The scaling region of the tricritical Ising model

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- A. Cortes-Cubero, R. Konik, ML, G. Mussardo, G. Takács, "Duality and form factors in the thermally deformed two-dimensional tricritical Ising model", SciPost Phys. 12 (2022) 162, arXiv2109.09767
- ML, G. Mussardo, G. Takács, "Confinement in the tricritical Ising model", Phys.Lett.B 828 (2022) 137008, arXiv:2111.05360
- ML. G. Mussardo, G. Takács, "Variations on vacuum decay: The scaling Ising and tricritical Ising field theories", Phys.Rev.D 106 (2022) 10, 105003, arXiv:2208.02273
- ML, A. Miscioscia, G. Mussardo, G. Takács, "Multicriticality in Yang-Lee edge singularity", accepted by JHEP, arXiv:2211.01123

E_8 experiment I.,$CoNb_2O_6$

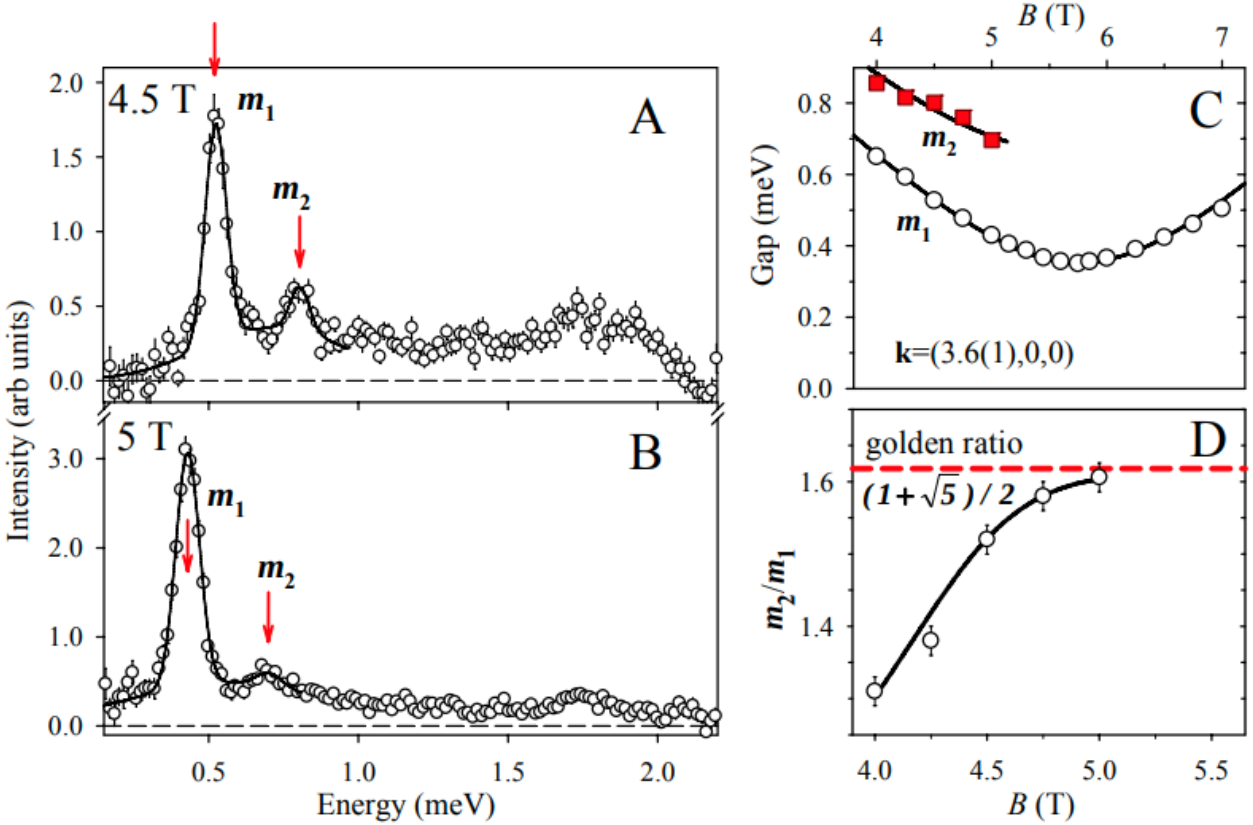

Figure 1: E_8 experiment, Picture from Coldea et.al. Science 327, 177 (2010); See also Zamolodchikov 1989

E_8 experiment II., $BaCo_2V_2O_8$

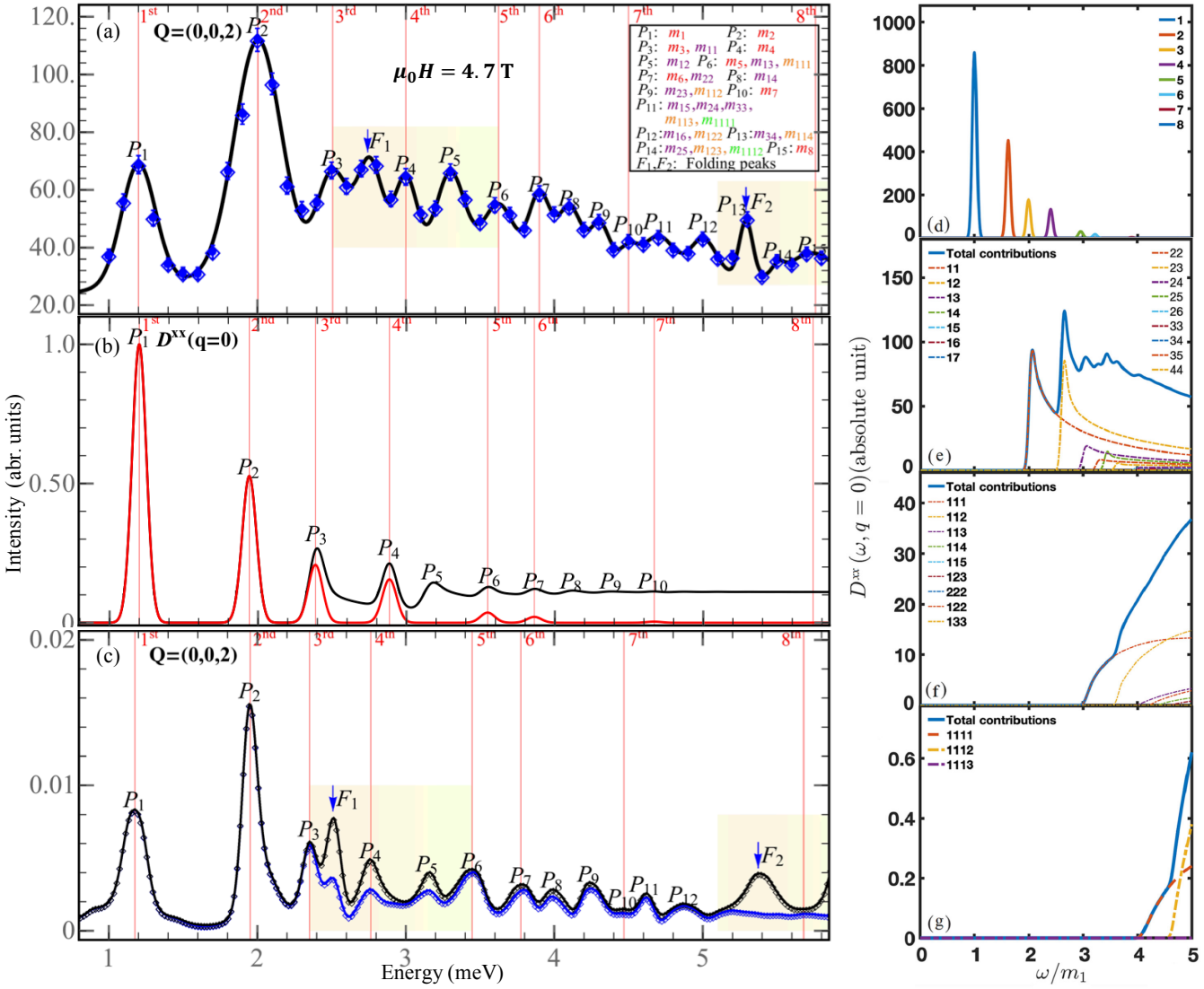

Figure 2: E_8 experiment, Picture from Zou et al. PRL 127 (2021)

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Ising Model (IM)

- 2D square lattice, $s_i = \pm 1$

$$\mathcal{H} = -\tilde{J} \sum_{\langle i,j \rangle} s_i s_j$$

- Quantum Ising model (transverse field, quantum spin chain)

$$H(\sigma, h) = -J \sum_i \sigma_i^z \sigma_{i+1}^z - Jh \sum_i \sigma_i^x$$

- Z_2 symmetry: simultaneous flip of all spins
- Quantum critical point $h = 1$, ferromagnetic/paramagnetic phase
- Transverse field is related to the temperature: $h - 1 \propto T - T_c$
- Order parameter: σ^z

IM: Disorder operator, Duality

- Introducing

$$\begin{aligned} \mu_i^z &= \prod_{j=-\infty}^{i} \sigma_j^x \\ \mu_i^x &= \sigma_i^z \sigma_{i+1}^z \end{aligned}$$

one can check that the operators $\mu^{x,z}$ satisfy the same algebra as $\sigma^{x,z}$, moreover

$$H(\sigma, h) \propto H(\mu, h^{-1}) \tag{1}$$

- This is the Kramers–Wannier duality, and μ^z is called the disorder operator.

IM scaling limit: free fermions

- In the scaling limit: $J \to \infty, h \to 1$ with fixed

$$M = 2J|1-h|$$

scales to a free Majorana fermion field theory

$$H = \int_{-\infty}^{\infty} dx \frac{1}{2\pi} \left[\frac{i}{2} \left(\psi(x)\partial_x \psi(x) - \bar{\psi}(x)\partial_x \bar{\psi}(x) \right) - iM\bar{\psi}(x)\psi(x) \right]$$

- The KW duality:

$$M \to -M; \quad \psi \to \psi; \quad \bar{\psi} \to -\bar{\psi}$$

- Critical point: $M = 0$, massless free fermion, $\mathcal{M}_{3,4}$ minimal CFT with $c = 1/2$ three primaries

$$\mathbf{1}, \quad \varepsilon, \quad \sigma/\mu$$

- Manifestation of the duality:

$$\varepsilon \to -\varepsilon; \quad \sigma \to \mu$$

Ising Field Theory

- General perturbation of the Ising fixed point:

$$\mathcal{A} = \mathcal{A}_{CFT} + \lambda \int d^2x \varepsilon + h_{\|} \int d^2x \sigma$$

- $h_{\|} = 0$, $\lambda > 0$ Free fermion, disordered phase (1GS+pt.), $\lambda < 0$ ordered phase (2GS+kinks)
- $\lambda = 0$ $h_{\|} \neq 0$ Zamolodchikov's E_8 integrable model
- General: non-integrable, inelastic scattering, confinement, Yang–Lee singularity:
- "Ising spectroscopy" Zamolodchikov et al. 2001, 2006, 2011, 2013, 2022

Confinement in the Ising field theory

- Ordered phase: 2-fold degenerate ground state, kink excitations of mass m_k
- $h_{\|} \neq 0$: explicit symmetry breaking, lifts GS degeneracy
- False vacuum: finite energy density

$$\Delta\mathcal{E} \approx 2h_{\|}\langle\sigma\rangle_{h_{\|}=0}$$

- "Meson" masses in the simplest approximation: 1D linear potential McCoy, Wu 1978

$$M_n \approx m_k \left[2 + \left(\frac{\Delta\mathcal{E}}{m_k}\right)^{2/3} z_n\right]$$

  where $-z_n$ are the zeroes of the Airy function.
- There are better approximations Fonseca, Zamolodchikov 2006; Rutkevich 2017

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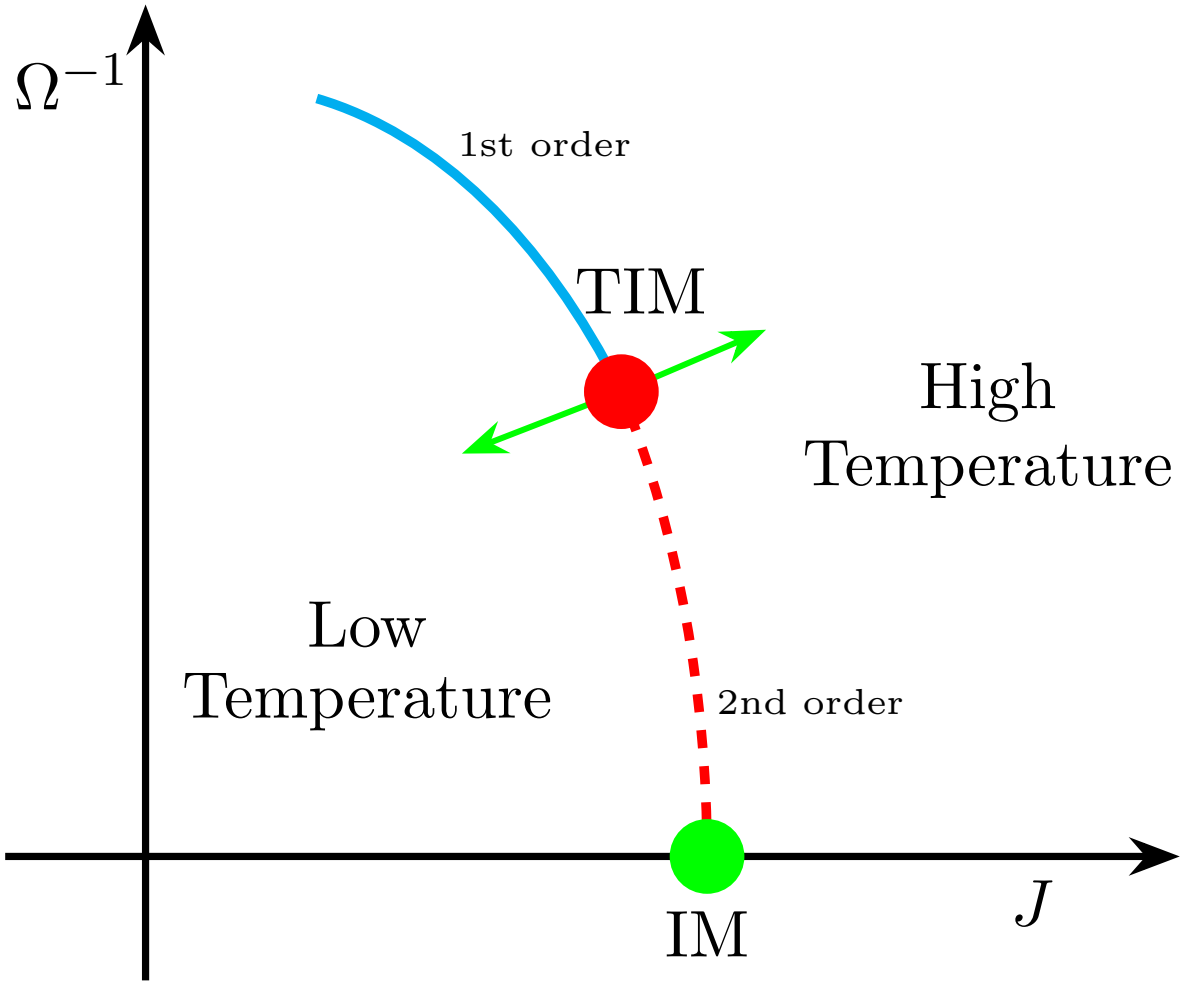
Tricritical Ising Model (TIM): IM with vacancies

- Ising spins with vacancies

$$\begin{aligned}\mathcal{H} &= -J\sum_{\langle i,j\rangle}^{N} s_i s_j t_i t_j - \Omega\sum_{i=1}^{N} t_i - H\sum_{i=1}^{N} s_i t_i \\ &\quad -H_3\sum_{\langle i,j\rangle}^{N}(s_i t_i t_j + s_j t_j t_i) - K\sum_{\langle i,j\rangle}^{N} t_i t_j \ ,\end{aligned}$$

  where $t_i = 0, 1$ and $s_i = \pm 1$.
- The general phase diagram is quite complicated...
- The tricritical point is located on the $H = H_3 = K = 0$ $J - \Omega$ plane.

TIM Qualitative Phase Diagram



TIM: Scaling Limit, Tricritical Ising CFT

- The tricritical point is described by the $\mathcal{M}_{4,5}$ CFT minimal model, with $c = 7/10$
- Operator content:

conformal weights	field	physical role	Landau-Ginzburg field
$(0,0)$	$\mathbb{I}$		identity
$(\frac{3}{80}, \frac{3}{80})$	σ	magnetisation	Φ
$(\frac{1}{10}, \frac{1}{10})$	ε	energy	$:\Phi^2:$
$(\frac{7}{16}, \frac{7}{16})$	σ'	submagnetisation	$:\Phi^3:$
$(\frac{3}{5}, \frac{3}{5})$	t	chemical potential	$:\Phi^4:$
$(\frac{3}{2}, \frac{3}{2})$	ε''	(irrelevant)	$:\Phi^6:$

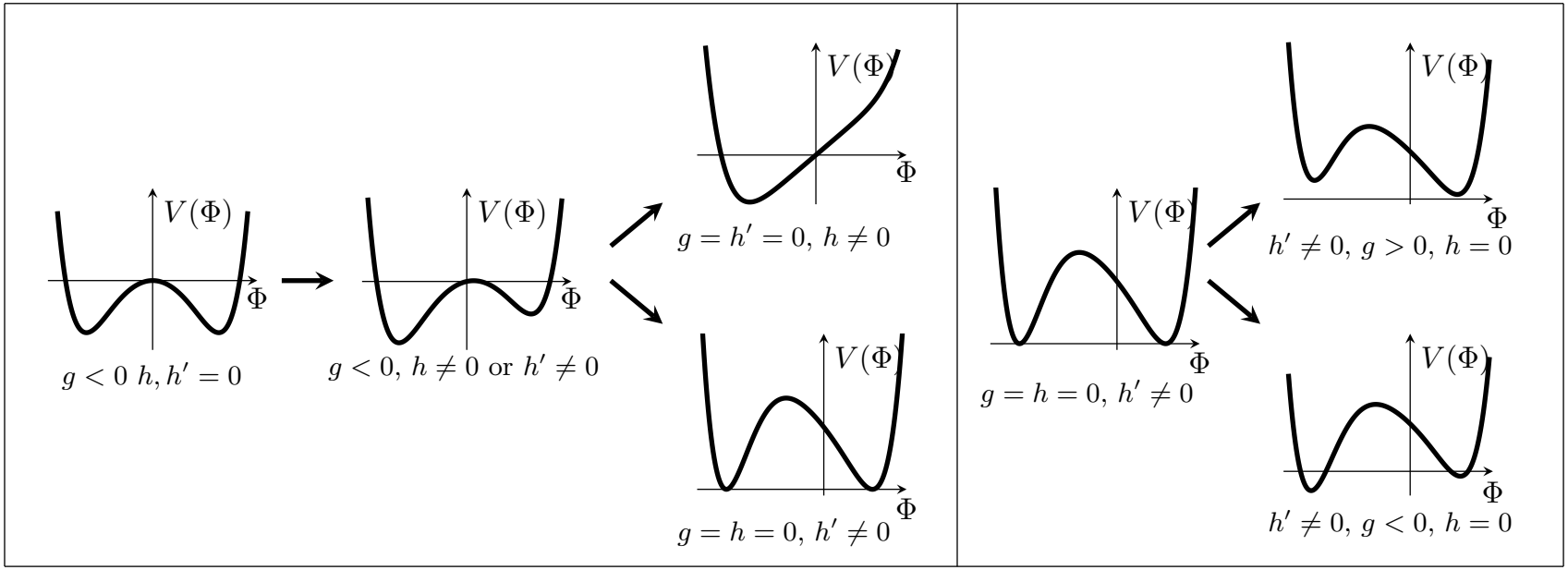
Around the tricritical point

- We study the following perturbations:

$$\mathcal{A}[g,h,h',z] = \mathcal{A} + g\int d^2x\varepsilon + h\int d^2x\sigma + h'\int d^2x\sigma' + z\int d^2xt$$

- $\mathcal{A}[g,0,0,0]$: E_7 model Christe, Mussardo 1990, Fateev, Zamolodchikov 1990
  - $g > 0$ Disordered phase, 1GS, 7 particles
  - $g < 0$ Ordered phase, 2GS 3 kinks-antikinks, 4 bound states of them
- $\mathcal{A}[0,h,0,0]$: non-integrable, 3 particles Lässig, Mussardo, Cardy 1991
- $\mathcal{A}[0,0,h',0]$: integrable $\mathcal{A}_3$ model, 2GS, kink-antikink, bound state of them Colomo, Koubek, Mussardo 1992
- $\mathcal{A}[0,0,0,z]$: $\mathcal{A}_4$, massive or massless integrable. The massive has 3-fold degenerate ground state. Reshetikin, Smirnov 1990, Bernard, Leclair 1990

Around the tricritical point: effective potential



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Duality

- One can introduce the formal operator D, implementing the following duality transformation:
- The magnetisation order parameters (odd fields) change into the disorder operators:

$$\mu = D^{-1}\sigma D, \quad \mu' = D^{-1}\sigma' D\,.$$

- Even fields transform to themselves:

$$D^{-1}\varepsilon D = -\varepsilon, \quad D^{-1}tD = t, \quad D^{-1}\varepsilon'' D = -\varepsilon''$$

- Therefore the thermal deformation ($g \propto T - T_c$)

$$\mathcal{A} = \mathcal{A}_{TIM} + g \int d^2x\, \varepsilon(x)$$

has a low-T/high-T duality.

TIM: Thermal deformation is the integrable E_7 model

- The thermal deformation is an interacting integrable QFT
- It is the E_7 Toda theory
- There are 7 stable particles, of masses m_i, scattering, bound state structure is known
- Z_2 symmetry: $1, 3, 6$ odd, $2, 4, 5, 7$ even particles
- High-T: unique ground state, 7 particles
- Low-T: 2-fold degenerate ground state, odd particles are kinks, even particles are bound states of them (particles on top of a ground state)

Form Factors: Matrix elements of local operators

- Form factors are matrix elements of local operators

$$F^{\mathcal{O}}_{\{a_i\}}(\theta_1, \theta_2, \ldots, \theta_n) = \langle 0|\mathcal{O}(0,0)|A_{a_1}(\theta_1)A_{a_2}(\theta_2)\ldots A_{a_n}(\theta_n)\rangle$$

- Form Factor bootstrap equations:

$$\begin{aligned} \mathcal{S}F_{\{n\}} &= F_{\{n\}} \\ \mathcal{B}F_{\{n+1\}} &= F_{\{n\}} \\ \mathcal{K}F_{\{n+2\}} &= F_{\{n\}} \end{aligned}$$

- Solution (VEV, 1 and 2 particles):

$$F_0 = \mathrm{const}; \quad F_a(\theta) = \mathrm{const};$$

$$F_{ab}(\theta_1, \theta_2) = \frac{Q_{ab}(\theta_1 - \theta_2)}{D_{ab}(\theta_1 - \theta_2)} F^{\min}_{ab}(\theta_1 - \theta_2)$$

- F_0 are known Fateev, Lukyanov, Zamolodchikov, Zamolodchikov 1998, D_{ab}, F_{ab} are fixed by scattering data, we are looking for Q
- Cortés Cubero, Konik, ML, Mussardo, Takacs SciPost Phys. 12 (2022)

FF: Asymtotics

- Q is given as

$$Q_{ab}(\theta) = \sum_{i=0}^{N_{ab}} a_{a,b}^{k} \cosh^{k} \theta$$

- OK, but which operator?
- If the conformal dimension of the operator is $\Delta_{\mathcal{O}}$, then

$$\lim_{|\theta_i| \to \infty} F^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1, \ldots, \theta_n) \sim e^{y_{\mathcal{O}}|\theta_i|}$$

  where $y_{\mathcal{O}} \leq \Delta_{\mathcal{O}}$, fixes N_{ab}
- This is usually enough, but in TIM gives the same for N_{ab} for σ/μ and σ'/μ'!!
- Bootstrap eqs. lead to an incomplete set of equations for F_i and a!!

FF eqs. for σ/σ'

$$F_6^\Phi = 0.115722 F_1^\Phi + 0.587743 F_3^\Phi$$

F_{12}:	$a_{12}^0 = 3.06131 F_1^\Phi - 19.8715 F_3^\Phi$	$a_{12}^1 = -8.95069 F_1^\Phi - 30.9146 F_3^\Phi$
F_{14}:	$a_{14}^0 = -160.899 F_1^\Phi - 1600.15 F_3^\Phi$	$a_{14}^1 = -626.504 F_1^\Phi - 3040.25 F_3^\Phi$
	$a_{14}^2 = -456.311 F_1^\Phi - 1437.25 F_3^\Phi$	
F_{15}:	$a_{15}^0 = 32.1365 F_1^\Phi + 177.579 F_3^\Phi$	$a_{15}^1 = 70.7301 F_1^\Phi + 289.789 F_3^\Phi$
	$a_{15}^2 = 37.5681 F_1^\Phi + 114.447 F_3^\Phi$	
F_{32}:	$a_{23}^0 = -38.6198 F_1^\Phi - 337.751 F_3^\Phi$	$a_{23}^1 = -142.958 F_1^\Phi - 621.037 F_3^\Phi$
	$a_{23}^2 = -87.8337 F_1^\Phi - 266.777 F_3^\Phi$	
F_{34}:	$a_{34}^0 = -493.626 F_1^\Phi - 2722.44 F_3^\Phi$	$a_{34}^1 = -1617.98 F_1^\Phi - 6682.94 F_3^\Phi$
	$a_{34}^2 = -1495.64 F_1^\Phi - 5104.39 F_3^\Phi$	$a_{34}^3 = -399.244 F_1^\Phi - 1174.92 F_3^\Phi$

FF: Duality, Clustering

- High-T: σ/σ': Z_2 odd, μ/μ': Z_2 even
- One can exploit the duality:

$$\lim_{\theta\to\infty} F_{ij}^{\Phi}(\theta) = \frac{\omega_{ij}}{F_0^{\Phi}}(F_i^{\Phi}F_j^{\Phi}) \tag{2}$$

  where odd FFs correspond to σ/σ' even ones correspond to μ/μ'
- For example:

$$0.851179 - 2.97692\frac{F_2^{\Phi}}{F_0^{\Phi}} = \omega_{22}\left(\frac{F_2^{\Phi}}{F_0^{\Phi}}\right)^2$$

- $\omega_{22} = \pm 1$, turns out that $\omega_{22} = -1$ is consistent with the, Δ-theorem, and the two solutions tell the two operators apart, other clustering eqs. fix all the one-particle form factors.

Check 1: Δ-theorem

- The Δ-theorem Delfino, Simonetti, Cardy 1996 gives the the UV-conformal weight of an operator:

$$\Delta_\Phi^{\mathrm{uv}} = -\frac{1}{2\langle\Phi\rangle}\int_0^\infty dr\, r\,\langle\Theta(r)\Phi(0)\rangle$$

- Inserting a complete set and truncating to the 1 and 2 particle contributions that we calculated lead to[1]:

	sum	exact
μ	0.0367	0.0375
μ'	0.3824	0.4375

[1] Form factors of Θ are known Acerbi, Valleriani, Mussardo 1996

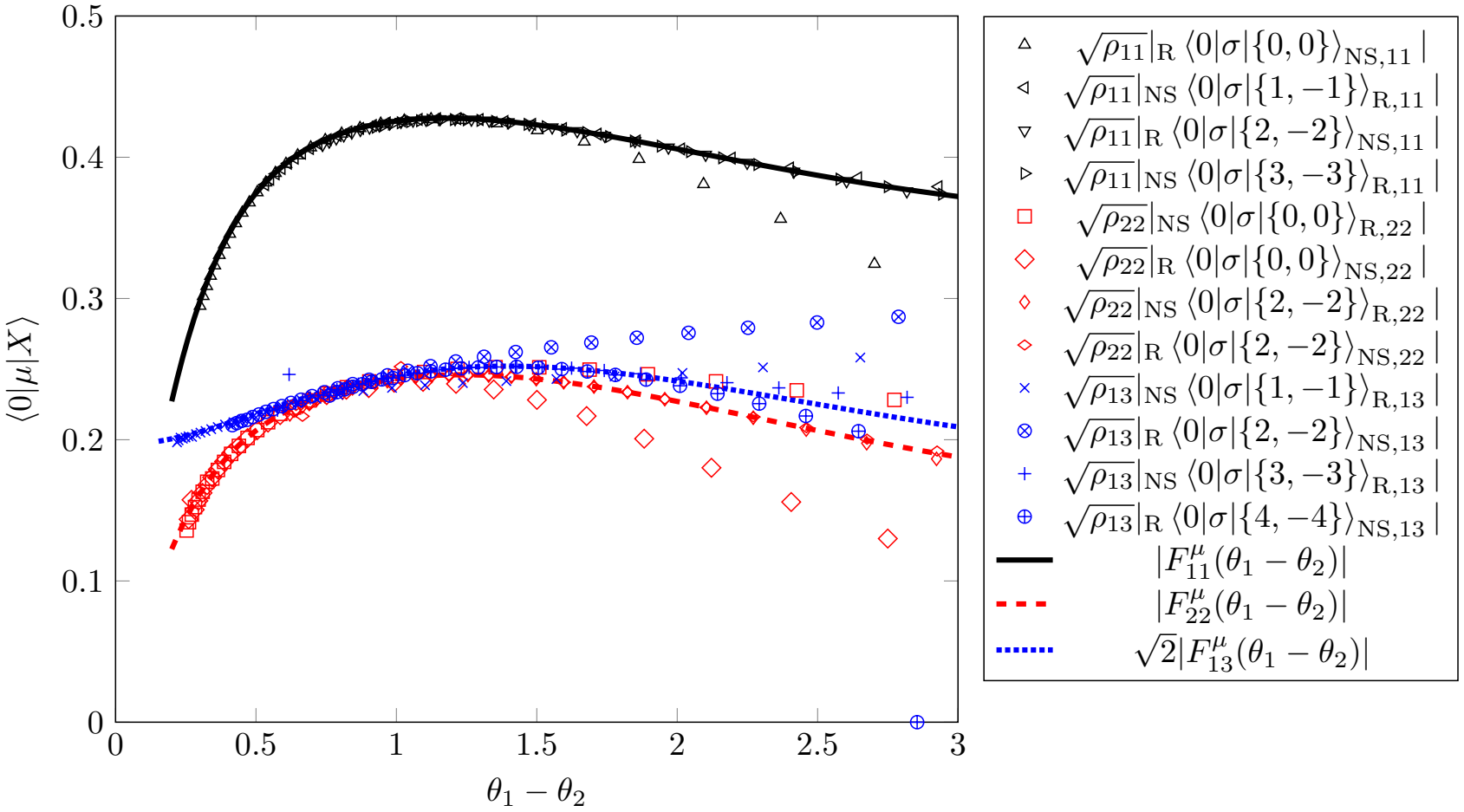
Check 2: Truncated Conformal Space

- The Hamiltonian:

$$H = H_{CFT} + V$$

- Diagonalize H using $\mathcal{H}_{CFT}$ as a basis (matrix elements of V can be calculated)
- Finite volume = discrete spectrum, truncation = finite matrix
- Identify the ground state, 1- and 2-particle states, and calculate matrix elements
- Relate finite-volume form factors to infinite volume ones Pozsgay, Takács 2008
- Duality: σ/σ' in low-T is μ/μ' in high-T!!

Check 2: Truncated Conformal Space



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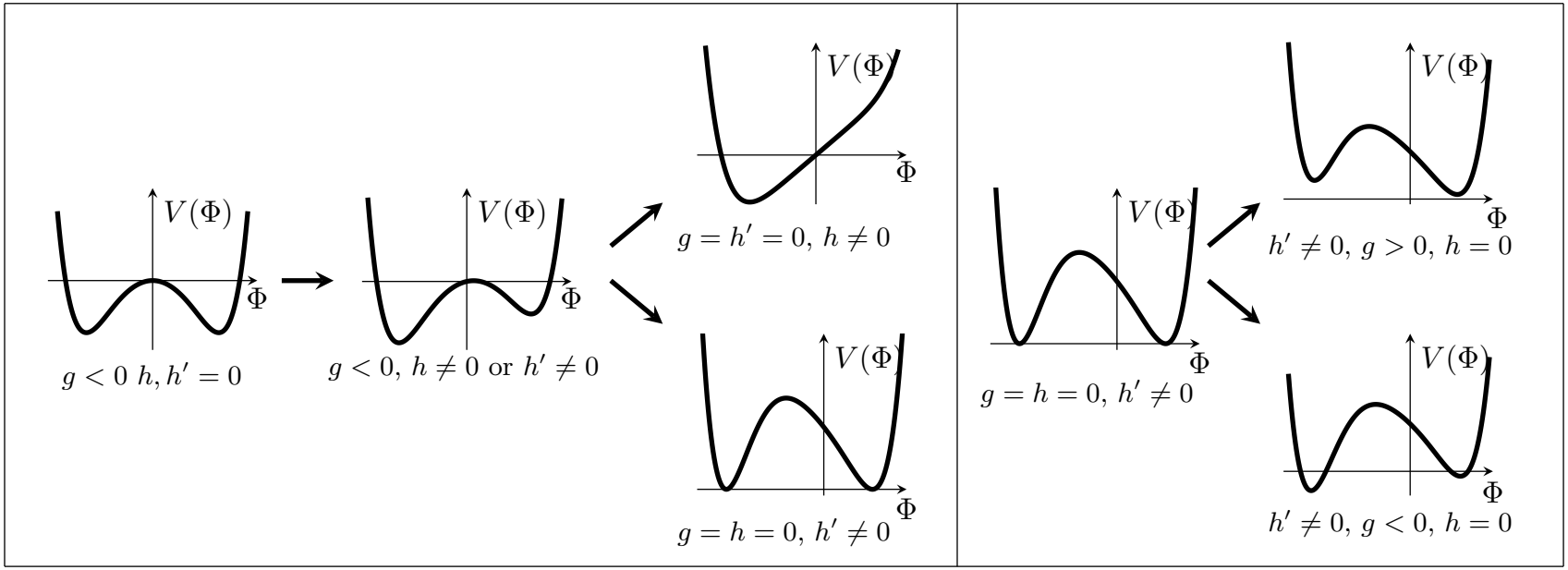
Around the tricritical point: effective potential



Confinement of kinks I.: E_7

- There is a false vacuum, with finite energy density, $\Delta\mathcal{E}$ where

$$\Delta\mathcal{E} = 2h\langle\sigma\rangle$$

for σ, and

$$\Delta\mathcal{E} = 2h'\langle\sigma'\rangle$$

for σ' perturbation

- Kinks-antikinks are confined due to a linear potential
- Semi-classical approximation: two kinks in linear potential

$$\Big[2\omega(\hat{p}) - E_n + \Delta\mathcal{E}|x|\Big]\phi_n^+(x) = 0 \quad , \quad \hat{p} = -i\partial_x \ ,$$

for identical kinks, and

$$\Big(\omega_a(p) + \omega_b(p) - E_n + \Delta\mathcal{E}|x|\Big)\phi_n^+(x) = 0$$

for non-identical kinks. $\omega_a(p) = \sqrt{m_a^2 + p^2}$

Confinement of kinks I.: E_7

- Solution can be given using saddle point, leading to the quantization conditions:

$$\sinh 2\theta_n - 2\theta_n = \frac{\Delta\mathcal{E}}{m^2}\left[2\pi\left(n+\frac{1}{4}\right) + i\log S(2\theta_n)\right]$$

for identical, and

$$m_a^2\left(\frac{1}{2}\sinh 2\theta_n - \theta_n\right) - m_b^2\left(\frac{1}{2}\sinh 2\tilde{\theta}_n - \tilde{\theta}_n\right)$$
$$= \Delta\mathcal{E}\left[2\pi\left(\frac{n}{2}+\frac{1}{4}\right) + i\log S_{ab}(\theta_n - \tilde{\theta}_n)\right]$$
$$\text{where } \tilde{\theta}_n = -\sinh^{-1}\left(\frac{m_a\sinh\theta_n}{m_b}\right)$$

for non-identical kink-antikink pairs The mass is given by

$$M_n = 2m\cosh\theta_n \qquad M_n = m_a\cosh\theta_n + m_b\cosh\tilde{\theta}_n$$

- Expected to work better for larger n

E_7 Confinement: comparison to TCSA

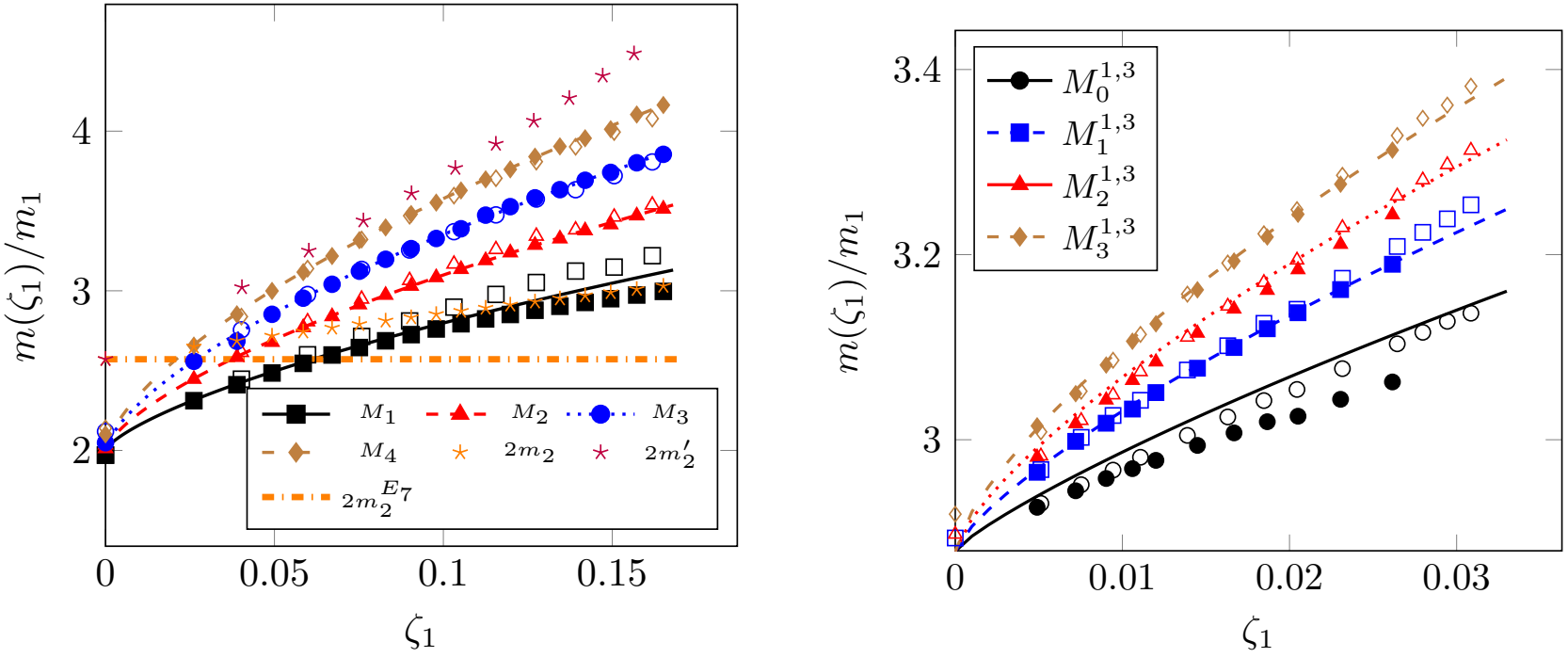
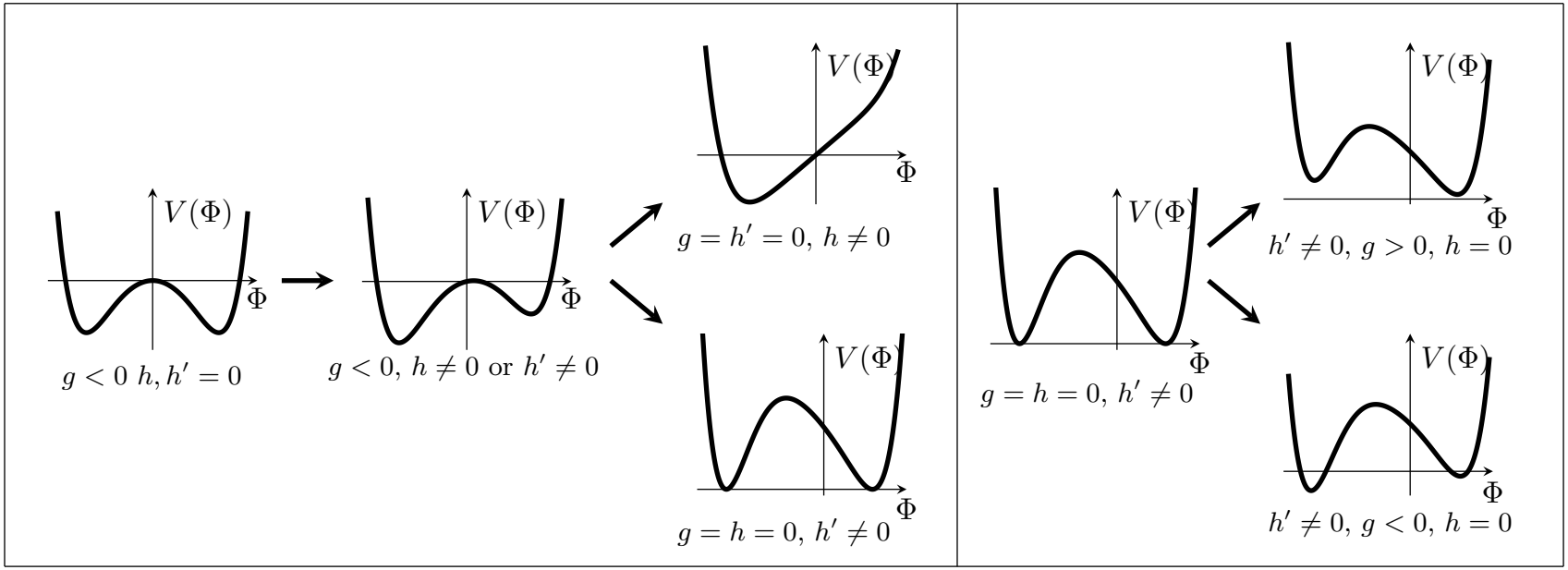


Figure 3: Meson masses in magnetic perturbations of the E_7 model. $\zeta_1 = \Delta\mathcal{E}/m_1^2$

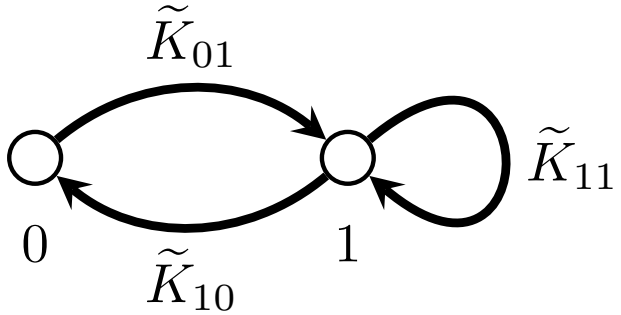
ML, Mussardo, Takács, Phys.Lett.B 828 (2022)

Around the tricritical point: effective potential



Confinement of kinks II.: $\mathcal{A}_3$ model

- Kink structure



- Non-diagonal scattering:

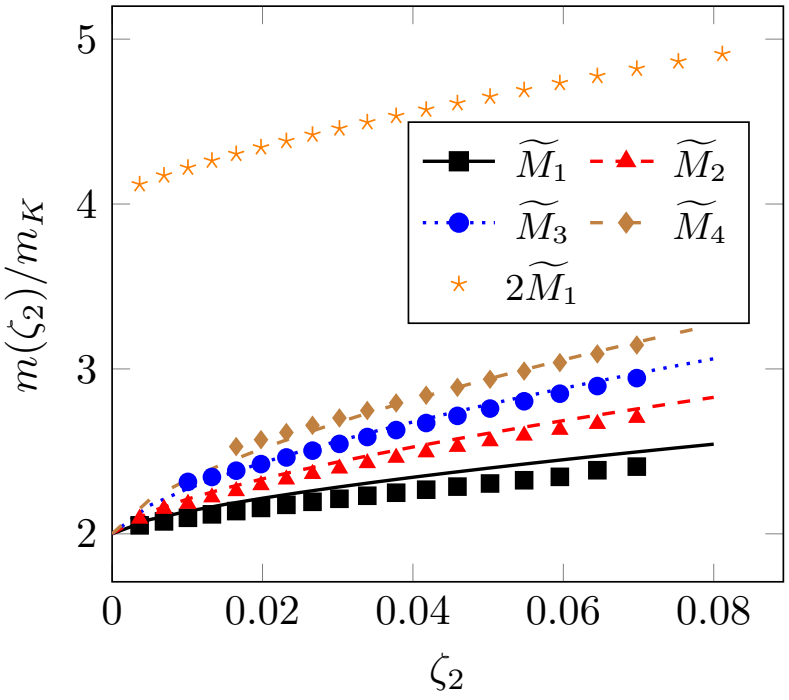
$$\begin{aligned} \widetilde{K}_{01}\widetilde{K}_{10} &\rightarrow \widetilde{K}_{01}\widetilde{K}_{10} \\ \widetilde{K}_{10}\widetilde{K}_{01} &\rightarrow \widetilde{K}_{10}\widetilde{K}_{01} + \widetilde{K}_{11}\widetilde{K}_{11} \end{aligned}$$

- The energy density of the false vacuum

$$\widetilde{\Delta\mathcal{E}} = |g(\langle\varepsilon\rangle_1 - \langle\varepsilon\rangle_0)|,$$

- Confinement is expected only when 1 is lifted up, $g < 0$ (this is verified by TCSA)
- Semi-classics can be used here as well

$\mathcal{A}_3$ Confinement: comparison to TCSA



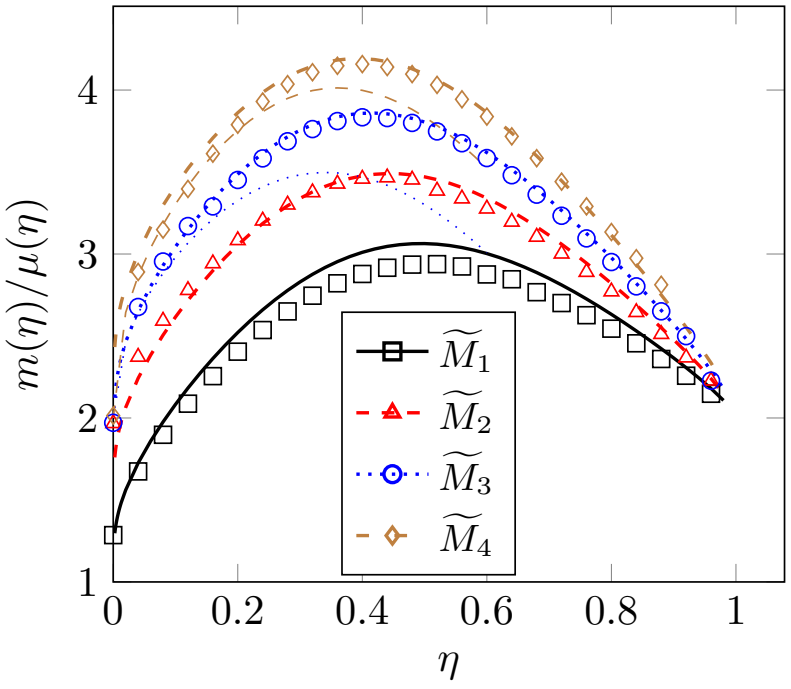


Figure 4: Meson masses in the thermal perturbation of the $\mathcal{A}_3$ model. $\zeta_2 = \widetilde{\Delta\mathcal{E}}/m_K^2$, $\eta = |h'|^{9/5}/\left(|h'|^{9/5} + |g|^{9/8}\right)$

ML, Mussardo, Takács, Phys.Lett.B 828 (2022)

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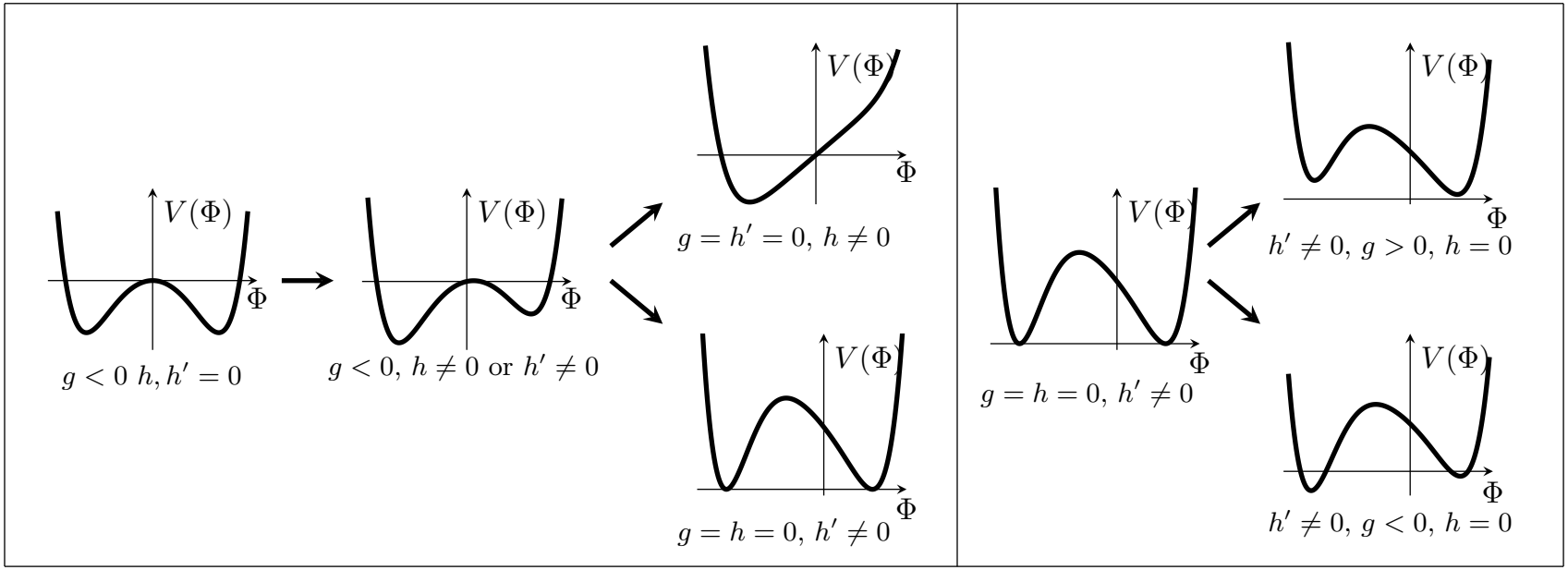
Around the tricritical point: effective potential



Fate of the false vacuum as a quantum quench

- On spin chains: Lagnese, Surace, Kormos, Calabrese, 2021
- Consider a model with degenerate ground states
- Introduce a perturbation which unbalances them
- Pick the true vacuum state as an initial state
- Time evolve it with a Hamiltonian which unbalances in the other way
- E.g. Ising model in the ordered phase, $h_{\|} \to -h_{\|}$
- Expectation for time evolution of operators

$$f^{\mathcal{O}}(t) = \frac{\langle \mathcal{O}(t) \rangle + \langle \mathcal{O}(0) \rangle}{2\langle \mathcal{O}(0) \rangle} \propto e^{-\gamma V t}$$

where

$$\gamma = \mathcal{C} \Delta \mathcal{E} \exp\left[-\frac{\pi M^2}{\Delta \mathcal{E}} \right]$$

Realization in the Ising model

Ising model in the ferromagnetic phase with longitudinal magnetic field

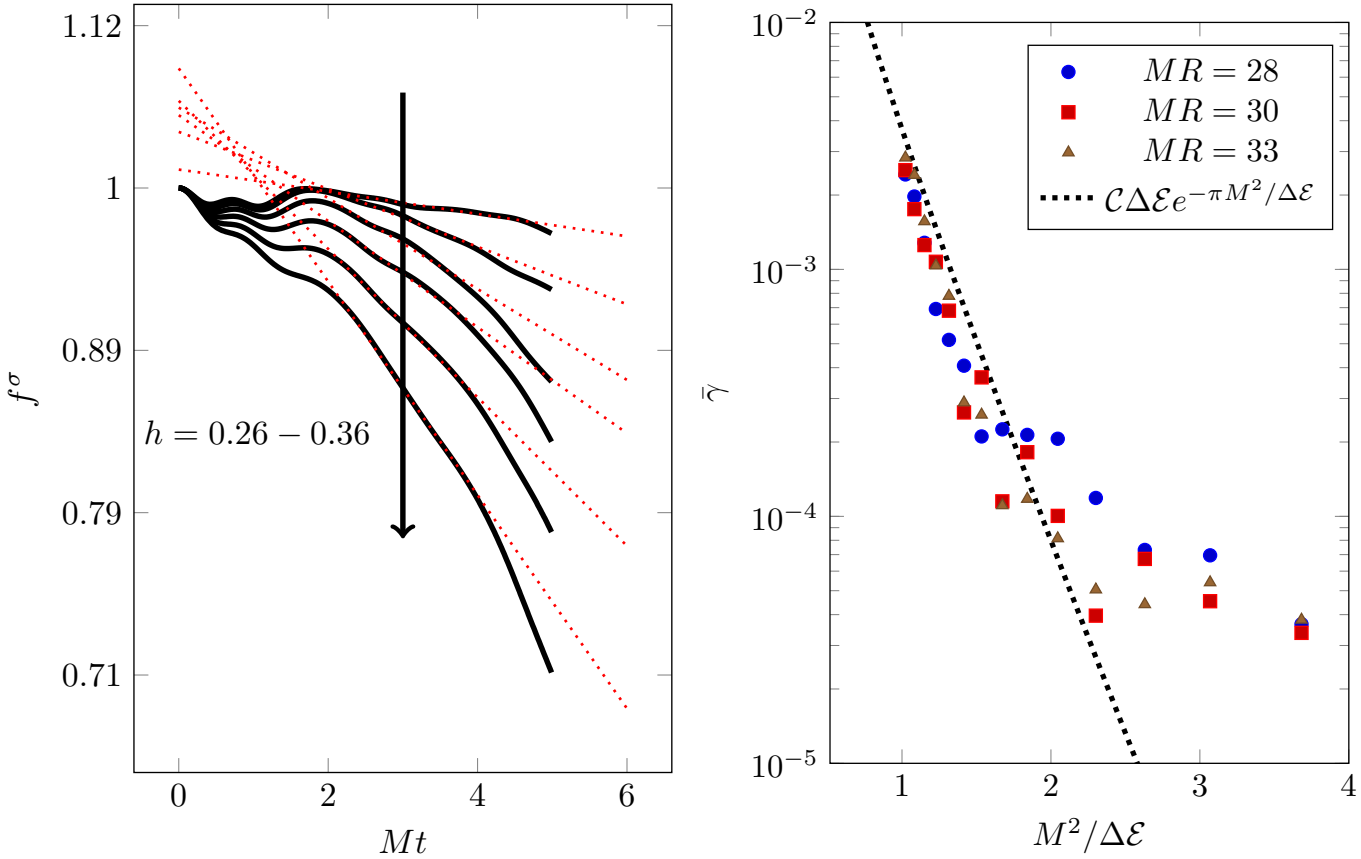


Figure 5: Time evolution and decay rate in the Ising model with magnetic field. ML, Mussardo, Takács, PRD 106 (2022); see also Szász-Schagrin, Takács, 2022

Decay in the tricritical Ising model

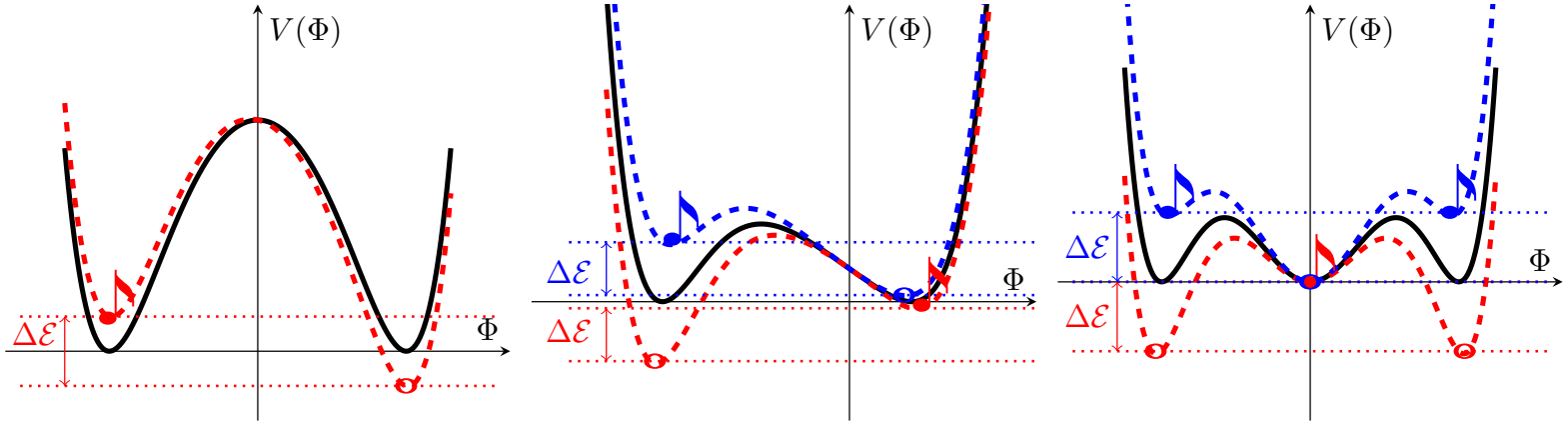


Figure 6: Effective potential changes in various quenches (E_7, $\mathcal{A}_3$, $\mathcal{A}_4$)

- From the E_7 model, we found that large oscillations dominate
- From $\mathcal{A}_3$ depending on the sign of the coupling: visible or large oscillaitons
- From $\mathcal{A}_4$: with magnetic field: oscillations, with ε, visible
- Oscillations correspond to particles above the false vacuum!

Decay rates in the tricritical Ising model

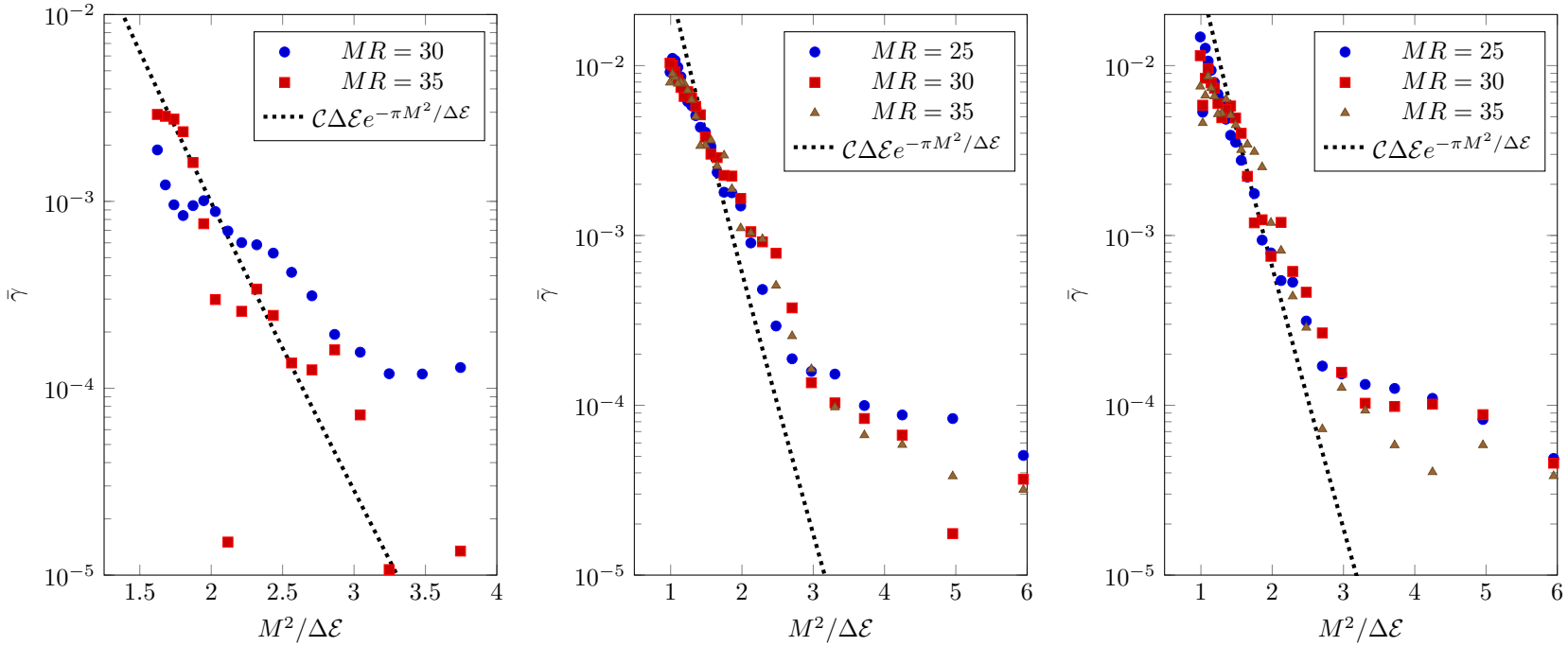


Figure 7: Decay rates in different quenches ($\mathcal{A}$ to para,$\mathcal{A}_4$ from anti-symmetric/symmteric)

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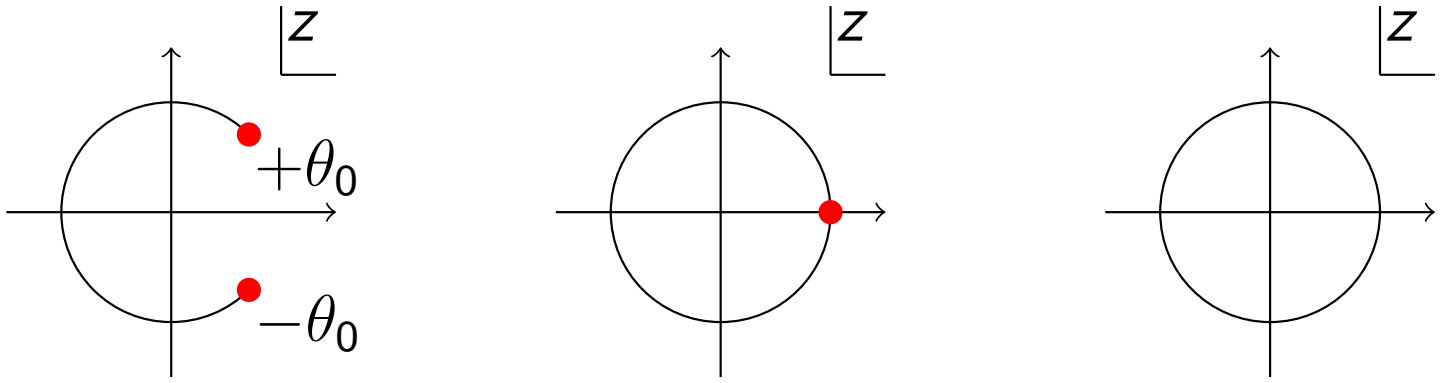
Yang–Lee singularity: Ising model

- 2D Ising model

$$H = -J \sum_{\langle i,j \rangle} s_i s_j + h \sum_i s_i; \qquad s_i = \pm 1$$

- Let $z = e^{-\beta h} = e^{i\theta}$, then the zeroes of the partition function $Z(z)$ are located on the unit circle Lee,Yang 1952.



$T > T_c$ $\qquad$ $T = T_c$ $\qquad$ $T < T_c$

- $T < T_c$: density of zeroes is anomalous Kortmann, Griffith 1971:

$$\eta(\theta) \overset{\theta \to \theta_0}{\sim} |\theta - \theta_0|^{\mu}$$

Yang–Lee singularity: Ising model

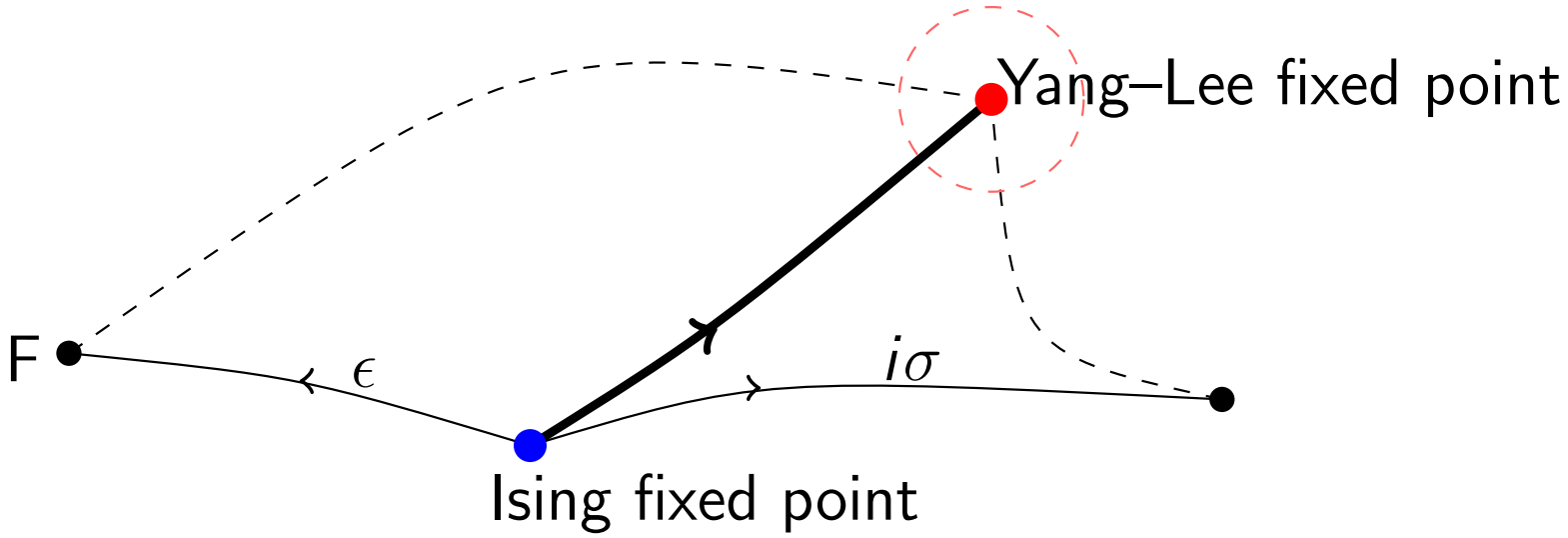
- Fisher, 1978

$$\mathcal{L}_{YL} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + (h - ih_c)\varphi + i\gamma\varphi^3 + \dots$$

- Cardy, 1985 0 Non-unitary minimal conformal field theory $\mathcal{M}(2,5)$, $c_{eff} = 2/5$

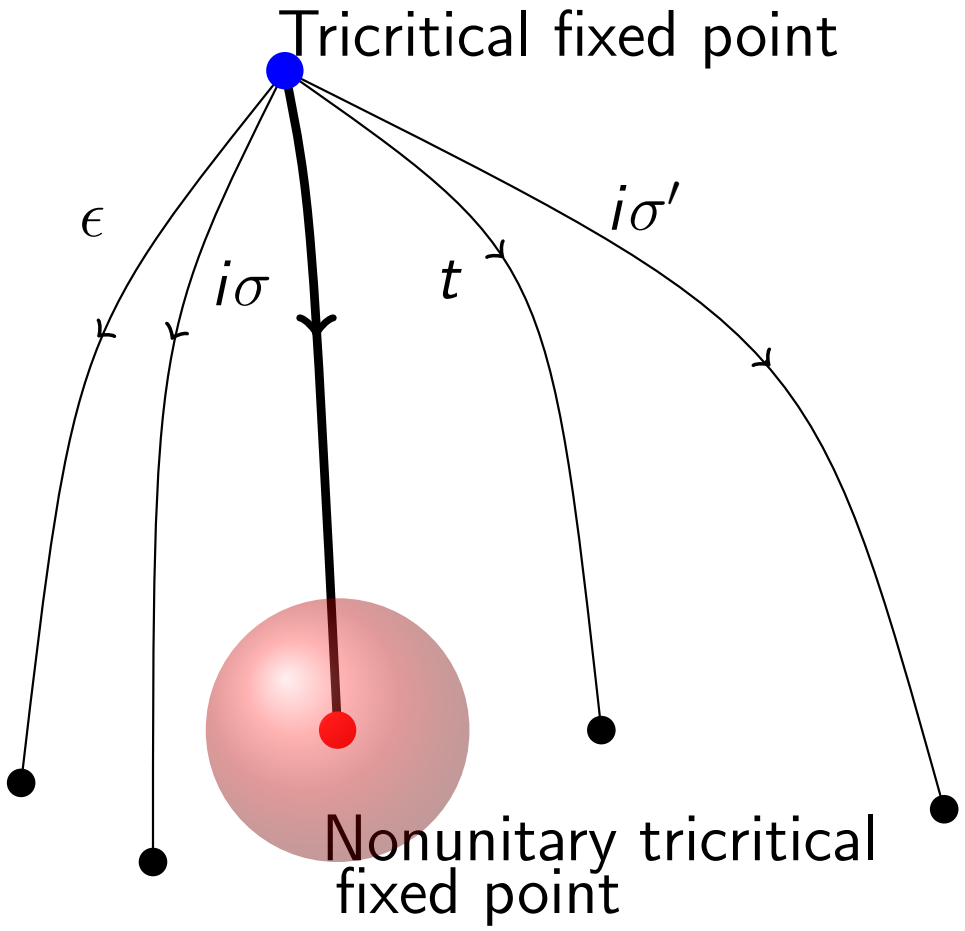
$$\{\mathbb{I}, \phi\}; \qquad \phi \times \phi = \mathbb{I} + \phi$$

- Ising model in imaginary magnetic field, spontaneously broken PT symmetry Fonseca, Zamolodchikov 2001, Xu, Zamolodchikov 2022



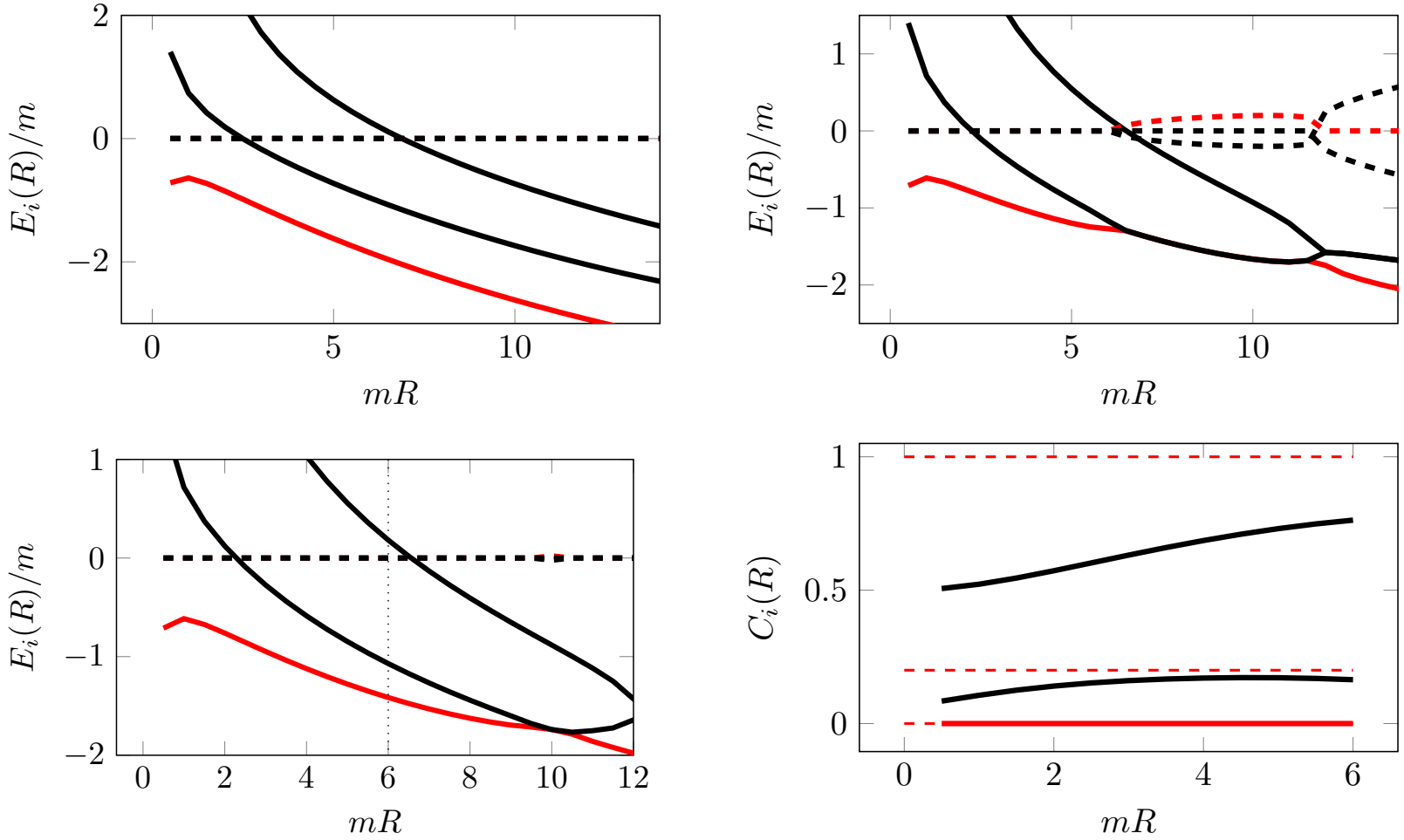
Tricritical point

- $\mathcal{M}(4,5)$ unitary minimal model
- 2 nontrivial relavant even fields: ϵ,t
- 2 odd fields: σ,σ'



Results from truncated space: Ising model

We tune the imaginary magnetic field with fixed ϵ coupling, in finite volume.



Lesson from $z = 0$

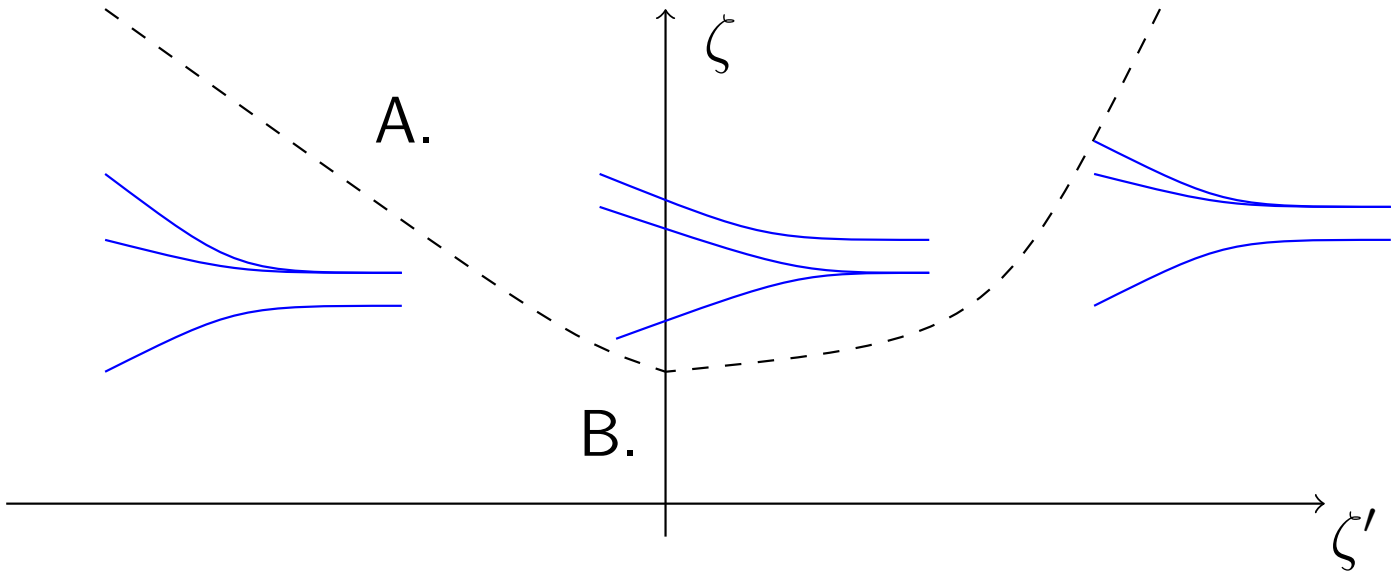


Figure 8: Sketch of the $z = 0$ section of the phase diagram. ζ/ζ' are dimensionless couplings of the leading/subleading magnetization.

One has to tune z to locate the tricritical point!

Results from truncated space: Tricritical Ising model

Critical YL: $\mathcal{M}(2,5)$; Tricritical YL: $\mathcal{M}(2,7)$ see also von Gehlen 1994

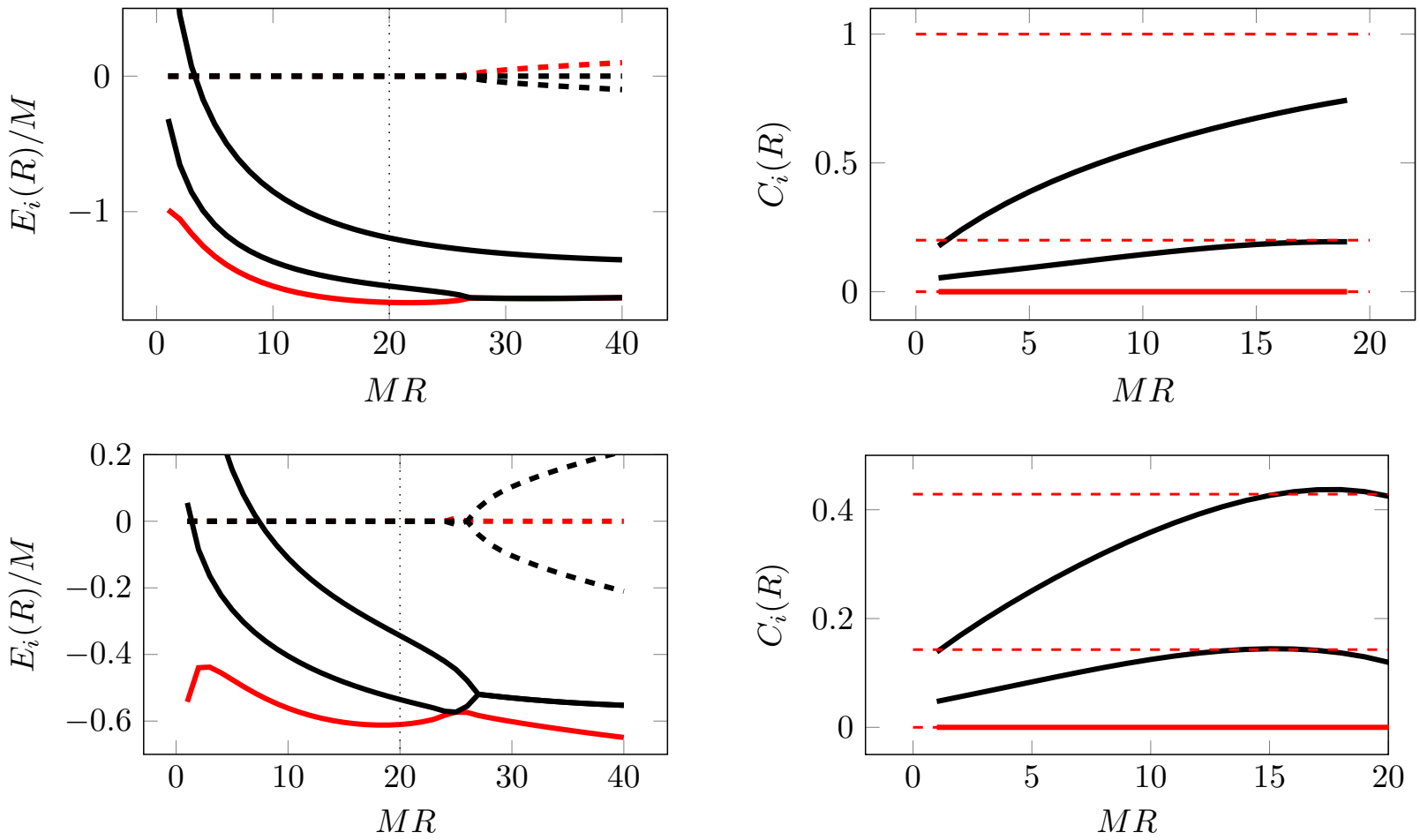


Table of Contents

Conclusion

- E_7 magnetic form factors
  - Tricritical Ising model has a low-T/high-T duality
  - Form factor bootstrap for order/disorder operators is ambiguous
  - Duality, clustering property and the Δ-theorem fix the ambiguity
- Confinement
  - Confinement of non-identical kinks
  - Confinement induced by thermal field
  - $\mathcal{A}_3$ semi-classic meson quantization works surprisingly well
- False vacuum decay: TCSA results are consistent with theory for the decay rate; the decay is obscured by oscillations when there are particles above the false vaccum
- Proposal: Multicritical Yang–Lee: $\mathcal{M}(p, p+1)$+imaginary coupled odd fields $\longrightarrow$ multicritical surface $\mathcal{M}(2, 2n+3)$, where $1 < n < p-1$