n)} Z_{\psi_0,\rho}(J) \right) \Big|_{J=0}$$

n-field correlator, and their collection $G_{\psi_0,\rho} := \left(G^{(0)}_{\psi_0,\rho}, G^{(1)}_{\psi_0,\rho}, \ldots, G^{(n)}_{\psi_0,\rho}, \ldots \right) \in \bigoplus_{n \in \mathbb{N}_0} \overset{n}{\otimes} \mathbb{F}$.

Above Feynman type quantum expectation value expressable as:

$$\left(\, J_1 \otimes \ldots \otimes J_n \; \middle| \; G^{(n)}_{\psi_0,\rho} \, \right)$$

Problem: no “Lebesgue” measure $[\mathrm{d}\psi]_\rho$ in infinite dimensions.

Neither $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)}[\mathrm{d}\psi]_\rho$ is meaningful. (Can be given some meaning in Euclidean signature.)

Neither formal Fourier transformation of this undefined measure is meaningful.

In usual QFT literature, $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)}[\mathrm{d}\psi]_\rho$ is handled as if it existed
as finite measure, with finite moments and analytic Fourier transform.

Formally playing with Fourier transform, one infers:

$Z:\,\mathbb{F}^*\to\mathbb{C}$ Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)}[\mathrm{d}\psi]_\rho$ “$\Leftrightarrow$” satisfies master-Dyson-Schwinger eq:

$$\Big(\,\mathbf{E}\big((-\mathrm{i})D_J+\psi_0\big)\;+\;\hbar\,J\,\Big)Z(J)\quad=\quad 0\quad(\forall J\in\mathbb{F}^*)$$

with $E(\psi):=DS(\psi)$ being the Euler-Lagrange functional at $\psi\in F$.
Looks kind of weird to interpret beyond formality. Does it have a meaning?
Yes, when expressed via field correlators $G=\big(G^{(0)},G^{(1)},...,G^{(n)},...\big)$.

Classical field theory

Let $\mathcal{M}$ be a smooth manifold (wannabe spacetime, but no metric, yet).

Let $V(\mathcal{M})$ be vector bundle over it (its smooth sections are matter fields – also metric).

On this, one has the covariant derivation operators, they form a $DV(\mathcal{M})$ affine bundle.
(Mediator fields – gauge fields.) Affine bundle over $T^*(\mathcal{M})\otimes V(\mathcal{M})\otimes V^*(\mathcal{M}) =: CV(\mathcal{M})$.

In total:
$\underbrace{(v,\nabla)}_{=:\psi} \in \underbrace{\Gamma\big(V(\mathcal{M})\times_{\mathcal{M}} DV(\mathcal{M})\big)}_{=:F}$ is a field configuration.
These with the $\mathcal{E}$ smooth function topology form a real topological affine space.

$\underbrace{(\delta v,\delta C)}_{=:\delta\psi} \in \underbrace{\Gamma\big(V(\mathcal{M})\times_{\mathcal{M}} CV(\mathcal{M})\big)}_{=:\mathbb{F}}$ is a field variation (difference of two field configurations).
These with the $\mathcal{E}$ smooth function topology form a real topological vector space.

Let a Lagrange form be given, which is

$$\mathrm{L}:\ V(\mathcal{M})\ \oplus\ T^*(\mathcal{M})\otimes V(\mathcal{M})\ \oplus\ T^*(\mathcal{M})\wedge T^*(\mathcal{M})\otimes V(\mathcal{M})\otimes V^*(\mathcal{M})\ \longrightarrow\ \overset{\dim(\mathcal{M})}{\wedge}T^*(\mathcal{M})$$

pointwise vector bundle homomorphism.

Lagrangian expression:

$$\Gamma\big(V(\mathcal{M})\times_{\mathcal{M}} DV(\mathcal{M})\big)\ \longrightarrow\ \Gamma\big(\overset{\dim(\mathcal{M})}{\wedge}T^*(\mathcal{M})\big),\quad (v,\nabla)\ \longmapsto\ \mathrm{L}(v,\nabla v,P(\nabla))$$

where $P(\nabla)$ is the curvature tensor.

Action functional:

$$S:\quad \underbrace{\Gamma\big(V(\mathcal{M})\times_{\mathcal{M}} DV(\mathcal{M})\big)}_{=:F}\ \longrightarrow\ \mathrm{Rad}(\mathcal{M},\mathbb{R}),\quad \underbrace{(v,\nabla)}_{=:\psi}\ \longmapsto\ \Big(\mathcal{K}\mapsto S_{\mathcal{K}}(v,\nabla)\Big)$$

where $S_{\mathcal{K}}(v,\nabla):=\int\limits_{\mathcal{K}}\mathrm{L}(v,\nabla v,P(\nabla))$ for all $\mathcal{K}\subset\mathcal{M}$ compact.

[Achtung: $S_{\mathcal{M}}(v,\nabla)$ generally not finite, e.g. already for stationary fields etc.]

Action functional $S : F \to \mathrm{Rad}(\mathcal{M}, \mathbb{R})$ Fréchet differentiable, its Fréchet derivative

$$DS : \quad F \times \mathbb{F} \longrightarrow \mathrm{Rad}(\mathcal{M}, \mathbb{R}), \quad (\psi, \delta\psi) \longmapsto \Big(\mathcal{K} \mapsto \big(DS_{\mathcal{K}}(\psi) \,\big|\, \delta\psi\big)\Big)$$

is the usual Euler-Lagrange integral on $\mathcal{K}$ $+$ usual boundary integral on $\partial\mathcal{K}$.
Jointly continuous in its variables, linear in second variable.

Let $\mathbb{F}_T$ be the compactly supported field variations from $\mathbb{F}$ with usual $\mathcal{D}$ test function topology.
(space of test field variations)

Euler-Lagrange functional:
We restrict DS in its second variable to $\mathbb{F}_T$, to make the EL integral over full $\mathcal{M}$ finite.

$$E : \quad F \times \mathbb{F}_T \longrightarrow \mathbb{R}, \quad (\psi,\, \delta\psi_T) \longmapsto \big(E(\psi) \,\big|\, \delta\psi_T\big) := \big(DS_{\mathcal{M}}(\psi) \,\big|\, \delta\psi_T\big)$$

Bulk Euler-Lagrange integral remains, no boundary term. Meaningful on full $\mathcal{M}$, real valued.
Jointly sequentially continuous, linear in second variable. (Also, $E : F \to \mathbb{F}_T^*$ continuous.)

The one-liner (field equation):

$$\psi \in F\,? \qquad \forall\, \delta\psi_T \in \mathbb{F}_T : \; \big(E(\psi) \,\big|\, \delta\psi_T\big) = 0.$$

Observables are the $O : F \to \mathbb{R}$ continuous maps.

Example: φ^4 theory.

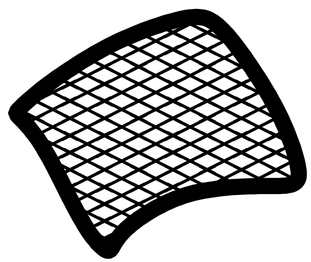
$\mathcal{M}$ is Minkowski spacetime, v is volume measure, $\Box$ is wave operator.

$F := \mathbb{F} := C^\infty(\mathcal{M},\mathbb{R})$ and $\mathbb{F}_T := C^\infty_c(\mathcal{M},\mathbb{R})$.

Euler-Lagrange functional is

$$E: \quad F\times\mathbb{F}_T \longrightarrow \mathbb{R}, \quad (\psi,\,\delta\psi_T)\longmapsto \int_{\mathcal{M}} \delta\psi_T\,\Box\psi\,\mathrm{v} \;+\; \int_{\mathcal{M}} \delta\psi_T\,\psi^3\,\mathrm{v}.$$

Field equation selects physically realizable fields over spacetime manifold.



$\mathcal{M}$

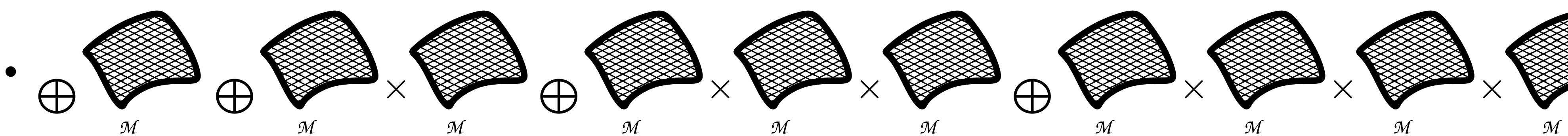
Rigorous form of the master Dyson-Schwinger equation

The fundamental objects of interest is the collection of n-field correlators.

They sit in the tensor algebra $\mathcal{T}(\mathbb{F}) := \bigoplus_{n\in\mathbb{N}_0} \overset{n}{\otimes}\,\mathbb{F}$ of field variations.

More precisely, they sit in a graded-symmetrized subspace, e.g. $\bigvee(\mathbb{F})$ or $\bigwedge(\mathbb{F})$ of $\mathcal{T}(\mathbb{F})$.

They are tuples $G = \big(G^{(0)}, G^{(1)}, G^{(2)}, ..., G^{(n)}, ...\big)$ of n-variate classical fields.
(Over n copies of spacetime manifold, $n = 0, 1, 2, ...$)



The multivariate fields $\overset{n}{\otimes}\,\mathbb{F}$ inherit a natural $\mathcal{E}$ smooth function topology from $\mathbb{F}$.
(Schwartz kernel theorem.)

The tensor algebra $\mathcal{T}(\mathbb{F})$ inherits a natural (Tychonoff) topology from each $\overset{n}{\otimes}\,\mathbb{F}$, entrywise.
(Theorem: similar in nature to $\mathcal{E}$ smooth function topology.)

It is a theorem that $\big(\mathcal{T}(\mathbb{F})\big)^{*} \equiv \mathcal{T}_a(\mathbb{F}^{*})$ and that $\big(\mathcal{T}(\mathbb{F})\big)^{**} \equiv \mathcal{T}(\mathbb{F})$.

By construction it is unital algebra, so e.g. left-multiplication $L_{\delta\psi}$ by some $\delta\psi \in \mathbb{F}$ meaningful.

Theorem: left-insertion ι_p (tracing out) by some $p \in \big(\mathcal{T}(\mathbb{F})\big)^* \equiv \mathcal{T}_a(\mathbb{F}^*)$ also meaningful.

As usual

$$\Big(\iota_p \, L_{\delta\psi} \;\pm\; L_{\delta\psi} \, \iota_p \Big) \, G \quad = \quad (p|\delta\psi) \, G \qquad (\forall p \in \mathbb{F}^* \text{ and } \delta\psi \in \mathbb{F} \text{ and } G \text{)}$$

graded-commutation relation.

[Important properties behave analogously as if $\mathbb{F}$ were finite dimensional.]

Take a classical observable $O:\ F\to\mathbb{R},\ \psi\mapsto O(\psi)$, let $O_{\psi_0}:=O\circ(\mathrm{I}_{\mathbb{F}}+\psi_0)$.

[One has $O_{\psi_0}(\psi-\psi_0)=O(\psi)\quad\forall\psi\in F$, with some fixed reference field $\psi_0\in F$.]

We say that O is multipolynomial iff for some $\psi_0\in F$ there exists $\mathbf{O}_{\psi_0}\in\mathcal{T}_a(\mathbb{F}^*)$, such that

$$\forall\,\psi\in F:\quad \underbrace{O_{\psi_0}(\psi{-}\psi_0)}_{=O(\psi)} \;=\; \Big(\mathbf{O}_{\psi_0}\,\Big|\,\big(1,\,\overset{1}{\otimes}(\psi{-}\psi_0),\,\overset{2}{\otimes}(\psi{-}\psi_0),\,...\big)\Big).$$

Similarly $E:\, F\to\mathbb{F}^*_T,\, \psi\mapsto E(\psi)$, let $E_{\psi_0}:=E\circ(\mathrm{I}_{\mathbb{F}}+\psi_0)$ the same re-expressed on $\mathbb{F}$.

[One has $E_{\psi_0}(\psi-\psi_0)=E(\psi) \quad \forall\psi\in F$, with some fixed reference field $\psi_0\in F$.]

We say that E is multipolynomial iff $\exists\,\mathbf{E}_{\psi_0}\in\mathcal{T}_a(\mathbb{F}^*)\otimes\mathbb{F}^*_T$, such that

$$\forall\,\psi\in F,\,\delta\psi_T\in\mathbb{F}_T:\ \underbrace{\Big(E_{\psi_0}(\psi-\psi_0)\,\Big|\,\delta\psi_T\Big)}_{=\big(E(\psi)\,\big|\,\delta\psi_T\big)} = \Big(\mathbf{E}_{\psi_0}\,\Big|\,\big(1,\,\overset{1}{\otimes}(\psi-\psi_0),\,\overset{2}{\otimes}(\psi-\psi_0),\,\ldots\big)\otimes\delta\psi_T\Big).$$

For fixed $\delta\psi_T\in\mathbb{F}_T$, one has $(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)\in\mathcal{T}_a(\mathbb{F}^*)$.

So one can left-insert with it on the field correlator algebra:

$\iota_{(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)}$ meaningfully acting on $\mathcal{T}(\mathbb{F})$.

The master Dyson-Schwinger (MDS) operator is:

$$(G, \delta\psi_T) \quad \longmapsto \quad \Big(\boldsymbol{\iota}_{(\mathbf{E}_{\psi_0} \,|\, \delta\psi_T)} \;-\; \mathrm{i}\,\hbar\, L_{\delta\psi_T} \Big)\, G$$

The master Dyson-Schwinger (MDS) equation is:

we search for (ψ_0, G_{ψ_0}) such that:

$$\underbrace{G^{(0)}_{\psi_0}}_{=:\, b\, G_{\psi_0}} \quad = \quad 1,$$

$$\forall\, \delta\psi_T \in \mathbb{F}_T \, : \qquad \underbrace{\Big(\boldsymbol{\iota}_{(\mathbf{E}_{\psi_0} \,|\, \delta\psi_T)} \;-\; \mathrm{i}\,\hbar\, L_{\delta\psi_T} \Big)}_{=:\, \mathbf{M}_{\psi_0,\, \delta\psi_T}} G_{\psi_0} \quad = \quad 0.$$

We argue that this is the tentative “one-liner” of QFT.
It says $\sim$ spontaneous local excitations decay in all modes according to classical EL.

[Feynman type quantum expectation value is then $\mu_{(\psi_0, G_{\psi_0})}(O) \;:=\; \big(\mathbf{O}_{\psi_0} \,\big|\, G_{\psi_0} \big)$.]

Example: ϕ^4 model.

Euler-Lagrange functional is

$$E: \quad F \times \mathbb{F}_T \longrightarrow \mathbb{R}, \quad (\psi,\, \delta\psi_T) \longmapsto \int\limits_{\mathcal{M}} \delta\psi_T\, \Box\psi\, \mathrm{v} \;+\; \int\limits_{\mathcal{M}} \delta\psi_T\, \psi^3\, \mathrm{v}.$$

$$\Downarrow$$

MDS operator is

$$\big(\mathbf{M}_{\psi_0,\delta\psi_T}\, G\big)^{(n)}(x_1, \ldots, x_n) =$$

$$\int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, \Box_y G^{(n+1)}(y, x_1, \ldots, x_n)\, \mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, G^{(n+3)}(y, y, y, x_1, \ldots, x_n)\, \mathrm{v}(y)$$

$$-\mathrm{i}\,\hbar\, n\, \tfrac{1}{n!} \textstyle\sum_{\pi\in\Pi_n} \delta\psi_T(x_{\pi(1)})\, G^{(n-1)}(x_{\pi(2)}, \ldots, x_{\pi(n)})$$

Pretty much well-defined, and clear recipe, if field correlators were functions.

Theorem: no solutions with high differentiability.
Theorem: for free Minkowski case, distributional solution only.
Dang!

Distributional solution to free MDS equation: $G_{\psi_0} = \exp(K_{\psi_0})$ where

$$\begin{aligned} K_{\psi_0}^{(0)} &= 0, \\ K_{\psi_0}^{(1)} &= 0, \\ K_{\psi_0}^{(2)} &= \mathrm{i}\,\hbar\,\mathtt{K}_{\psi_0}^{(2)} && \longleftarrow \text{(symmetrized propagator)} \\ K_{\psi_0}^{(n)} &= 0 && (n \geq 2) \end{aligned}$$

So we expect distributional solutions only, at best.

How can one evaluate on distributions interaction term like $G^{(n+3)}(y, y, y, x_1, ..., x_n)$?

With a sufficient condition called Hörmander's criterion? (Theorem: no.)

Via approximation with functions? (Theorem: no.)

Surprising solution by physicists: Wilsonian regularization.

Feynman integral "$\Longleftrightarrow$" MDS equation.

Wilsonian regularized Feynman integral:
integrate not on $\mathbb{F}$, only on the image space $C_\kappa[\mathbb{F}]$ of a smoothing operator $C_\kappa : \mathbb{F} \to \mathbb{F}$.

[Smoothing operator: $\sim$ convolution, can be generalized to manifolds. Does UV damping.]

Wilsonian regularized Feynman integral "$\Longleftrightarrow$" regularized MDS equation:

we search for $(\psi_0, G_{\psi_0,\kappa})$ such that:
$$\underbrace{G^{(0)}_{\psi_0,\kappa}}_{=:\, b\, G_{\psi_0,\kappa}} = 1,$$

$$\forall\, \delta\psi_T \in \mathbb{F}_T : \quad \underbrace{\left(\boldsymbol{\ell}_{(\mathbf{E}_{\psi_0} \,|\, \delta\psi_T)} - \mathrm{i}\,\hbar\, L_{C_\kappa \delta\psi_T} \right)}_{=:\, \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}} G_{\psi_0,\kappa} = 0.$$

Removes no-go theorem for solutions with high differentiability.
Thus, brings back the problem from distributions to smooth functions.

Smooth function solution to free regularized MDS equation: $G_{\psi_0,\kappa} = \exp(K_{\psi_0,\kappa})$ where

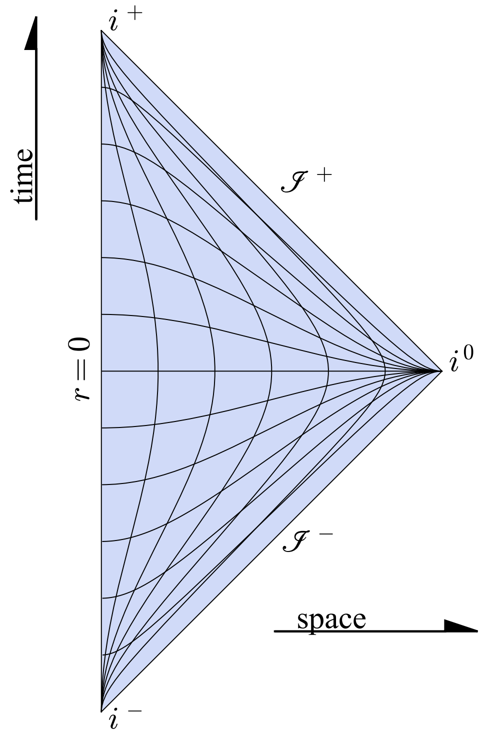
$$
\begin{aligned}
K^{(0)}_{\psi_0,\kappa} &= 0, \\
K^{(1)}_{\psi_0,\kappa} &= 0, \\
K^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\,\mathtt{K}^{(2)}_{\psi_0,\kappa} \qquad \longleftarrow \text{(smoothed symmetrized propagator)} \\
K^{(n)}_{\psi_0,\kappa} &= 0 \qquad (n \geq 2)
\end{aligned}
$$

No problem to evaluate on interaction term like $G^{(n+3)}(y, y, y, x_1, ..., x_n)$.

What we do with κ dependence? (Renormalization, work in progress with Zs.Tarcsay.)

Existence condition for regularized MDS solutions

If Euler-Lagrange functional $E: F \to \mathbb{F}^*_T$ conformally invariant:
re-expressable on Penrose conformal compactification.



That is always a compact manifold with boundary, with sufficiently regular boundary.

$E: F \to \mathbb{F}^*_T$ reformulable over this compact base manifold with regular boundary.

So, one can assume $\mathcal{M}$ compact with nice enough boundary.

In such situation, $\mathbb{F} = \mathbb{F}_T$ and have nice properties:
countably Hilbertian nuclear Fréchet space.

$$F_0 \;\supset\; F_1 \;\supset\; \dots \;\supset\; F_k \;\supset\; \dots \;\supset\; \mathbb{F}$$

(Intersection of shrinking Hilbert spaces F_k.)

Theorem:
without punishment, one can equip $\mathcal{T}(\mathbb{F})$ with a better topology, inheriting CHNF topology.

$$H_0 \;\supset\; H_1 \;\supset\; \dots \;\supset\; H_k \;\supset\; \dots \;\supset\; \mathcal{T}_h(\mathbb{F})$$

Regularized MDS operator is then a Hilbert-Schmidt linear map

$$\mathbf{M}_{\psi_0,\kappa} : \quad H_k \otimes F_k \longrightarrow H_0, \quad G \otimes \delta\psi_T \longmapsto \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}\, G$$

Theorem: one can legitimately trace out $\delta\psi_T$ variable to form

$$\hat{\mathbf{M}}^2_{\psi_0,\kappa} : \quad H_k \longrightarrow H_k, \quad G \longmapsto \sum_{i\in\mathbb{N}_0} \mathbf{M}^\dagger_{\psi_0,\kappa,\delta\psi_T\,i}\, \mathbf{M}_{\psi_0,\kappa,\delta\psi_T\,i}\, G$$

By construction: G is κ-regularized MDS solution $\iff$ $b\,G = 1$ and $\hat{\mathbf{M}}^2_{\psi_0,\kappa} G = 0$.

Theorem (A.László):

(i) the iteration

$$G_0 \;:=\; \mathbb{1} \;\text{ and }\; G_{k+1} \;:=\; G_k - \tfrac{1}{T}\,\hat{\mathbf{M}}^2_{\psi_0,\kappa} G_k \qquad (k = 0, 1, 2, ...)$$

is always convergent if $T > 0$ large enough.

(ii) the κ-regularized MDS solution space is nonempty iff

$$\lim_{k\to\infty} b\,G_k \;\neq\; 0.$$

(iii) and in this case

$$\lim_{k\to\infty} G_k$$

is an MDS solution, up to normalization factor.

Use for lattice-like numerical method in Lorentz signature?

Summary

- Feynman integral has no rigorous definition in Lorentz signature.
- Can be substituted by master Dyson-Schwinger (MDS) equation.
- Function spaces and operators for MDS equation are well defined, in suitable variables.
- Wilsonian regularized version of MDS equation is well defined, in suitable variables.
- Does not need a pre-arranged fixed causal structure.
- A necessary and sufficient existence condition was proved for MDS equation.
- Provides a convergent iterative approximation algorithm. (Lattice is Lorentz signature?)
- Outlook: rigorous definition of Wilsonian renormalization.

Backup slides

Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Solution space to the equation turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space:
  - Elements of solution space X are elementary events.
  - Collection of Borel sets Σ of X are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is classical probability space.

Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Finite charge weak solution space to the equation turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space:
  - One dimensional subspaces of the solution space $\mathcal{H}$ are elementary events, X.
  - Collection of all closed subspaces Σ of $\mathcal{H}$ are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is quantum probability space.

Fréchet derivative in top.vector spaces

Let F and G real top.affine space, Hausdorff.
Subordinate vector spaces: $\mathbb{F}$ and $\mathbb{G}$.

A map $S:\ F\to G$ is Fréchet-Hadamard differentiable at $\psi\in F$ iff:
there exists $DS(\psi):\ \mathbb{F}\to\mathbb{G}$ continuous linear, such that for all sequence $n\mapsto h_n$ in $\mathbb{F}$, and nonzero sequence $n\mapsto t_n$ in $\mathbb{R}$ which converges to zero,

$$(\mathbb{G})\lim_{n\to\infty}\left(\frac{S(\psi+t_n\,h_n)-S(\psi)}{t_n}-DS(\psi)\,h_n\right)\quad=\quad 0$$

holds.

Fréchet derivative of action functional

Fréchet derivative of $S:\ F\longrightarrow \mathrm{Rad}(\mathcal{M},\mathbb{R})$ is

$$DS:\ F\times\mathbb{F}\longrightarrow \mathrm{Rad}(\mathcal{M},\mathbb{R}),\ (\psi,\delta\psi)\longmapsto\Big(\mathcal{K}\mapsto\big(DS_{\mathcal{K}}(\psi)\big|\delta\psi\big)\Big)$$

For $\underbrace{(v,\nabla)}_{=:\psi}\in F$ given,

$$\underbrace{(\delta v,\delta C)}_{=:\delta\psi}\ \mapsto\ \big(DS_{\mathcal{K}}(v,\nabla)\,\big|\,(\delta v,\delta C)\big)=$$

$$\int_{\mathcal{K}}\Big(D_1\mathrm{L}(v,\nabla v,P(\nabla))\,\delta v+D_2^a\mathrm{L}(v,\nabla v,P(\nabla))\,(\nabla_a\delta v+\delta C_a v)+2\,D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))\,\tilde{\nabla}_{[a}\delta C_{b]}\Big)$$

$$=\int_{\mathcal{K}}\Big(D_1\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\dots c_m]}\,\delta v-\big(\tilde{\nabla}_a D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\dots c_m]}\big)\,\delta v\Big)+$$

$$\Big(D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\dots c_m]}\,\delta C_a v-2\,\big(\tilde{\nabla}_a D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\dots c_m]}\big)\,\delta C_b\Big)$$

$$+\ m\int_{\partial\mathcal{K}}\Big(D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[ac_1\dots c_{m-1}]}\,\delta v+2\,D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))_{[ac_1\dots c_{m-1}]}\,\delta C_b\Big)$$

$$(m:=\dim(\mathcal{M}))$$

[usual Euler-Lagrange bulk integral + boundary integral]

Distributions on manifolds

$V(\mathcal{M})$ vector bundle, $V^{\times}(\mathcal{M}) := V^{*}(\mathcal{M}) \otimes \overset{\dim(\mathcal{M})}{\wedge} T^{*}(\mathcal{M})$ its de Rham dual.
$V^{\times\times}(\mathcal{M}) \equiv V(\mathcal{M})$.

Correspondingly: $\mathbb{F}^{\times}$ and $\mathbb{F}_{T}^{\times}$ are de Rham duals of $\mathbb{F}$ and $\mathbb{F}_{T}$.

$\mathbb{F} \times \mathbb{F}_{T}^{\times} \to \mathbb{R}$, $(\delta\psi, p_{T}) \mapsto \int\limits_{\mathcal{M}} \delta\psi\, p_{T}$ and $\mathbb{F}_{T} \times \mathbb{F}^{\times} \to \mathbb{R}$, $(\delta\psi_{T}, p) \mapsto \int\limits_{\mathcal{M}} \delta\psi_{T}\, p$ jointly sequentially continuous.

Therefore, $\mathbb{F} \subset \left(\mathbb{F}_{T}^{\times}\right)^{*}$ and $\mathbb{F}_{T} \subset \left(\mathbb{F}^{\times}\right)^{*}$ continuous dense injections in $\mathcal{E} \to \mathcal{D}^{*}$ and $\mathcal{D} \to \mathcal{E}^{*}$ sense.
(distribution valued sections)

Let $A : \mathbb{F} \to \mathbb{F}$ continuous linear.
It has formal transpose iff there exists $A^{t} : \mathbb{F}_{T}^{\times} \to \mathbb{F}_{T}^{\times}$ continuous linear, such that $\forall \delta\psi \in \mathbb{F}$ és $p_{T} \in \mathbb{F}_{T}^{\times}$: $\int\limits_{\mathcal{M}} (A\,\delta\psi)\, p_{T} = \int\limits_{\mathcal{M}} \delta\psi\, (A^{t}\, p_{T})$.
Topological transpose of formal transpose $\left(A^{t}\right)^{*} : \left(\mathbb{F}_{T}^{\times}\right)^{*} \to \left(\mathbb{F}_{T}^{\times}\right)^{*}$ is the distributional extension of A.
Not always exists.

Fundamental solution on manifolds

Let $E\,:\; F\times\mathbb{F}_T\to\mathbb{R}$ be Euler-Lagrange functional, and $J\in\mathbb{F}_T^*$.

$\mathtt{K}_{(J)}\in F$ is solution with source J, iff $\forall\delta\psi_T\in\mathbb{F}_T\;:\;\left(E(\mathtt{K}_{(J)})\,|\,\delta\psi_T\right)=(J|\delta\psi_T)$.

Specially: one can restrict to $J\in\mathbb{F}_T^\times\subset\mathbb{F}^\times\subset\mathbb{F}_T^*$.

A continuous map $\mathtt{K}\,:\;\mathbb{F}_T^\times\to F$ is fundamental solution, iff for all $J\in\mathbb{F}_T^\times$ the field $\mathtt{K}(J)\in F$ is solution with source J.

May not exists, and if does, may not be unique.

If $\mathtt{K}_{\psi_0}\,:\;\mathbb{F}_T^\times\to\mathbb{F}$ vectorized fundamental solution is linear (e.g. for linear $E_{\psi_0}\,:\;\mathbb{F}\to{\mathbb{F}_T}^*$): $\mathtt{K}_{\psi_0}\in\mathcal{L}in(\mathbb{F}_T^\times,\mathbb{F})\subset(\mathbb{F}_T^\times)^*\otimes(\mathbb{F}_T^\times)^*$ is distribution.

Generalization of convolution to manifolds

A continuous linear map $C:\ (\mathbb{F}^{\times})^{*} \to \mathbb{F}$ is called smoothing operator.
Schwartz kernels theorem: $C \ \longleftrightarrow\ \kappa \in \mathbb{F} \boxtimes \mathbb{F}^{\times}$, i.e. κ is field over $\mathcal{M}{\times}\mathcal{M}$.

$C_{\kappa}:\ \mathbb{F}_{T} \to \mathbb{F}$ cont.lin., where $\big(C_{\kappa}\,\delta\psi_{T}\big)(x) := \int_{y\in\mathcal{M}} \kappa(x,y)\,\delta\psi_{T}(y)\ \ \forall \delta\psi_{T} \in \mathbb{F}_{T}, x \in \mathcal{M}$.

$C^{t}_{\kappa}:\ \mathbb{F}^{\times}_{T} \to \mathbb{F}^{\times}$ cont.lin., where $\big(C^{t}_{\kappa}\,p_{T}\big)(y) := \int_{x\in\mathcal{M}} p_{T}(x)\,\kappa(x,y)\ \ \forall p_{T} \in \mathbb{F}^{\times}_{T}, y \in \mathcal{M}$.

partially compactly supported κ: $\forall\,\mathcal{K} \subset \mathcal{M}$ compact
$\{(x,y) \in \mathcal{M}{\times}\mathcal{M} \,|\, x \in \mathcal{K},\ \kappa(x,y) \neq 0\}$ and $\{(x,y) \in \mathcal{M}{\times}\mathcal{M} \,|\, y \in \mathcal{K},\ \kappa(x,y) \neq 0\}$
has compact closure.

Then C_{κ} cont.lin. $\mathbb{F} \to \mathbb{F}$ and C^{t}_{κ} cont.lin. $\mathbb{F}^{\times} \to \mathbb{F}^{\times}$.
Then C_{κ} cont.lin. $\mathbb{F}_{T} \to \mathbb{F}_{T}$ and C^{t}_{κ} cont.lin. $\mathbb{F}^{\times}_{T} \to \mathbb{F}^{\times}_{T}$.
On distributions, the transpose of these.

E.g. ordinary convolution over Minkowski:
$\mathcal{M}$ (affine space), T (subordinate vector space), v (constant volume form).
Let $\eta:\ T \to \mathbb{R}$ smooth compactly supported.

Let $(x,y) \mapsto \kappa(x,y) := \eta(x{-}y)\,\mathrm{v}(y)\,I$.

Then $\delta\psi \in \mathbb{F}$: $\ C_{\kappa}\,\delta\psi = \eta \star \delta\psi$.

Particular solutions to the free MDS equation

Distributional solutions to free MDS equation: $G_{\psi_0} = \exp(K_{\psi_0})$ where

$$\begin{aligned} K^{(0)}_{\psi_0} &= 0, \\ K^{(1)}_{\psi_0} &= 0, \\ K^{(2)}_{\psi_0} &= \mathrm{i}\,\hbar\,\mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

Smooth function solutions to free regularized MDS equation: $G_{\psi_0} = \exp(K_{\psi_0,\kappa})$ where

$$\begin{aligned} K^{(0)}_{\psi_0,\kappa} &= 0, \\ K^{(1)}_{\psi_0,\kappa} &= 0, \\ K^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\,(C_\kappa \otimes C_\kappa)\mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0,\kappa} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

[Here $C_\kappa(\cdot) := \eta \star (\cdot)$ is convolution by a test function η.]

Renormalization from functional analysis p.o.v.

Let $\mathbb{F}$ and $\mathbb{G}$ real or complex top.vector space, Hausdorff loc.conv complete.

Let $M : \mathbb{F} \rightarrowtail \mathbb{G}$ densely defined linear map (e.g. MDS operator).

Closed: the graph of the map is closed.

Closable: there exists linear extension, such that its graphs closed (unique if exists).

Closable $\Leftrightarrow$ where extendable with limits, it is unique.

Multivalued set:
$\mathrm{Mul}(M) := \left\{ y \in \mathbb{G} \,\middle|\, \exists\, (x_n)_{n\in\mathbb{N}} \right.$ in $\mathrm{Dom}(M)$ such that $\lim\limits_{n\to\infty} x_n = 0$ and $\left. \lim\limits_{n\to\infty} M x_n = y \right\}$.

$\mathrm{Mul}(M)$ always closed subspace.

Closable $\Leftrightarrow \mathrm{Mul}(M) = \{0\}$.

Maximally non-closable $\Leftrightarrow \mathrm{Mul}(M) = \overline{\mathrm{Ran}(M)}$. Pathological, not even closable part.

Polynomial interaction term of MDS operator maximally non-closable!

MDS operator:

$$\mathbf{M}: \quad \mathbb{F}_T \otimes \mathcal{T}(\mathbb{F}) \to \mathcal{T}(\mathbb{F}_{\mathbb{C}}), \quad G \mapsto \mathbf{M}\,G$$

linear, everywhere defined continuous on $\mathcal{D} \otimes \mathcal{E} \to \mathcal{E}$. So,

$$\mathbf{M}: \quad \mathcal{T}(\mathbb{F}_T^{\times\,*}) \rightarrowtail \mathbb{F}_T^* \otimes \mathcal{T}(\mathbb{F}_T^{\times\,*}), \quad G \mapsto \mathbf{M}\,G$$

linear, densely defined as $\mathcal{D}^* \rightarrowtail \mathcal{D}^* \otimes \mathcal{D}^*$.

Similarly: $\mathbf{M}_\kappa$ regularized MDS operator (κ: a fix regularizator).

Not good equation:

$$G \in \mathcal{T}(\mathbb{F}_T^{\times *}) \;? \qquad G^{(0)} = 1 \text{ and } \exists\, G_\kappa \to G \text{ approximator sequence, such that } : \lim_{\kappa \to \delta_0} \mathbf{M}\, G_\kappa = 0.$$

All G would be selected, because $\mathrm{Mul}()$ set of interaction term is full space.

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Can be good:

$$G \in \mathcal{T}(\mathbb{F}_T^{\times\,*})\;? \qquad G^{(0)} = 1 \text{ and } \exists\, G_\kappa \to G \text{ approximator sequence, such that :}$$
$$\forall\, \kappa:\; \mathbf{M}_\kappa\, G_\kappa = 0.$$

That is, as implicit function of κ, not as operator closure kernel.

Running coupling:
If in $\mathbf{M}_\kappa$ EL terms are combined with weights $g(\kappa)$.
(Not just with real factors.)
E.g.:

$$(g, G) \in \mathcal{T}(\mathbb{F}_T^{\times\,*})\;? \qquad G^{(0)} = 1 \text{ and } \exists\, G_\kappa \to G \text{ approximator sequence, such that :}$$
$$\forall\, \kappa:\; \mathbf{M}_{g(\kappa),\kappa}\, G_\kappa = 0.$$