Anomalous dimensions of composite operators for exclusive hard scattering processes

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Theory seminar Wigner RCP

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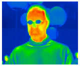
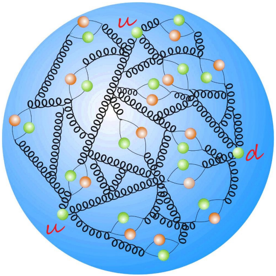


How to gain insight into the structure of hadrons

- Important question: How do hadronic properties emerge from the properties of the constituent partons?
- Experimentally: Perform high-energy scattering experiments that can resolve the inner hadron structure (e.g. scatter electrons off a proton)
- Depending on the kinematics (inclusive vs. exclusive): Different properties of hadron structure
- In QCD: Factorization between short range and long range physics
- Long range functions provide information on partons within proton



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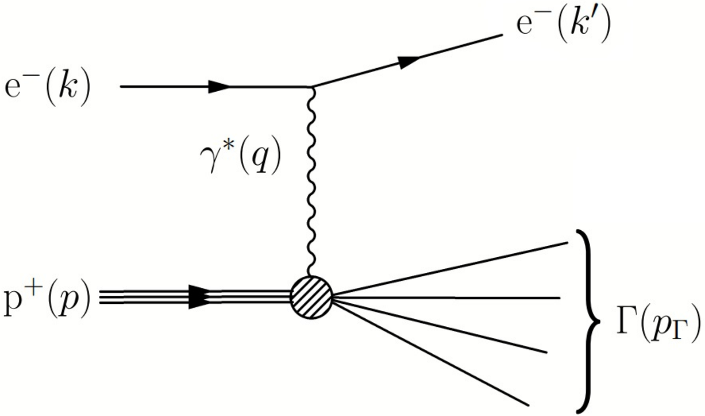
Part I: Probing proton structure inclusively (DIS)



Assumptions:

- Photon highly virtual, $Q^2 \equiv -q^2 \gg p^2$
- $s \gg m_p^2$

The DIS cross section

The physical cross section of DIS is proportional to

$$\frac{1}{q^4} L_{\mu\nu} W^{\mu\nu}$$

Here, $L_{\mu\nu}$ represents the leptonic tensor and $W_{\mu\nu}$ the hadronic one.

- $L_{\mu\nu}$ encodes the polarization information of the electrons and the off-shell photon. Applying standard techniques it is easy to find that

$$L_{\mu\nu} = \frac{1}{2} Tr[\not{k}' \gamma_\mu \not{k} \gamma_\nu].$$

- $W^{\mu\nu}$ encodes the information of the $\gamma^* p^+ \to \Gamma$ process, the amplitude of which is

$$\mathcal{M}(\gamma^* p^+ \to \Gamma) \sim \langle \Gamma | \, J_\mu \, \big| p^+(p) \big\rangle$$

  with

$$J_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f \text{ the electromagnetic current.}$$

The DIS hadronic tensor

The hadronic tensor appearing in the DIS cross section can then be written as

$$W_{\mu\nu} = \int \mathrm{d}^4x \, e^{iq\cdot x} \left\langle p^+(p) \right| J_\mu(x) J_\nu(0) \left| p^+(p) \right\rangle .$$

Note that this is independent of the final states Γ.

Hence, the calculation of the hadronic tensor of DIS boils down to calculating the product of two current operators.

The standard formalism to deal with this type of problem is the operator product expansion (OPE).

The OPE

The OPE was first introduced by Wilson [Wilson, 1969] and later proven in perturbation theory by Zimmerman [Zimmermann, 1973].

The main idea is that the time-ordered product of two local operators $J(x)$ and $J'(y)$ can be expanded in a series of regular operators, multiplied by functions (called Wilson coefficients) encoding the singularity of the operator product as $x = y$

$$\mathcal{T} J(x) J'(y) = \sum_{n=0}^{\infty} C_n(x-y) \mathcal{O}_n\Big(\frac{x-y}{2}\Big).$$

To apply the OPE to the DIS hadronic tensor, we use the optical theorem to relate the rate of $\gamma^* p^+ \to \Gamma$ to the imaginary part of the forward scattering rate $\gamma^* p^+ \to \gamma^* p^+$:

$$W_{\mu\nu} = 2\,\mathrm{Im}\, T_{\mu\nu},$$

$$T_{\mu\nu} = i \int \mathrm{d}^4 x\, e^{iq\cdot x} \left\langle p^+(p) \right| \mathcal{T} J_\mu(x) J_\nu(0) \left| p^+(p) \right\rangle .$$

Application of the OPE to DIS

$T_{\mu\nu}$ can be explicitly calculated as the forward matrix element for Compton scattering, $\gamma^* q \to \gamma^* q$ (photon off-shell and no polarizations included). This gives

$$T_{\mu\nu} \sim -\bar{u}(p) \frac{\gamma_\mu (\not{p} + \not{q}) \gamma_\nu}{(p+q)^2} u(p).$$

As we are interested in the regime of large Q^2, we expand the denominator for $Q^2 \gg p^2$

$$\frac{1}{(p+q)^2} = -\frac{1}{Q^2} \sum_n \left(\frac{2p \cdot q}{Q^2} \right)^n$$

such that

$$T_{\mu\nu} \sim \frac{1}{Q^2} \bar{u}(p) \gamma_\mu (\not{p} + \not{q}) \gamma_\nu u(p) \sum_n \left(\frac{2p \cdot q}{Q^2} \right)^n .$$

Application of the OPE to DIS

The ingredients of the OPE, i.e. the Wilson coefficients and the operators, can be read of from the momentum expansion in a relatively straightforward manner:

- Factors of p_μ should come from factors of $i\partial_\mu$ from the operators, acting on the external states
- The dependence on the short-distance scale should be incorporated into the Wilson coefficients

This implies that the Wilson coefficients for DIS will be of the following form

$$C^{\mu_1 \dots \mu_n} \sim \frac{2^n}{Q^{2n+1}} q^{\mu_1 \dots \mu_n}.$$

Application of the OPE to DIS

For the extraction of the operators, it is customary to use a basis of gauge-invariant operators, meaning that ordinary derivatives are replaced by covariant ones

$$\partial_\mu \to D_\mu = \partial_\mu - ig_s A_\mu.$$

Furthermore, the OPE is dominated by leading-twist operators, where *twist = dimension - spin*. These operators are symmetric in the Lorentz indices and traceless.

Hence, the operators appearing in the OPE for DIS are gauge-invariant leading-twist spin-N operators (focus on flavor non-singlet operators in this talk)

$$\mathcal{O}^{\mathrm{NS}}_{\mu_1 \ldots \mu_N} = \mathcal{S}\bar{\psi}' \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} \psi.$$

PDFs and DIS

Finally, one has to consider the forward matrix element of these operators

$$\left\langle p^+(p)\right| \mathcal{O}_{\mu_1\dots\mu_N} \left|p^+(p)\right\rangle \sim \mathcal{M}_N(Q)\, p_{\mu_1}\dots p_{\mu_N}.$$

The functions $\mathcal{M}_N$ are directly related to the parton distribution functions (PDFs)

$$f_q(x) \sim \sum_n \frac{\mathrm{Im}\,\mathcal{M}_n}{x^n},$$

which can be interpreted to give the probability to find a quark inside the proton with momentum xp $(0 \leq x \leq 1)$. They encode the longitudinal momentum/polarization carried by partons within fast-moving hadrons.

Since the PDFs are defined in terms of hadronic states, they are non-perturbative
$\Rightarrow$ Direct extraction from experimental data (see e.g. [Brock et al., 1995]) or using lattice QCD (see e.g. [Alexandrou et al., 2020], [Ji et al., 2021], [Wang et al., 2021])

Scale dependence of PDFs

Besides the PDFs themselves, phenomenologically it is also important to understand how PDFs vary when we change the energy scale of the process. Because of their direct relation to operator matrix elements, the scale dependence of the distributions is determined by the scale dependence of the operators

$$\frac{\mathrm{d}[\mathcal{O}]}{\mathrm{d}\ln\mu^2} = \gamma[\mathcal{O}], \quad \gamma \equiv a_s\gamma^{(0)} + a_s^2\gamma^{(1)} + \ldots$$

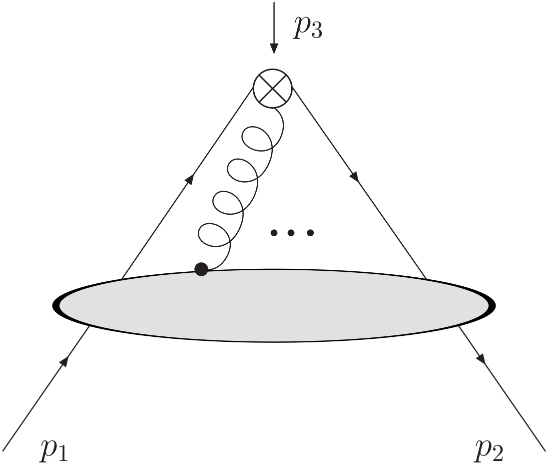
The operator anomalous dimension can be calculated perturbatively!

Explicitly, we gain access to the operator anomalous dimensions by considering the forward partonic matrix elements of the operators

$$\langle\psi(p)|O^{NS}_{\mu_1\ldots\mu_N}(0)|\overline{\psi}(p)\rangle$$

and renormalizing.

Operator anomalous dimensions



In the present case we have $p_1 = p_2$ and $p_3 = 0$. For practical convenience we consider $\mathcal{O}_N \equiv \Delta^{\mu_1} \dots \Delta^{\mu_N} O^{NS}_{\mu_1 \dots \mu_N}$ with $\Delta^2 = 0$.

Operator anomalous dimensions

In forward kinematics, the operators simply renormalize multiplicatively

$$\mathcal{O}_{N+1} = Z_{N,N}[\mathcal{O}_{N+1}].$$

The anomalous dimensions are extracted from the Z-factors as

$$\gamma_{N,N} = -\frac{1}{Z_{N,N}}\frac{\mathrm{d}Z_{N,N}}{\mathrm{d}\ln\mu^2}$$

and are related to the splitting functions by a Mellin transformation

$$\gamma_{N,N} = -\int_0^1 \mathrm{d}x\, x^N P_{NS}(x).$$

The latter determine the scale dependence of the PDFs through the DGLAP equation [Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]

$$\frac{\mathrm{d}f(x,\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} P(y) f\left(\frac{x}{y},\mu^2\right).$$

Operator anomalous dimensions

A special case is $N = 0$, which is just the vector current $\bar{\psi}\gamma\psi$ (conserved quantity!)

$$\mathcal{O}_1 = Z_{0,0}[\mathcal{O}_1] = 1 \times [\mathcal{O}_1]$$

These forward anomalous dimensions are known completely up to the 3-loop level, and in certain limits up to the 5-loop level [Gross and Wilczek, 1973], [Floratos et al., 1977], [Gracey, 1994], [Moch et al., 2004], [Velizhanin, 2012], [Ruijl et al., 2016], [Moch et al., 2017], [Herzog et al., 2019], [Velizhanin, 2020]

At the l-loop level, the forward anomalous dimensions in general consist of harmonic sums of maximum weight $w = 2l - 1$ and denominators in $N + \alpha$ (with $\alpha \in \mathbb{N}$) up to the same maximum power.

Harmonic sums at argument N are recursively defined by [Vermaseren, 1999, Blümlein and Kurth, 1999]

$$S_{\pm m}(N) = \sum_{i=1}^{N} (\pm 1)^i \, i^{-m}, \quad S_{\pm m_1, m_2, \ldots, m_d}(N) = \sum_{i=1}^{N} (\pm 1)^i \, i^{-m_1} \, S_{m_2, \ldots, m_d}(i) \, .$$

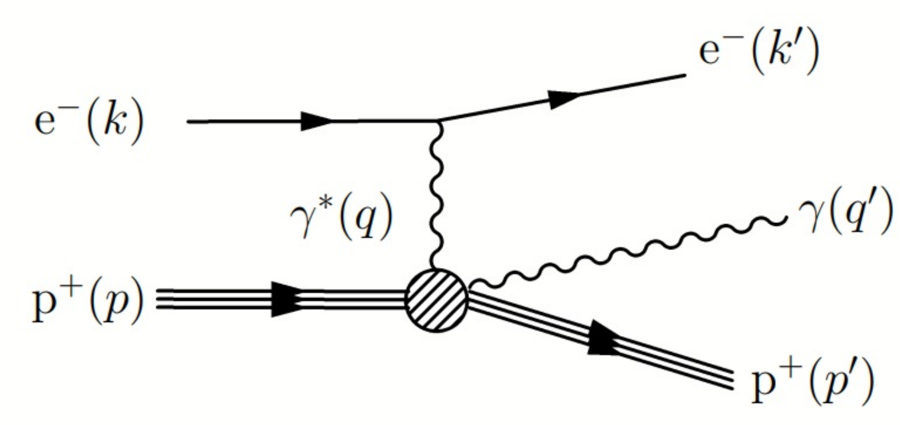
Summary of part I

- We can learn about the structure of the proton by scattering an electron off it.
- In inclusive DIS, this information is summarized in the hadronic tensor

$$W_{\mu\nu} = \int \mathrm{d}^4x\, e^{iq\cdot x} \left\langle p^+(p)\right| J_\mu(x)J_\nu(0) \left|p^+(p)\right\rangle .$$

- Through the OPE, this tensor was related to the hadronic matrix elements of leading-twist spin-N operators $\langle p^+(p)|\, \mathcal{O}_{\mu_1\ldots\mu_N}\, |p^+(p)\rangle$. These are directly related to the PDFs
  - ◇ Probability to find a quark inside the proton with momentum $xp\ (0 \leq x \leq 1)$
  - ◇ Encode the longitudinal momentum/polarization carried by partons within the proton
  - ◇ Studied in detail using e.g. HERA data [Abramowicz et al., 2015], [Accardi et al., 2016]
  - ◇ Scale dependence determined by anomalous dimensions of defining operators

Part II: Probing the proton exclusively (DVCS)



DVCS vs DIS

We can play the same game as for DIS, with a few important differences

- The hadronic tensor will now be related to the following time-ordered product of currents

$$T_{\mu\nu} = i \int \mathrm{d}^4x \, e^{i(q+q')\cdot x} \left\langle p' \right| \mathcal{T} J_\mu(x) J_\nu(0) \left| p \right\rangle .$$

- Again applying the machinery of the OPE, one finds that the same operators as those in DIS appear. However, because $p - p' \neq 0$, one now also has to take into account total derivative operators.
- Similarly as in DIS, hadronic matrix elements of the operators in DVCS are related to generalized parton distributions (GPDs), which also depend on the momentum transfer between the initial and final state proton.
  $\rightarrow$ Transverse distributions of partons + contributions partonic orbital angular momentum to total hadronic spin
  $\rightarrow$ Will be measured with unprecedented precision at the future EIC

[Boer et al., 2011], [Abdul Khalek et al., 2021]

Scale dependence of GPDs

Like PDFs, GPDs are defined as hadronic matrix elements of QCD operators, and thus cannot be calculated in perturbation theory.

$\Rightarrow$ Direct extraction from experimental data (see e.g. [Brock et al., 1995]) or using lattice QCD (see e.g. [Alexandrou et al., 2020], [Ji et al., 2021], [Wang et al., 2021])

Their scale dependence however is directly determined by the scale dependence of the defining operators, which is characterized by the operator anomalous dimensions

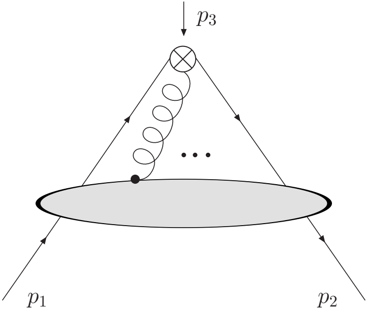
$$\frac{\mathrm{d}[\mathcal{O}]}{\mathrm{d}\ln\mu^2} = \gamma[\mathcal{O}], \quad \gamma \equiv a_s\gamma^{(0)} + a_s^2\gamma^{(1)} + \ldots$$

The operator anomalous dimensions can be calculated perturbatively!

Explicitly, we gain access to the operator anomalous dimensions by considering the non-forward partonic matrix elements of the operators

$$\langle \psi(p') | O^{NS}_{\mu_1 \ldots \mu_N}(p - p') | \overline{\psi}(p) \rangle$$

and renormalizing.



For simplicity we take $p_1 = p, p_2 = 0$ and $p_3 = -p$. For practical convenience we consider $\mathcal{O}_N \equiv \Delta^{\mu_1} \ldots \Delta^{\mu_N} O^{NS}_{\mu_1 \ldots \mu_N}$ with $\Delta^2 = 0$.

Determination of non-forward anomalous dimensions

The calculations follow a standard workflow:

- Generate the Feynman diagrams using *QGRAF* [Nogueira, 1993]
- Feed the output to a *FORM* [Vermaseren, 2000],[Kuipers et al., 2013] program to determine the topologies and color factors
  [Larin et al., 1997],[van Ritbergen et al., 1999],[Herzog et al., 2016]
- Perform the actual diagram calculations with the *FORCER* program [Ruijl et al., 2020], which can efficiently deal with massless propagator-type diagrams in $d = 4 - 2\epsilon$
- Extract the anomalous dimensions from the $1/\epsilon$-pole through renormalization

Operator renormalization: Non-forward kinematics

$$\langle \psi(p_1) | \mathcal{O}(p_3) | \overline{\psi'}(p_2) \rangle$$

In non-forward kinematics ($p_3 \neq 0$), there is mixing with total derivative operators

$$\begin{pmatrix} \mathcal{O}_{N+1} \\ \partial \mathcal{O}_N \\ \vdots \\ \partial^N \mathcal{O}_1 \end{pmatrix} = \begin{pmatrix} Z_{N,N} & Z_{N,N-1} & \dots & Z_{N,0} \\ 0 & Z_{N-1,N-1} & \dots & Z_{N-1,0} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & Z_{0,0} \end{pmatrix} \begin{pmatrix} [\mathcal{O}_{N+1}] \\ [\partial \mathcal{O}_N] \\ \vdots \\ [\partial^N \mathcal{O}_1] \end{pmatrix}$$

Hence we now also have an anomalous dimension matrix (ADM)

$$\hat{\gamma} = -\frac{\mathrm{d} \ln \hat{Z}}{\mathrm{d} \ln \mu^2} = \begin{pmatrix} \gamma_{N,N} & \gamma_{N,N-1} & \cdots & \gamma_{N,0} \\ 0 & \gamma_{N-1,N-1} & \cdots & \gamma_{N-1,0} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \gamma_{0,0} \end{pmatrix}.$$

Operator renormalization: Non-forward kinematics

The elements of the ADM determine the scale dependence of non-forward (exclusive) distributions through the ERBL equation

[Efremov and Radyushkin, 1980a],[Efremov and Radyushkin, 1980b],[Lepage and Brodsky, 1979],

[Lepage and Brodsky, 1980]

$$\frac{\mathrm{d}H(x,\chi,t,\mu^2)}{\mathrm{d}\ln\mu^2} = \frac{1}{|\chi|}\int_{-1}^{1}\mathrm{d}y\; V(x,y)H(y,\chi,t,\mu^2).$$

$$\sum_{k=0}^{N}\gamma_{N,k}y^k = -\int_0^1 \mathrm{d}x\; x^N V(x,y).$$

We now have to choose a basis for the additional operators. We briefly discuss two possibilities which have appeared in the literature.

- Diagonal elements = forward anomalous dimensions
- Triangular

The Gegenbauer basis

In this basis the operators are expanded in terms of Gegenbauer polynomials

$$\mathcal{O}^{\mathcal{G}}_{N,k} = (\Delta \cdot \partial)^k \overline{\psi'} \not{\Delta} C_N^{3/2}\left(\frac{\overleftarrow{D}\cdot\Delta - \Delta\cdot\overrightarrow{D}}{\overleftarrow{\partial}\cdot\Delta + \Delta\cdot\overrightarrow{\partial}}\right)\psi$$

with [Olver et al., 2010]

$$C_N^{\nu}(z) = \frac{\Gamma(\nu + 1/2)}{\Gamma(2\nu)} \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{(N+l+2)!}{(l+1)!} \left(\frac{1}{2} - \frac{z}{2}\right)^l.$$

- This choice of operator basis is used in conformal symmetry studies
  [Efremov and Radyushkin, 1980a], [Belitsky and Müller, 1999], [Braun et al., 2017]
- The anomalous dimensions of the Wilson operators in this basis are known up to $O(a_s^3)$ [Braun et al., 2017].

The total derivative basis

In this approach we identify operators by counting powers of derivatives

$$\mathcal{O}^{\mathcal{D}}_{k,N-k} = (\Delta\cdot\partial)^k\{\overline{\psi'}\not{\Delta}(\Delta\cdot D)^{N-k}\psi\}$$

E.g. for $N = 1$ we have

$$\left\{\overline{\psi'}\not{\Delta}(\Delta\cdot D)\psi, (\Delta\cdot\partial)\overline{\psi'}\not{\Delta}\psi\right\}.$$

- This choice of operator basis is used for hadronic studies on the lattice, see e.g. [Göckeler et al., 2005] and [Gracey, 2009]
- In this basis, the Wilson anomalous dimensions for low-N operators are known up to $O(a_s^3)$ (see [Gracey, 2009] for analytical results and [Kniehl and Veretin, 2020] for a numerical extension of these).

What are the goals of our work?

Hence we see that knowledge of the mixing matrices is limited in the total derivative basis. Furthermore, because of the different bases the above results cannot be directly compared. The goals of the thesis can now be summarized by the following two questions:

1. Can we extend and formalize the calculations in the total derivative basis from first principles?
2. Can we relate the calculations of anomalous dimensions in different operator bases to each other?

Relating operator bases

Can we relate the calculations of anomalous dimensions in different operator bases to each other?

To relate operator bases to the total derivative one, we consider a generalization of the $\mathcal{D}$-basis

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = (\Delta\cdot\partial)^{p}\left\{(\Delta\cdot D)^{q}\overline{\psi'}\,\not{\Delta}(\Delta\cdot D)^{r}\psi\right\}$$

Assuming the chiral limit ($m_q = 0$), the following relations hold

- Total derivatives act as

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = \mathcal{O}^{\mathcal{D}}_{p-1,q+1,r} + \mathcal{O}^{\mathcal{D}}_{p-1,q,r+1},$$

- Left- and right-derivative operators renormalize with the same renormalization constants

$$\mathcal{O}^{\mathcal{D}}_{k,0,N} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j}\,[\mathcal{O}^{\mathcal{D}}_{k+j,0,N-j}]\,,$$

$$\mathcal{O}^{\mathcal{D}}_{k,N,0} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j}\,[\mathcal{O}^{\mathcal{D}}_{k+j,N-j,0}]$$

Relating the total derivative and Gegenbauer bases

Can we relate the calculations of anomalous dimensions in different operator bases to each other?

We have derived a relation between the operators in the Gegenbauer basis and the operators in the total derivative basis

$$\mathcal{O}^{\mathcal{G}}_{N,k} = \frac{1}{2N!} \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{(N+l+2)!}{(l+1)!} \mathcal{O}^{\mathcal{D}}_{k-l,0,l} \, .$$

Writing the corresponding relation for the renormalized operators, this in turn relates the anomalous dimensions in both bases to each other

$$\sum_{j=0}^{N} (-1)^j \frac{(j+2)!}{j!} \gamma^{\mathcal{G}}_{N,j} = \frac{1}{N!} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{(N+j+2)!}{(j+1)!} \sum_{l=0}^{j} \gamma^{\mathcal{D}}_{j,l}.$$

Constraints on the anomalous dimensions

Can we extend and formalize the calculations in the total derivative basis from first principles?

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = (\Delta \cdot \partial)^{p} \Big\{ (\Delta \cdot D)^{q} \overline{\psi'} \not{\Delta} (\Delta \cdot D)^{r} \psi \Big\}$$

Assuming the chiral limit ($m_q = 0$), the following relations hold

- Total derivatives act as

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} \; = \; \mathcal{O}^{\mathcal{D}}_{p-1,q+1,r} + \mathcal{O}^{\mathcal{D}}_{p-1,q,r+1},$$

- Left- and right-derivative operators renormalize with the same renormalization constants

$$\mathcal{O}^{\mathcal{D}}_{k,0,N} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j} \, [\mathcal{O}^{\mathcal{D}}_{k+j,0,N-j}] \, ,$$

$$\mathcal{O}^{\mathcal{D}}_{k,N,0} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j} \, [\mathcal{O}^{\mathcal{D}}_{k+j,N-j,0}]$$

Constraints on the anomalous dimensions

Can we extend and formalize the calculations in the total derivative basis from first principles?

Applying the first relation successively on $\mathcal{O}^{\mathcal{D}}_{N,0,0}$ we derived a recursion-type relation for the bare operators

$$\mathcal{O}^{\mathcal{D}}_{0,N,0} - (-1)^N \sum_{j=0}^{N} (-1)^j \binom{N}{j} \mathcal{O}^{\mathcal{D}}_{j,0,N-j} = 0.$$

Implementing the renormalization then led to a relation between the renormalization factors $Z^{\mathcal{D}}_{N,k}$ and hence between the anomalous dimensions

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,\,j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,\,j}$$

Constraints on the anomalous dimensions

Can we extend and formalize the calculations in the total derivative basis from first principles?

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,\, j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j}$$

- ✓ Order-independent consistency check
- ✓ Can be used to construct the full mixing matrix from the knowledge of the forward anomalous dimensions $\gamma_{N,N}$ (and a boundary condition, see below)

4-step algorithm for constructing the ADM

Can we extend and formalize the calculations in the total derivative basis from first principles?

1. Calculate

$$\binom{N}{k}\sum_{j=0}^{N-k}(-1)^j\binom{N-k}{j}\gamma_{j+k,\,j+k}$$

   and construct an Ansatz for the off-diagonal piece

2. Calculate

$$\sum_{j=k}^{N}(-1)^k\binom{j}{k}\sum_{l=j+1}^{N}(-1)^l\binom{N}{l}\gamma^{\mathcal{D}}_{l,\,j}$$

3. Substitute into the consistency relation $\Rightarrow$ System of equations, solution not necessarily unique $\Rightarrow$ Need boundary condition!
4. Determine all-N expression for $\gamma^{\mathcal{D}}_{N,0}$ from Feynman diagrams

Feynman diagrams and $\gamma^{\mathcal{D}}_{N,0}$

Can we extend and formalize the calculations in the total derivative basis from first principles?

From the general consistency relation for $k = 0$ we can derive a recursion-type relation for the last column

$$\gamma^{\mathcal{D}}_{N,0} = (-)^N \left[\sum_{i=0}^{N} \gamma^{\mathcal{D}}_{N,i} - \sum_{j=0}^{N-1} (-)^j \binom{N}{j} \gamma^{\mathcal{D}}_{j,0} \right]$$

So, if we can compute the first sum on the RHS, we can recursively build up the last column. For this, let us look at the renormalization pattern for the right-derivative operators

$$\mathcal{O}^{\mathcal{D}}_{N+1} = Z_{N,N}[\mathcal{O}_{N+1}] + Z^{\mathcal{D}}_{N,N-1}[\partial \mathcal{O}_N] + \cdots + Z^{\mathcal{D}}_{N,0}[\partial^N \mathcal{O}_1]$$

Feynman diagrams and $\gamma^{\mathcal{D}}_{N,0}$

Can we extend and formalize the calculations in the total derivative basis from first principles?

It then follows that there is a direct relation between the bare OMEs and the sum of anomalous dimensions

$$\frac{\mathcal{O}^{\mathcal{D}}_{N+1}}{\epsilon} \sim \sum_{i=0}^{N} \gamma^{\mathcal{D}}_{N,i} \equiv \mathcal{B}(N+1).$$

E.g. at $O(a_s)$ we simply have

$$\frac{\mathcal{O}^{\mathcal{D},(0)}_{N+1}}{\epsilon} = \sum_{i=0}^{N} \gamma^{\mathcal{D},(0)}_{N,i} = \mathcal{B}^{(0)}(N+1).$$

Hence, it follows that there is a direct relation between the bare OMEs and the last column of the mixing matrix

$$\gamma^{\mathcal{D}}_{N,0} = \sum_{i=0}^{N} (-1)^i \binom{N}{i} \mathcal{B}(i+1).$$

Structure of the anomalous dimensions and sums

As we saw before, the forward (diagonal) anomalous dimensions consist of harmonic sums and denominators. We expect the off-diagonal elements of the ADM to have a similar structure. So, to apply our algorithm, we need to calculate single and double sums over such structures.

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j}.$$

These types of sums appearing can be dealt with using the principles of symbolic summation, see e.g. [Graham et al., 1989, Kauers and Paule, 2011] for extensive overviews. In particular, the *MATHEMATICA* package *SIGMA* [Schneider, 2004], which uses creative telescoping to solve summation problems, is very helpful.

Structure of the anomalous dimensions and sums

Telescoping: Suppose we have a summation $\sum_{k=a}^{N} f(k)$

$\rightarrow$ Find function $g(N)$ such that the summand can be written as

$$f(k) = \Delta g(k) \equiv g(k+1) - g(k)$$

$$\begin{aligned} \Rightarrow \sum_{k=a}^{N} f(k) &= \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k) \\ &= g(N+1) - g(a) \end{aligned}$$

Structure of the anomalous dimensions and sums

Creative telescoping [Zeilberger, 1991]: Suppose we have the summation

$$\sum_{k=a}^{b} f(n,k) \equiv S(n)$$

$\rightarrow$ Find functions $c_0(n), ..., c_d(n)$ and $g(n,k)$ such that

$$g(n,k+1) - g(n,k) = c_0(n)f(n,k) + ... + c_d(n)f(n+d,k)$$

Now apply summation on both sides of the equation

$$\Rightarrow g(n,b+1) - g(n,a) = c_0(n)\sum_{k=a}^{b} f(n,k) + ... + c_d(n)\sum_{k=a}^{b} f(n+d,k)$$

$\Rightarrow$ Inhomogeneous recurrence for original sum

$$q(n) = c_0(n)S(n) + ... + c_d(n)S(n+d)$$

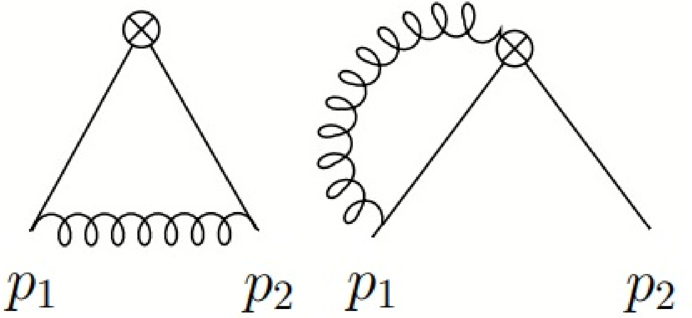
Calculating sums

- In this way, *SIGMA* generates and solves recurrence for given summation problem
- Solution consists of solution set for homogeneous recurrence + particular solution
- For final closed expression of summation: Determine linear combination of solutions that has same initial values as the given sum

Let us now, as an example, look at the computation of the 1-loop ADM

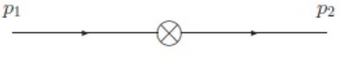
Example: 1-loop ADM

At the 1-loop level, there are only 2 Feynman diagrams

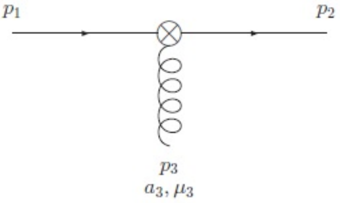


Feynman rules for operator insertions

- Contract OMEs with $\Delta_{\mu_1}...\Delta_{\mu_N}$, $\Delta^2 = 0$
- For $L = 1$: Need up to 1 gluon attached to operator



$$\not{\Delta}\,(\Delta\cdot p_2)^{N-1}$$



$$-g t^{a_3} \not{\Delta} \Delta^{\mu_3} \sum_{j_1=0}^{N-2} (p_2\cdot\Delta)^{N-2-j_1} (q\cdot\Delta - p_1\cdot\Delta)^{j_1}$$

See e.g. [Moch et al., 2017].

Example: 1-loop ADM

The calculation of the bare OMEs is then straightforward and we find

$$\mathcal{O}_{N+1}^{(0)} = 1 + \frac{a_s}{\epsilon} C_F \left(2S_1(N+1) - \frac{2}{N+2} - 1 \right).$$

The last column of the mixing matrix can be calculated as

$$\gamma_{N,0} = \sum_{i=0}^{N} (-)^i \binom{N}{i} \mathcal{B}(i+1)$$

which in this example implies

$$\gamma_{N,0}^{(0)} = C_F \sum_{i=0}^{N} (-)^i \binom{N}{i} \left(2S_1(i+1) - \frac{2}{i+2} - 1 \right).$$

Example: 1-loop ADM

Using [Vermaseren, 1999]

$$\sum_{j=0}^{N'}(-1)^j\binom{N'}{j} = \delta_{N',0}$$

$$\sum_{j=0}^{N'}(-1)^j\binom{N'}{j}\frac{1}{m+j} = \frac{1}{\binom{N'+m}{N'}}\frac{1}{m}$$

$$\sum_{j=0}^{N'}(-1)^j\binom{N'}{j}S_1(m+j) = \frac{-1}{\binom{N'+m}{N'}}\frac{1}{N'},$$

we find

$$\gamma_{N,0}^{(0)} = C_F\left(\frac{2}{N+2}-\frac{2}{N}\right).$$

Example: 1-loop ADM

Next we need the single sum of the forward anomalous dimensions

$$\binom{N}{k}\sum_{j=0}^{N-k}(-1)^j\binom{N-k}{j}\gamma^{(0)}_{j+k,j+k}.$$

Using *SIGMA* we find

$$\binom{N}{k}\sum_{j=0}^{N-k}(-1)^j\binom{N-k}{j}\gamma^{(0)}_{j+k,j+k} = C_F\left(\frac{-2(k+1)}{N+2}+\frac{2(k+2)}{N+1}-\frac{4}{N-k}\right).$$

Based on this, we then choose the following Ansatz for the off-diagonal piece

$$\gamma^{(0)}_{N,k} = C_F\left(\frac{a_1}{N+2}+\frac{a_2}{N+1}+\frac{a_3}{N-k}\right).$$

Example: 1-loop ADM

Again using *SIGMA* for the double sum of the Ansatz

$$\frac{1}{C_F}\sum_{j=k}^{N}(-1)^k\binom{j}{k}\sum_{l=j+1}^{N}(-1)^l\binom{N}{l}\gamma_{l,j}^{(0)} =$$
$$a_1\left[\frac{-2-k}{N+1}+\frac{2+k}{N+2}\right]-\frac{a_2}{N+1}-\frac{a_3}{N-k}.$$

We now substitute the results into the consistency relation

$$\left(\frac{a_1}{N+2}+\frac{a_2}{N+1}+\frac{a_3}{N-k}\right)=\left(\frac{-2(k+1)}{N+2}+\frac{2(k+2)}{N+1}-\frac{4}{N-k}\right)$$
$$+\,a_1\left[\frac{-2-k}{N+1}+\frac{2+k}{N+2}\right]-\frac{a_2}{N+1}-\frac{a_3}{N-k}$$

$$\Rightarrow \gamma_{N,k}^{(0)} = C_F\left(\frac{2}{N+2}-\frac{2}{N-k}\right).$$

Results and discussion

- Wilson anomalous dimensions:
  - To $O(a_s^5)$ in the leading-n_f approximation
  - $O(a_s^2)$ expression in leading-color limit
  - Complete $N = 4$ matrix in full QCD, with some entries of the anomalous dimension matrix to $O(a_s^5)$
  - See [Moch and Van Thurenhout, 2021] for the explicit expressions

Focused in this talk on Wilson operators. However, nowhere in the analysis do we actually refer to the Dirac structure of the operators, and hence the methods and our algorithm also hold for operators with $\gamma_\mu \to \Gamma$

- Transversity ($\gamma_\mu \to \sigma_{\mu\nu}$) anomalous dimensions:
  - To $O(a_s^4)$ order in the leading-n_f approximation
  - See [Van Thurenhout, 2022] for the explicit expressions
- Extend expressions in Gegenbauer basis to $O(a_s^4)$ in leading-n_f limit

Leading-n_f limit: Consider generic flavor group SU(n_f) and send $n_f \to \infty$
Leading-color limit: Consider generic color group SU(N_c) and send $N_c \to \infty$

Results and discussion

$$\gamma^{\mathcal{D}}_{3,2} = -2.13333\, a_s + (-23.1028 + 2.14519\, n_f)\, a_s^2 + (-405.973 + 67.6368\, n_f + 0.51977\, n_f^2)\, a_s^3$$
$$+(-7071.4 + 1807.36\, n_f - 74.6209\, n_f^2 - 1.23872\, n_f^3)\, a_s^4 + O(a_s^5)$$
$$\gamma^{\mathcal{D}}_{3,1} = -0.8\, a_s + (-8.66775 + 0.693333\, n_f)\, a_s^2 + (-142.057 + 25.4966\, n_f + 0.289975\, n_f^2)\, a_s^3 + O(a_s^4)$$
$$\gamma^{\mathcal{D}}_{3,0} = -0.355556\, a_s + (-5.27039 + 0.209383\, n_f)\, a_s^2 + (-95.1612 + 12.9537\, n_f + 0.193624\, n_f^2)\, a_s^3 + O(a_s^4)$$
$$\gamma^{\mathcal{D}}_{2,1} = -2\, a_s + (-22.5556 + 1.96296\, n_f)\, a_s^2 + (-385.466 + 66.1992\, n_f + 0.532922\, n_f^2)\, a_s^3$$
$$+(-6437.94 + 1751.72\, n_f - 71.158\, n_f^2 - 1.1502\, n_f^3)\, a_s^4$$
$$+(-147044 + 43307.7\, n_f - 3728.02\, n_f^2 - 15.8471\, n_f^3 + 0.293057\, n_f^4)\, a_s^5 + O(a_s^6)$$
$$\gamma^{\mathcal{D}}_{2,0} = -0.666667\, a_s + (-9.50617 + 0.481481\, n_f)\, a_s^2 + (-170.654 + 24.8232\, n_f + 0.3107\, n_f^2)\, a_s^3 + O(a_s^4)$$
$$\gamma^{\mathcal{D}}_{1,0} = -1.77778\, a_s + (-24.1646 + 1.58025\, n_f)\, a_s^2 + (-429.724 + 66.7191\, n_f + 0.61454\, n_f^2)\, a_s^3$$
$$+(-8331.61 + 1873.78\, n_f - 58.8907\, n_f^2 - 0.954217\, n_f^3)\, a_s^4$$
$$+(-200373 + 51918.3\, n_f - 3724.15\, n_f^2 - 30.7867\, n_f^3 + 0.171853\, n_f^4)\, a_s^5 + O(a_s^6)$$

Blue results: New

Green results: Agreement with previous calculations [Gracey, 2003], [Kniehl and Veretin, 2020]

Cross-checks

✓ Gauge independence of the anomalous dimensions
→ When performing Feynman diagram computations: Work in general covariant gauge, keep gauge parameter

$$= -\frac{i}{p^4}(p^2 g_{\mu\nu} - \xi p_\mu p_\nu)\delta_{ab}$$

✓ Agreement with previous computations in the same operator basis

✓ Cross-check with results in different bases using derived basis transformation formulae

✓ Leading-n_f results agree with all-order computation based on conformal symmetry, see [Van Thurenhout and Moch, 2022] for details.

Summary and outlook

We can gain insight into proton structure by scattering elementary particles off protons. The relevant information is then typically represented by parton distributions

- Inclusive processes (forward kinematics): PDFs
- Exclusive processes (non-forward kinematics): GPDs.

These distributions are defined in terms of hadronic matrix elements of QCD operators and hence non-perturbative objects. However, their scale dependence is determined by the anomalous dimensions of the corresponding operators, which can be calculated order per order in perturbation theory.

The calculation of these anomalous dimensions is in principle straightforward in the forward case. For non-forward kinematics life becomes more complicated because of mixing with total derivative operators.

Summary and outlook

- New method for calculating off-diagonal elements of the mixing matrix, based purely on renormalization structure in chiral limit
- Independent check of previous calculations in different bases
- New results, e.g. 5-loop anomalous dimensions in leading-n_f limit
- Nice advantage of our method: Well-suited for automation with computer algebra programs
- Generalize method to different operators (e.g. flavor singlet operators) in QCD and different models altogether $\rightarrow$ needs to be studied

End

Thank you for your attention!

Appendices and references

Some comments on *Forcer*

- *Form* [Vermaseren, 2000], [Kuipers et al., 2013] program for the reduction of four-loop massless propagator-type integrals to master integrals
- Parametric IBP reductions
- Often possible to avoid explicit IBP reductions by reducing topologies to simpler ones (1-loop integrals, triangle rule, ...) $\rightarrow$ Automatized!
- Less diagrams for which actual IBP reductions are necessary, special rules for these

More details can be found in the original paper [Ruijl et al., 2020].

GPDs

$$F^q \equiv \int \frac{dz^-}{2\pi} e^{ix\chi^+ z^-} \langle p' | \overline{\psi}(-z/2)\gamma^+\psi(z/2) | p\rangle \sim H(x,\chi,t)\overline{\psi}(p')\gamma^+\psi(p)$$
$$+ E(x,\chi,t)\overline{\psi}(p')\frac{i\sigma^{+\nu}\tilde{\Delta}_\nu}{2m_p}\psi(p)$$
$$+ \text{higher twist}$$

$$\int dx\, x^N F^q \sim \overline{\psi}(0)\gamma^+ D^N \psi(0)$$

see e.g. [Diehl, 2003]. Here χ is the skewedness

$$\chi = \frac{p^+ - p'^+}{p^+ + p'^+}.$$

For some four-vector $v \equiv (v^0, v^1, v^2, v^3)$ light-cone coordinates are defined as

$$v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^3), \quad \vec{v} = (v^1, v^2).$$

ADM in the Gegenbauer basis

We start by introducing the renormalized non-local light-ray operators $[\mathcal{O}]$, which act as generating functions for local operators, see e.g. [Braun et al., 2017], as

$$[\mathcal{O}](x; z_1, z_2) = \sum_{m,k} \frac{z_1^m z_2^k}{m!\ k!} \left[\bar{\psi}(x)(\overleftarrow{D} \cdot n)^m \not{n}(n \cdot \overrightarrow{D})^k \psi(x)\right] .$$

The renormalization group equation for these light-ray operators can be written as

$$\left(\mu^2 \partial_{\mu^2} + \beta(a_s)\partial_{a_s} + \mathcal{H}(a_s)\right)[\mathcal{O}](z_1, z_2) = 0$$

The evolution operator $\mathcal{H}(a_s)$ is an integral operator and acts on the light-cone coordinates of the fields [Balitsky and Braun, 1989]

$$\mathcal{H}(a_s)[\mathcal{O}](z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, h(\alpha, \beta)[\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

with $z_{12}^\alpha \equiv z_1(1-\alpha) + z_2\alpha$ and the evolution kernel $h(\alpha, \beta)$. The moments of the evolution kernel correspond to the anomalous dimensions of the local operators.

ADM in the Gegenbauer basis

The light-ray operators admit an expansion in a basis of local operators in terms of Gegenbauer polynomials, see e.g. [Belitsky and Müller, 1999],

$$\mathcal{O}^{\mathcal{G}}_{N,k} = (\partial_{z_1} + \partial_{z_2})^k C_N^{3/2}\left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}}\right)\mathcal{O}(z_1, z_2)\Big|_{z_1=z_2=0},$$

where $k \geq N$ is the total number of derivatives. The Gegenbauer polynomials can be written as [Olver et al., 2010]

$$C_N^{\nu}(z) = \frac{\Gamma(\nu + 1/2)}{\Gamma(2\nu)} \sum_{l=0}^{N} (-1)^l \frac{\Gamma(2\nu + N + l)}{l!\,(N-l)!\,\Gamma(\nu + 1/2 + l)} \left(\frac{1}{2} - \frac{z}{2}\right)^l.$$

The renormalized operators $[\mathcal{O}^{\mathcal{G}}_{N,k}]$ obey the evolution equation

$$\left(\mu^2 \partial_{\mu^2} + \beta(a_s)\partial_{a_s}\right)[\mathcal{O}^{\mathcal{G}}_{N,k}] = \sum_{j=0}^{N} \gamma^{\mathcal{G}}_{N,j} [\mathcal{O}^{\mathcal{G}}_{j,k}].$$

ADM in the Gegenbauer basis

Beyond the one-loop level, also the mixing matrix in the Gegenbauer basis gets non-zero off-diagonal contributions. These can be calculated in general as [Müller, 1994, Braun et al., 2017]

$$\hat{\gamma}^{\mathcal{G}}(a_s) \;=\; \mathbf{G}\left\{[\hat{\gamma}^{\mathcal{G}}(a_s),\hat{b}]\left(\frac{1}{2}\hat{\gamma}^{\mathcal{G}}(a_s)+\beta(a_s)\right)+[\hat{\gamma}^{\mathcal{G}}(a_s),\hat{w}(a_s)]\right\}\,, \quad (3)$$

in terms of the matrix commutators denoted as $[*,*]$ and with

$$\mathbf{G}\{\hat{M}\}_{N,k} \;=\; -\frac{M_{N,k}}{a(N,k)}\,, \quad (4)$$

and

$$a(N,k) = (N-k)(N+k+3)\,, \quad (5)$$

$$\hat{b}_{N,k} = -2k\delta_{N,k} - 2(2k+3)\vartheta_{N,k}\,. \quad (6)$$

The discrete step-function in the last term is defined as

$$\vartheta_{N,k} \;\equiv\; \begin{cases} 1 & \text{if } N-k>0 \text{ and even} \\ 0 & \text{else.} \end{cases} \quad (7)$$

ADM in the Gegenbauer basis

The $N - k$ even condition originates from the fact that, in the Gegenbauer basis, only CP-even operators are considered. The conformal anomaly, $\hat{w}(a_s)$, can be written as a power series in the strong coupling

$$\hat{w}(a_s) = a_s \hat{w}^{(0)} + a_s^2 \hat{w}^{(1)} + \dots . \quad (8)$$

For the determination of the mixing matrix at order a_s^l, the conformal anomaly is only needed up to order a_s^{l-1} [Müller, 1991].

ADM in the Geyer basis

Here we define the operators as

$$\mathcal{O}^{\mathcal{B}}_{N,k} \equiv \overline{\psi}\gamma(\overrightarrow{D} + \overleftarrow{D})^{N-k}(\overleftarrow{D} - \overrightarrow{D})^{k}\psi,$$

see e.g. [Geyer, 1982] and [Blümlein et al., 1999]. The contraction with an arbitrary light-like vector is understood, i.e.

$$\gamma \equiv \Delta^{\mu}\,\gamma_{\mu},$$
$$D \equiv \Delta^{\mu}\,D_{\mu}$$

and $\Delta^2 = 0$. These operators, and correspondingly their anomalous dimensions, can be related to those in the derivative basis. For this the relation

$$\mathcal{O}^{\mathcal{D}}_{0,N-k,k} = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathcal{O}^{\mathcal{D}}_{j,N-j,0}$$

for the operators in the derivative basis is useful.

ADM in the Geyer basis

For the bare operators we then find (apply binomial theorem twice)

$$\mathcal{O}^{\mathcal{B}}_{N,k} = \sum_{i=0}^{N-k} \sum_{j=0}^{k} (-1)^i \binom{N-k}{i} \binom{k}{j} \sum_{l=0}^{i+j} (-1)^l \binom{i+j}{l} \mathcal{O}^{\mathcal{D}}_{l,N-l,0}$$

which leads to

$$[\mathcal{O}^{\mathcal{B}}_{N,k}] = \sum_{i=0}^{N-k} \sum_{j=0}^{k} (-1)^i \binom{N-k}{i} \binom{k}{j} \sum_{l=0}^{i+j} (-1)^l \binom{i+j}{l} \sum_{m=0}^{N-l} (-1)^m \, \gamma^{\mathcal{D}}_{N-l,m} \times \sum_{n=0}^{m} (-1)^n \binom{m}{n} [\mathcal{O}^{\mathcal{D}}_{N-m+n,0,m-n}]$$

for the renormalized ones.

ADM in the Geyer basis

The evolution of the local operators in this basis can be summarized as

$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}^{\mathcal{B}}_{N,N} = \sum_{k=0}^{N} \frac{1 \pm (-1)^k}{2} \gamma^{\mathcal{B}}_{N,k} [\mathcal{O}^{\mathcal{B}}_{k,N}].$$

The relation between the anomalous dimensions in the Geyer and derivative bases then becomes

$$\gamma^{(0)}_{N,N} + \sum_{j=0}^{N-1} \frac{1 \pm (-1)^j}{2} \gamma^{(0),\mathcal{B}}_{N,j} = \pm(-1)^N \sum_{l=0}^{N} 2^l (-1)^l \binom{N}{l} \gamma^{(0),\mathcal{D}}_{l,0}$$

at the 1-loop level with the upper (lower) sign for even (odd) N. This provides an additional consistency check at this order.

Five-loop ADM in the leading-n_f limit

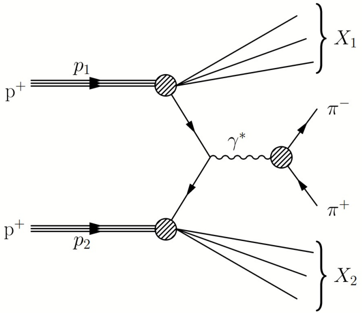
- By analyzing the leading-n_f anomalous dimensions up to 4 loops, it becomes clear that certain patterns start to emerge.
- The majority of terms in the L-loop anomalous dimensions can be deduced from the expression of the $(L-1)$-loop ones.
- What is left then is a small number of unknown terms, which can be fixed by using our consistency relation.
- This is how we determined the expression for the 5-loop Wilson anomalous dimensions. Note that this in principle can be extended to arbitrary orders.
- This method is also used to determine the leading-n_f mixing matrices for the transversity operators.

All-order results in the leading-n_f limit

- The forward anomalous dimensions in the leading-n_f limit have been determined to all orders in perturbation theory
- The computation was based on exact conformal symmetry at the Wilson-Fisher critical point, see [Gracey, 1994] and [Gracey, 2003]
- This technique was recently extended for the non-forward anomalous dimensions, and the results presented here agree with the more general ones.
- See [Van Thurenhout and Moch, 2022] for details and explicit expressions.

Example of an inclusive process

- Inclusive polarized Drell-Yan



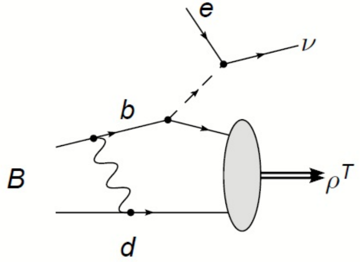
Distributions: Transversity distributions (TDFs) $h^T(x, \mu_f^2)$

[Ralston and Soper, 1979], [Artru and Mekhfi, 1990], [Jaffe and Ji, 1991], [Jaffe and Ji, 1992], [Cortes et al., 1992]

- ◇ Difference in probabilities of finding a parton in a transversly polarized nucleon polarized parallel to the nucleon spin and an oppositely polarized one
- ◇ Studied e.g. by the STAR experiment at RHIC [Adamczyk et al., 2015]

Example of an exclusive process

- Exclusive production of transversely polarized ρ-meson



Distributions: Transverse distribution amplitudes (DAs) $\phi(x, \mu_F^2)$

[Lepage and Brodsky, 1980]

- ◇ Measure parton distributions within mesons
- ◇ Important input for e.g. LHCb [Aaij et al., 2013]

Relations in x-space

We start from the evolution kernel in x-space, $V(x, y)$. The Mellin transform of this kernel with respect to x is given by a polynomial of degree N in y, the coefficients of which are the N-space anomalous dimensions

$$-\int \mathrm{d}x\, x^N V(x, y) = \sum_{k=0}^{N} \gamma_{N,k}\, y^k. \tag{9}$$

In the limit of $y \to 1$, the RHS of Eq.(9) simply reduces to a sum of anomalous dimensions

$$-\lim_{y\to 1} \int \mathrm{d}x\, x^N V(x, y) = \sum_{k=0}^{N} \gamma_{N,k} \tag{10}$$

which is equivalent to the binomial transform of the elements in the last column of the ADM in the total derivative basis

$$-\lim_{y\to 1} \int \mathrm{d}x\, x^N V(x, y) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \gamma_{k,0}. \tag{11}$$

Furthermore, from Eq.(9) it follows that the last column corresponds to the $y \to 0$ limit of the Mellin transform

$$\gamma_{N,0} = -\lim_{y\to 0} \int \mathrm{d}x\, x^N V(x,y). \tag{12}$$

Hence, we find the following relation for the x-space evolution kernel

$$\lim_{y\to 1} \int \mathrm{d}x\, x^N V(x,y) = \lim_{y\to 0} \sum_{k=0}^{N} (-1)^k \binom{N}{k} \int \mathrm{d}x\, x^k V(x,y). \tag{13}$$

Relations in x-space

We can also relate sums of anomalous dimensions multiplied by the summation parameter to derivatives of the Mellin transform of the kernel, e.g.

$$\sum_{k=1}^{N} k\gamma_{N,k} = -\lim_{y\to 1}\frac{\mathrm{d}}{\mathrm{d}y}\int \mathrm{d}x\, x^N V(x,y). \tag{14}$$

In the total derivative basis, this sum is related to the binomial transform of the first column

$$\sum_{k=1}^{N} k\gamma_{N,k} = -\sum_{k=1}^{N}(-1)^k\binom{N}{k}\gamma_{k,1}. \tag{15}$$

Hence using the conjugation property of the binomial transform we could write

$$\gamma_{N,1} = \lim_{y\to 1}\frac{\mathrm{d}}{\mathrm{d}y}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\int \mathrm{d}x\, x^j V(x,y). \tag{16}$$

Relations in x-space

We can also access $\gamma_{N,1}$ directly from Eq.(9) as

$$\gamma_{N,1} = -\lim_{y\to 0} \frac{\mathrm{d}}{\mathrm{d}y} \int \mathrm{d}x\, x^N V(x,y). \tag{17}$$

Combining Eqs.(16) and (17) then gives

$$\lim_{y\to 1} \frac{\mathrm{d}}{\mathrm{d}y} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \int \mathrm{d}x x^j V(x,y) = -\lim_{y\to 0} \frac{\mathrm{d}}{\mathrm{d}y} \int \mathrm{d}x x^N V(x,y). \tag{18}$$

Relations in x-space

This type of relation can be generalized to have an arbitrary number of derivatives acting on the Mellin transform of the kernel. First we can write

$$\lim_{y\to 1} \frac{\mathrm{d}^k}{\mathrm{d}y^k} \int \mathrm{d}x\, x^N V(x,y) = -\sum_{j=k}^{N} (j)_k \gamma_{N,j} \tag{19}$$

$$= -k! \sum_{j=k}^{N} \binom{j}{k} \gamma_{N,j} \tag{20}$$

with

$$(k)_n = k(k-1)(k-2)\dots(k-n+1) \tag{21}$$

the Pochhammer symbol.

Relations in x-space

Multiplying twice with a factor of $(-1)^k$ and using that the anomalous dimensions in the total derivative basis satisfy

$$\sum_{j=k}^{N}(-1)^k\binom{j}{k}\gamma_{N,j} = \sum_{j=k}^{N}(-1)^j\binom{N}{j}\gamma_{j,k} \tag{22}$$

we can rewrite this as

$$\lim_{y\to 1}\frac{\mathrm{d}^k}{\mathrm{d}y^k}\int \mathrm{d}x\, x^N V(x,y) = -(-1)^k k! \sum_{j=k}^{N}(-1)^j\binom{N}{j}\gamma_{j,k} \tag{23}$$

$$= -(-1)^k k! \sum_{j=0}^{N}(-1)^j\binom{N}{j}\gamma_{j,k}. \tag{24}$$

The second equality again follows from the triangularity of the ADM.

Relations in x-space

Applying the binomial transform then yields

$$\lim_{y\to 1} \frac{\mathrm{d}^k}{\mathrm{d}y^k} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \int \mathrm{d}x\, x^j V(x,y) = -(-1)^k k!\, \gamma_{N,k}. \qquad (25)$$

Finally, from Eq.(9) it follows that

$$\gamma_{N,k} = -\frac{1}{k!} \lim_{y\to 0} \frac{\mathrm{d}^k}{\mathrm{d}y^k} \int \mathrm{d}x\, x^N V(x,y) \qquad (26)$$

such that

$$\lim_{y\to 1} \frac{\mathrm{d}^k}{\mathrm{d}y^k} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \int \mathrm{d}x\, x^j V(x,y) = (-1)^k \lim_{y\to 0} \frac{\mathrm{d}^k}{\mathrm{d}y^k} \int \mathrm{d}x\, x^N V(x,y). \qquad (27)$$

This identity is valid for all $k \geq 0$.

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