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New renormalons from analytic trans-series

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January 13, 2023

WORK WITH

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1. Review on renormalons and the OPE prediction

2. Renormalons in integrable QFTs
(Work with Marcos Mariño and Tomás Reis)

Review on renormalons and the OPE prediction

Perturbative expansions and trans-series

- Perturbative expansions of relevant quantities in a QFT are factorially divergent. For example, a propagating particle ϕ expanded in the coupling constant $\alpha > 0$:

$$\langle \phi(x)\phi(y)\rangle(\alpha) \propto \int \mathcal{D}\phi \, \mathrm{e}^{-(S_{\text{free}}[\phi]+\alpha S_{\text{int}}[\phi])} \phi(x)\phi(y)$$

$$\sim \sum_{n\geq 0} \frac{\alpha^n}{n!} \int \mathcal{D}\phi \, \mathrm{e}^{-S_{\text{free}}[\phi]} \underbrace{S_{\text{int}}[\phi]^n}_{\text{Polynomial in } \phi} \phi(x)\phi(y) = \sum_{n\geq 0} a_n \alpha^n$$

$$\text{where } a_n \overset{n\to\infty}{\sim} K r^n n^b n!$$

  S is the action of the QFT and the coefficients a_n can be computed from Feynman diagrams.

- To fully describe a quantity in QFT, we need to extend the perturbative series to a **trans-series**:

$$\langle \phi(x)\phi(y)\rangle(\alpha) \sim \underbrace{\sum_{n\geq 0} a_n \alpha^n}_{\substack{\text{Factorially divergent} \\ \text{(Accesible from} \\ \text{perturbation theory)}}} + \mathrm{e}^{-2/\alpha}\alpha^{-b} \underbrace{\sum_{n\geq 0} b_n \alpha^n}_{\text{Factorially divergent}} + \cdots$$

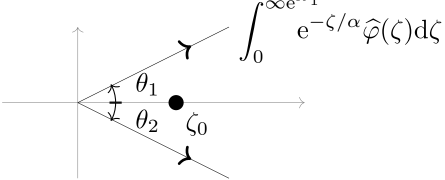
Borel summation and imaginary ambiguities

$$\widetilde{\varphi}(\alpha) = \sum_{n\geq 0} a_n \alpha^{n+1} \xrightarrow{\text{Borel t.}} \widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n \xrightarrow{\text{Laplace t.}} \int_0^{\infty e^{i\theta}} e^{-\zeta/\alpha}\widehat{\varphi}(\zeta)d\zeta.$$

Large order behavior of the perturbative series | Singularity in the Borel transform | Imaginary ambiguity in the Borel sum

$$a_n \sim \left(\frac{1}{\zeta_0}\right)^n n! \xrightarrow{\text{Borel t.}} \frac{1}{(1-\zeta/\zeta_0)} \xrightarrow{\text{Laplace t.}} \pm i\pi e^{-\zeta_0/\alpha}$$

Borel plane ζ



- Quantities we compute in QFT must be real (and unambiguous) for $\alpha > 0$.
- This indicates we need to add imaginary ambiguous exponentials to the perturbative result:

  Borel sum $\left(\sum a_n \alpha^{n+1}\right) \mp i\pi e^{-\zeta_0/\alpha}$

  is non-ambiguous.

Instantons and renormalons

Factorial divergence	$\longrightarrow$	Ambiguity in the Borel sum	$\longrightarrow$	Addition of exponentially small terms (imaginary ambiguous)

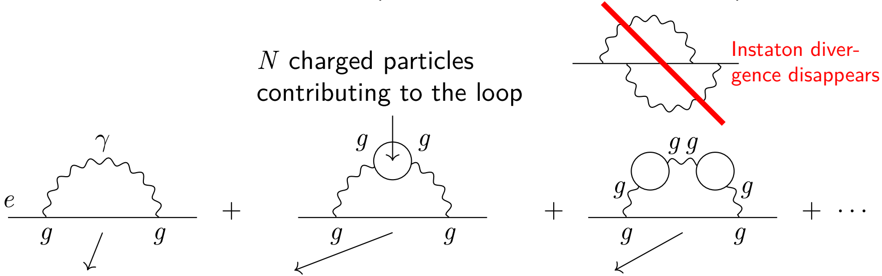
- 2 types of factorial divergence, classified according to their origin:

  instantons and **renormalons**

- **Instanton** factorial divergence arises from increasing number of diagrams.
- However, we know that **renormalon** factorial divergences are also present in many QFTs, including realistic models like QED and QCD.

Renormalon definition

Renormalons are the factorial divergence arising from small and large momentum in loop integrals.
Example of renormalon divergence (propagating electron at large N):



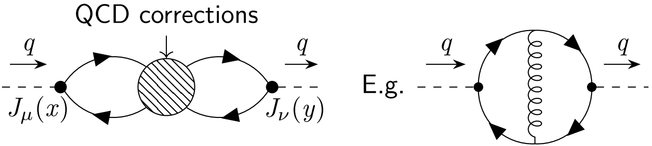
$$\color{red}{a_{0,0}}\color{black}{g^2 + \left(a_{1,0} + }\color{red}{a_{1,1}}\color{black}{N\right) g^4 + \left(a_{2,0} + a_{2,1}N + }\color{red}{a_{2,2}}\color{black}{N^2\right) g^6 + \cdots}$$

- We only select diagrams that give the highest power of N at each fixed order in the coupling g^2 (N is the number of particles contributing to the loops).
- When we compute the loop integrals, the diagram at order g^{2m} gives a coefficient that goes like $m!$.
- This factorial divergence is what we call renormalon, and is unrelated to the increase in number of diagrams.

The operator product expansion (OPE)

- Renormalons are usually understood through the large N limit and the operator product expansion (OPE).
- Renormalon locations in the Borel plane can be derived from general grounds in asymptotically free QFTs, by using the OPE (the arguments go back to Parisi, 't Hooft in 1970-1980).
- Typical example: The **Adler function** $D(q^2)$ in massless QCD:

$$\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}(x-y)q} \langle J_\mu(x) J_\nu(y) \rangle, \qquad \text{quark current } J_\mu(x) = \overline{q}(x)\gamma_\mu q(x),$$



- OPE: $A(x)[= J_\mu(x)]$ and $B(y)[= J_\nu(y)]$ two local operators

$$A(x)B(0) = \sum_i c_i(x) O_i(0), \quad x \to 0,$$

where $O_i(0)$ are also local operators and $c_i(x)$ are the Wilson coefficients.

The operator product expansion (OPE)

- To construct the OPE of the Adler function, we have to list all local operators that can be build from the fields:

  q_i quark field (dimension 3/2), $\mathcal{A}_\mu^a$ gluon field (dimension 1).

- The OPE of the Adler function becomes an expansion at large q, when going to momentum space

$$D(q^2) = \overbrace{c_{\mathbb{I}}(\alpha)}^{\text{Perturbative part}} + \underbrace{\frac{0}{q^{1,2,3}}}_{\text{No operators of dim 1,2,3}} + \overbrace{c_{GG}(\alpha)}^{\text{Expansion in the coupling}} \frac{\langle G_{\mu\nu} G^{\mu\nu} \rangle}{q^4} + \underbrace{\mathcal{O}(1/q^6)}_{\text{Operators of higher dimension}}$$

$\longrightarrow$ Increasing dimension of the operators

  where $G_{\mu\nu}$ (dim 2) is the field strength tensor of the gluon.

- The lowest dimensional local operator, which is Lorentz and gauge invariant, and respects the symmetries of massless QCD, is $G_{\mu\nu} G^{\mu\nu}$ (dim 4).
- $\mathcal{A}_\mu^a \mathcal{A}_\mu^a$ (dim 2) excluded from gauge invariance.
- $\overline{q}_i q_i$ (dim 3) breaks chiral symmetry.

OPE renormalon prediction

- The OPE is an expansion at large q^2, but we can rewrite it as an expansion in the **running** coupling:

$$\alpha(\mu^2 = q^2) = \frac{1}{\beta_0 \log(q^2/\Lambda^2)} \implies \frac{\Lambda^{2n}}{q^{2n}} = \overbrace{\mathrm{e}^{-n/(\beta_0 \alpha)}}^{\beta_0>0 \text{ in QCD (asymptotic freedom)}}$$

where $\beta_0 = \frac{1}{4\pi}\left(11 - \frac{2N}{3}\right)$ is the first coefficient of the QCD beta function, N is the number of quarks and Λ is the QCD scale parameter (perturbative approximations are not valid at energies $E < \Lambda$).

- In terms of the coupling, the OPE of the Adler function becomes a trans-series

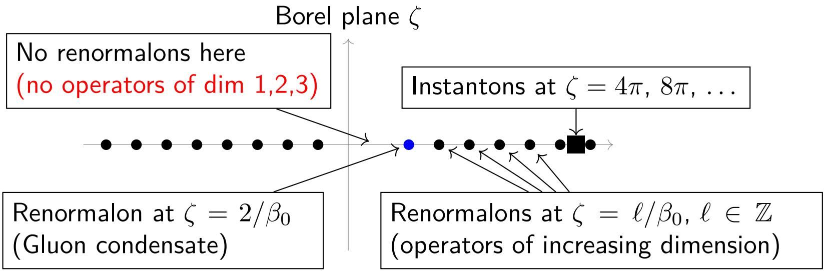
$$D(q^2) = c_{\mathbb{I}}(\alpha) + \overbrace{0}^{\text{No operators of dim 1,2,3}} + c_{GG}(\alpha)\frac{\langle G_{\mu\nu} G^{\mu\nu}\rangle}{\Lambda^4}\mathrm{e}^{-2/(\beta_0\alpha)} + \mathcal{O}\left(\mathrm{e}^{-3/(\beta_0\alpha)}\right).$$

- The first exponential correction has an imaginary ambiguity (in $\langle G_{\mu\nu} G^{\mu\nu}\rangle$) that will fix the first singularity in the Borel plane.

OPE renormalon prediction

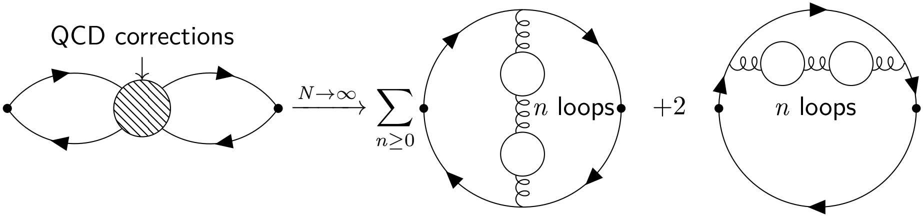
Operator of dimension d $\longrightarrow$ $\frac{1}{q^d} \sim \mathrm{e}^{-d/(2\beta_0 \alpha)}$ ambiguity in the OPE $\longrightarrow$ Factorial divergence $a_n \sim n! \left(\frac{1}{d/(2\beta_0)} \right)^n$ Renormalon singularity at $\zeta = \frac{d}{2\beta_0}$

- Predicted renormalons for the Adler function $D(q^2)$, according to the OPE:



OPE renormalon prediction

The renormalon locations are **confirmed** in the large N limit of the Adler function (including the absence of renormalons close to the origin):



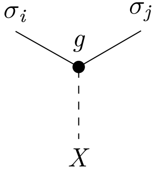
Renormalons in integrable QFTs (Work with Marcos Mariño and Tomás Reis)

Asymptotically free integrable QFTs

Asymptotically free integrable 2-dimensional QFTs are very rich and exactly solvable, which makes them great toy models. E.g.:

- $O(N)$ non-linear sigma model: N scalar particles $\boldsymbol{\sigma}(x) = \big(\sigma_1(x), \ldots, \sigma_N(x)\big)$ satisfying the constraint $\boldsymbol{\sigma}(x) \cdot \boldsymbol{\sigma}(x) = 1$:

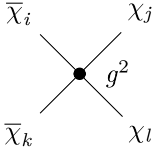
$$\mathcal{L}(\boldsymbol{\sigma}, X, g) = \frac{1}{g^2}\left\{\frac{1}{2}\partial^\mu \boldsymbol{\sigma} \cdot \partial_\mu \boldsymbol{\sigma} + X\big(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} - 1\big)\right\}$$



  (X is a Lagrange multiplier that imposes the constraint $\boldsymbol{\sigma}(x) \cdot \boldsymbol{\sigma}(x) = 1$).

- $O(N)$ Gross-Neveu model: N fermions $\boldsymbol{\chi}(x) = \big(\chi_1(x), \ldots, \chi_N(x)\big)$ with a 4 vertex interaction:

$$\mathcal{L}(\boldsymbol{\chi}, g) = \frac{\mathrm{i}}{2}\overline{\boldsymbol{\chi}} \cdot (\gamma^\mu \partial_\mu \boldsymbol{\chi}) + \frac{g^2}{8}(\overline{\boldsymbol{\chi}} \cdot \boldsymbol{\chi})^2$$



The free energy $F(h)$

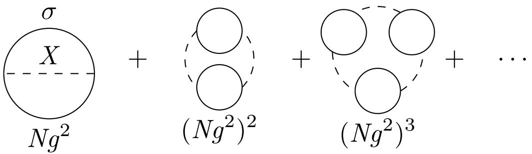
- In order to use integrability to our advantage, we add a chemical potential h coupled to a conserved charge Q such that it excites a single species of particles of the lowest mass m in the ground state

$$\mathcal{L} \xrightarrow[\text{formalism}]{\text{Hamiltonian}} \boxed{\mathsf{H} \mapsto \mathsf{H} - h\mathsf{Q}} \xrightarrow[\text{formalism}]{\text{Lagrangian}} \mathcal{L}(h) = \mathcal{L} + \begin{cases} 2\mathrm{i}h(\sigma_1\partial_0\sigma_2 - \sigma_2\partial_0\sigma_1) \\ \qquad\qquad + h^2(\sigma_1^2 + \sigma_2^2), \\ h\,\overline{\chi}_1\gamma_0\chi_1. \end{cases}$$

- We are interested in the **free energy** per unit volume:

$$F(h) = -\lim_{V,\beta\to\infty} \frac{1}{V\beta} \log \mathrm{Tr}\, \mathrm{e}^{-\beta(\mathsf{H} - h\mathsf{Q})} \propto \boxed{\begin{array}{c}\text{Lowest eigenvalue} \\ \text{of } \mathsf{H} - h\mathsf{Q}.\end{array}}$$

- The free energy can be computed perturbatively from a path integral corresponding to diagrams with no external edges (vacuum diagrams). E.g. in the large N limit, we would consider the diagrams



The free energy from the Bethe ansatz

- The free energy can also be computed from the Bethe ansatz.
- The Fermi density of Bethe roots $\epsilon(\theta)$ satisfies the integral equation

$$\epsilon(\theta) - \int_{-B}^{B} K(\theta - \theta')\epsilon(\theta')\mathrm{d}\theta' = h - \underset{\uparrow}{m}\cosh(\theta), \quad \epsilon(\pm B) = 0,$$

Mass gap $m \propto \mathrm{e}^{-1/(Ng^2)}$

where the kernel $K(\theta)$ is specified by the S-matrix of the excited particles:

$$K(\theta) = \frac{1}{2\pi\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\theta} \log S(\theta).$$

- The S-matrix can be derived from integrability.
  $O(N)$ non-linear sigma model:

$$S(\theta) = -\frac{\Gamma\left(1+\mathrm{i}\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\Delta+\mathrm{i}\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}-\mathrm{i}\frac{\theta}{2\pi}\right)\Gamma\left(\Delta-\mathrm{i}\frac{\theta}{2\pi}\right)}{\Gamma\left(1-\mathrm{i}\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\Delta-\mathrm{i}\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\mathrm{i}\frac{\theta}{2\pi}\right)\Gamma\left(\Delta+\mathrm{i}\frac{\theta}{2\pi}\right)}, \quad \Delta = \frac{1}{N-2}.$$

- B has the role of a coupling in the Bethe ansatz setting:

$$\frac{1}{B} = 2\beta_0 g^2 + \mathcal{O}\left(g^4\right),$$ (β_0 is the first coeff. of the beta function, $\beta_0 = \frac{N-2}{4\pi}$ for σ model).

Extracting renormalons from the Bethe ansatz

- The free energy can then be computed as

$$F(h) - F(0) = -\frac{m}{2\pi}\int_{-B}^{B} \epsilon(\theta)\cosh(\theta)\mathrm{d}\theta.$$

- The Bethe ansatz equations can be solved numerically for a given $B > 0$.

Important

The Bethe ansatz result contains "everything":

$$\text{perturbative expansion} + \underbrace{\text{exponential corrections.}}_{\text{ambiguity cancellation}+\text{other exponentials}}$$

- Extracting a perturbative expansion from the Bethe ansatz equations is a non-trivial exercise (involves treating the integral equations in Fourier space, using a Wiener-Hopf decompositon,...).
- Thanks to the work of D. Volin, it is possible to extract very long perturbative series for $F(h) - F(0)$ directly from the Bethe ansatz.

Exponential corrections from the Bethe ansatz

- We analyzed the integral equations with the Wiener-Hopf method, as in old work [Wiegmann, Hasenfratz, Niedermayer, Balog, Wiesz,...], but incorporating exponentially small corrections that were previously neglected.
- This leads to fully analytic results for the trans-series of $F(h) - F(0)$.
- For example, for the $O(N=3)$ non-linear sigma model, we found ($\alpha = g^2/(2\pi) + \mathcal{O}(g^4)$)

$$F(h) - F(0) = -\frac{h^2}{4\pi}\Bigg[\overbrace{\frac{1}{\alpha} - \frac{1}{2} + \mathcal{O}(\alpha)}^{\text{Perturbative series}} + \frac{32}{\mathrm{e}^2}\left(-\frac{2}{\alpha^3} + \frac{-\log(\alpha) - 3 + \gamma_E + 5\log(2)}{\alpha^2} + \mathcal{O}\left(\alpha^{-1}\right)\right)\mathrm{e}^{-2/\alpha}$$

$$+ \frac{512}{\mathrm{e}^4}\left(\frac{1 \overbrace{\pm \mathrm{i}}^{\text{cancels instanton ambiguity in pert. series}}}{\alpha^3} + \mathcal{O}\left(\alpha^{-2}\right)\right)\mathrm{e}^{-4/\alpha} + \mathcal{O}\left(\mathrm{e}^{-6/\alpha}\right)\Bigg] \overbrace{\mp \mathrm{i}\frac{m^2}{16}}^{\text{cancels renormalon } \propto\, \mathrm{e}^{-2/\alpha}}.$$

Explicit imaginary ambiguities cancel with the ambiguities emerging from the Borel sum of each divergent series.

Classifying the exponential corrections in instantons and renormalons

- Even more interesting is to compute the exponential corrections for general N. By then taking the large N limit, we can identify if the exponential corrections have an instanton or a renormalon origin.

Reminder

Instanton factorial divergences arise from the increasing number of diagrams. In the large N limit, we only consider a selected number of diagrams in which instantons disappear, but renormalons survive.

- We found the following exponential corrections:

$O(N)$ σ model: $\pm\mathrm{i}\,\exp\left(-\frac{2}{\alpha}(N-2)\ell\right) \xrightarrow{N\to\infty} 0$ Instanton

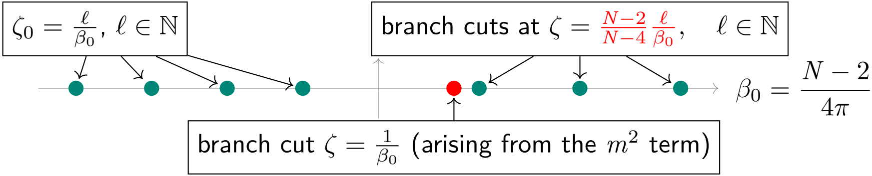
$O(N)$ G-N model: $\pm\mathrm{i}\,\exp\left(-\frac{2}{\alpha}\frac{N-2}{N-4}\ell\right) \xrightarrow{N\to\infty} \pm\mathrm{i}\,\exp\left(-\frac{2}{\alpha}\ell\right)$ Renorm.

where $\alpha = 2\beta_0 g^2 + \mathcal{O}(g^4)$ and $\ell = 1, 2, 3, \ldots$

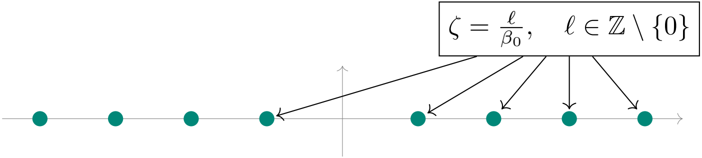
- The two models also have an exponential correction $\mathrm{e}^{-2/\alpha}$, arising from the m^2 term. This exponential correction is of renormalon origin.

Location of renormalons in integrable QFTs

- Let us focus on Gross-Neveu, where everything is renormalons:



- However, the complete renormalon prediction from the OPE would be:



- This picture matches our result at large N, but not at finite N.

Caveat

The free energy does not admit an OPE, but one expects the position of renormalons to be universal for all quantities computed in a given QFT.

Conclusions

- Quantities in QFTs have factorially divergent perturbative series. To fully describe a quantity, we have to also add exponentially supressed terms in the coupling:

$$\langle \phi(x)\phi(y)\rangle(\alpha) \sim \sum_{n\geq 0} a_n \alpha^n + \mathrm{e}^{-2/\alpha}\alpha^{-b}\sum_{n\geq 0} b_n\alpha^n + \cdots$$

- An important type of factorial divergences are **renormalons**, which can be represented as singularities in the Borel plane and are associated with exponential corrections. Renormalons are mostly understood through the large N limit and the OPE.
- By exploiting the integrability of some QFTs, we were able to test the OPE prediction about the position of renormalons. The OPE prediction seems to be a large N approximation.
- Is the original prediction really wrong? Is there an explanation for this discrepancy?

Many thanks!