Fractal Structure in QCD and Tsallis Statistics

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Summary

What is a fractal

Thermofractals

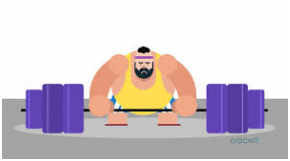
Yang-Mills fields and thermofractals

Dynamic of heavy-quarks in the QGP

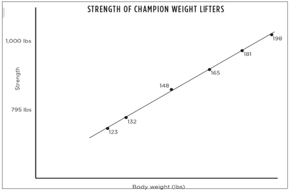
Thermofractal in hadron structure? z-Scaling.

Fractal and fractional derivatives.

Scale and Self-Similarity



SCALING



SELF-SIMILARITY

A.D. - PRD 93 (2016) 054001

AD, T. Frederico, E. Megías, D.P. Menezes, Entropy 20 (2018) 633

Thermofractals

A thermofractal is a system defined by the following properties:

I- It is a system in thermodynamical equilibrium with total energy given by $U = K + E$, where K corresponds to the kinetic energy of N constituent subsystems and E describes the internal energy of those subsystems, which are endowed with an internal substructure.

II- The constituent particles are thermofractals which can be divided in two sublasses: Type I and Type II. For each subclass, the corresponding associated ratio, E/K or E/U, can vary according to a self-similar distribution, $P(U)$. This means the distribution of the internal energy is independent of the hierarchic level of the fractal structure.

III- At some level n in the hierarchy of subsystems the phase space is so narrow that one can neglect their internal structure and assume the following expression to the probability: $P(U_n)\, dU_n = \rho\, dU_n$

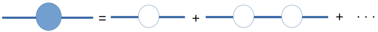
$$P(\varepsilon) = A\left[1 + (q-1)\frac{\varepsilon}{Nk_B\tau}\right]^{-\frac{1}{q-1}}$$

Fractals in Yang-Mill fields

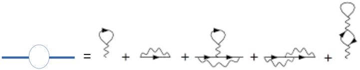
A.D., E. Megías,D.P. Menezes PRD (2020)

Complex structure of the effective parton:

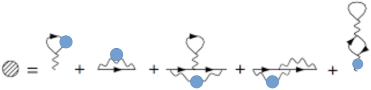
Self-energy interaction:



Self-energy diagrams:



Many-loops diagrams:



Yang-Mills theory is renormalizable: $\Gamma(p, m, g) = \lambda^{-D}\Gamma(p, \mu, \bar{g})$

F. Dyson, PR 75 (1949) 1736   Stuekelberg and Petermann, Helv. Phys. Acta 26 (1953) 499

M. Gell-Mann and F.E. Low, PR 95 (1954) 1300

Callan-Symanzik Equation
Renormalization group equation: $\left[M\frac{\partial}{\partial M} + \beta_g \frac{\partial}{\partial \bar{g}} + d\right]\Gamma = 0 \Rightarrow M\frac{d\Gamma}{dM} = 0$

Effective coupling constant $\bar{g}$ Effective mass μ

Scaling properties are present in YMF

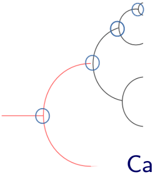
K. Symanzik, Comm. Math. Phys. 18 (1970) 227

C.G. Callan Jr., PRD 2 (1970) 1541

Non extensivity in gauge field theory

$$P(\varepsilon) = G^n[1-(q-1)\frac{\varepsilon}{\lambda}]^{\frac{1}{q-1}}$$

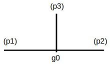
q - the number of internal degrees of freedom in the fractal structure that are relevant in the process of energy transfer to the effective parton

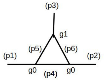


Describes how momentum and energy are distributed at each vertex:

$\bar{g} = \prod_i G\left[1-(q-1)\frac{\varepsilon_i}{k\tau}\right]^{\frac{1}{q-1}}$ effective coupling

Calculation of q from gauge field parameters: 1-loop approx. (QCD)





Fractal method: $\beta_g = -\frac{1}{q-1} g^{N'+1}$

CS equation: $\frac{1}{q-1} = d - \gamma$ $\Rightarrow \frac{1}{q-1} = \frac{11}{3}N_c - \frac{4}{3}N_f \Rightarrow q = 1.14$

QCD: $d - \gamma = \left[\frac{11}{3}c_1 - \frac{4}{3}c_2\right]$

Comparison with experiments

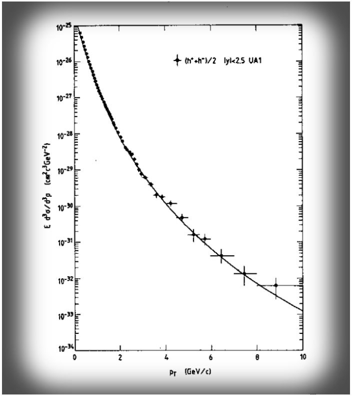
Extended Hagedorn theory to non extensive statistics: AD, Physica A 391 (2012) 6380

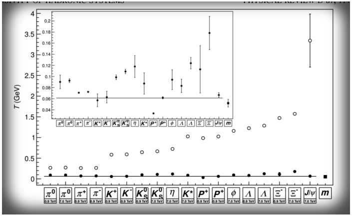
use of Tsallis factor: $P(\varepsilon) = A[1 + (q-1)\frac{\varepsilon}{k\tau}]^{-\frac{1}{q-1}}$

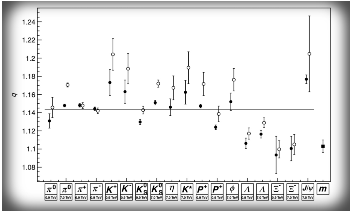
L. Marques, E. Andrade-II, AD, PRD 87 (2013) 114022

L. Marques, J. Cleymans, AD, PRD 91 (2015) 054025

Experimental value $q = 1.14 \pm 0.01$





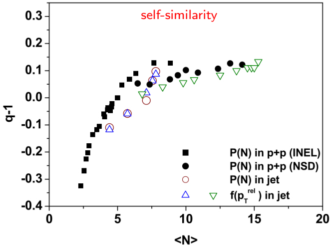


Experimental verification

Scale invariance of gauge theory

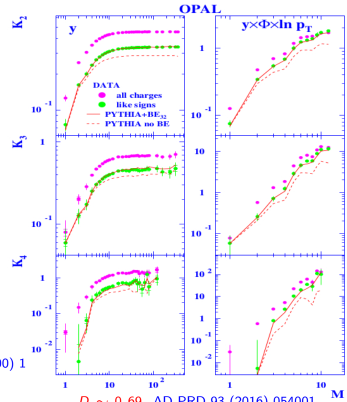
leads to fractal structure

fractal dimension in multiparticle production



G. Wilk and Z. Wlodarczyk, PLB 727 (2013) 163

fractal dimension - from intermittecy analysis



E. Sarkisyan-Grinbaum, PLB 477 (2000) 1

$D \sim 0.69$ AD PRD 93 (2016) 054001

Multiplicity and energy

Multiplicity as a manifestation of fractal aspects: AD, PRD 93 (2016) 054001

r is the scale in wich energies are measured.

$\varepsilon \sim r^{-D}$ is the scaling behavior of the individual parton energy.

$E \sim r^{-1}$ is the scaling behavior of the total energy.

$\mathcal{N}$ is the multiplicity.

R is the ratio between parton energy ε and its immediate parent energy

$\mathcal{N} r^{-D} \propto r^{-1} \rightarrow \mathcal{N} \propto r^{-1+D}$

$\mathcal{N} \propto E^{1-D}$

$D \sim 0.69$ from fractal dimension analysis and intermittence analysis

Theory: $1 - D \sim 0.31$

Experiment: $1 - D \sim 0.302$
E. Sarkisyan-Grinbaum et al. PRD 93 (2016) 054046

Generalization of Hagedorn's Thermodynamics

Hagedorn's Self-Consistent Thermodynamics

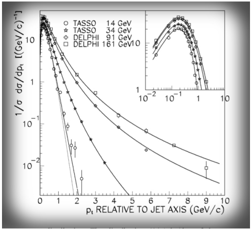
Limiting temperature T_H. R. Hagedorn, N. Cimento 3 (1965) 147

Hadron mass spectrum $\rho(m) = \frac{a}{m^{5/2}} exp\{m/T_H\}$

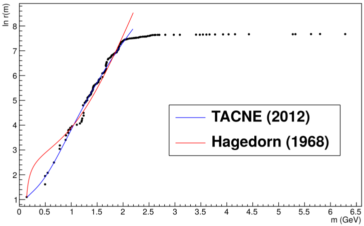
Nonextensive Self-Consistent Thermodynamics

Limiting Temperature

Hadron mass spectrum $\rho(m) = \frac{a}{m^{5/2}} exp_q\{m/T_H\}$



power-law distributions

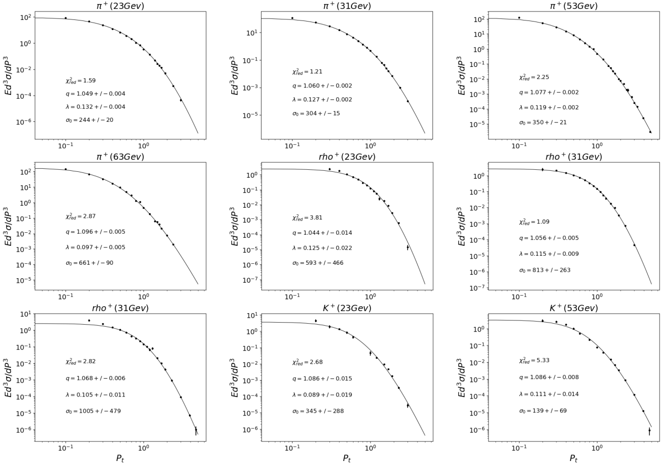


non extensive mass spectrum

Generalized Hagedorn Self-Consistent Thermodynamics

AD Physica A 391 (2012) 6380

Fractal structure of hadrons (in development)



Fractal structure of hadrons (in development)

Thermofractal structure of hadrons:

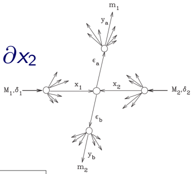
$$\frac{d^4\sigma}{dq^4} = |\langle\varphi\phi|\, g(p)\, |\phi_o\rangle|^2 \delta(q^2 - m_q)$$

$$E\frac{d^2\sigma}{2\pi q_t dq_t\ dq_z} = \frac{1}{2}|\langle\varphi\phi|\, g(E)\, |\phi_o\rangle|^2 \propto e_q(\varepsilon/\lambda)^2$$

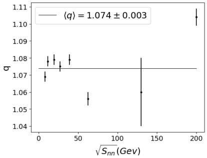
z-scaling: $\psi(z) = \frac{d\sigma(z)}{dz}$

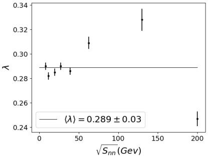
M. Tokarev, I. Zborovsky:

$$E\frac{d^2\sigma}{dq^2} = \frac{\sigma_0}{\pi s}\frac{d^2\sigma}{dx_1 dx_2} \qquad \frac{1}{1+(q-1)z}\frac{\partial z}{\partial x_1}\frac{\partial z}{\partial x_2} = C\frac{\partial^2 z}{\partial x_1}\partial x_2$$



$$z = f(x_1)f(x_2) - \frac{1}{q-1} \quad \Rightarrow \psi(z) = [1 + (q-1)z]^{\frac{-q}{q-1}}$$





$$\frac{1}{q'-1} = 2 \times \frac{1}{q-1} \Rightarrow q' = 1.07 \text{ if } q = 1.14$$

Dynamics of quarks in the medium

Boltzmann Equation:

$$\frac{\partial f(\mathrm{p},t)}{\partial t} = -\nabla_{\mathrm{p}} f(\mathrm{p},t) \cdot \mathrm{F} + C[f]$$

$$h[f(\mathrm{p}), f(\mathrm{q})] = f(\mathrm{p}) f(\mathrm{q})$$

Non-Additive Boltzmann Equation:

Correlation functional

$$h[f(\mathrm{p}), f(\mathrm{q})] = \left[f(\mathrm{p})^{1-q} + f(\mathrm{q})^{1-q} - 1\right]^{\frac{1}{1-q}}$$

Fokker-Planck Equation

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial p_i}\left[A_i f + \frac{\partial (B_{ij} f)}{\partial p_j}\right] = 0$$

Plastino-Plastino Equation

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial p_i}\left[A_i f + \frac{\partial (B_{ij} f^{2-q})}{\partial p_j}\right] = 0$$

Plastino-Plastino Equation was proved for the 1st time in
A.D., E. Megias, A. Golmankhaneh and R. Pasechnik in
PLB 839 (2023) 137752

Special Derivative: $\bar{\partial}^2 f / \partial x^2 \to \partial^2 f^{2-q} / \partial x^2$

comes from q-deformed derivative: $\bar{\partial} f / \partial x \to f^{1-q} \partial f / \partial x$

Fractal Derivatives

Haussdorff Geometry:

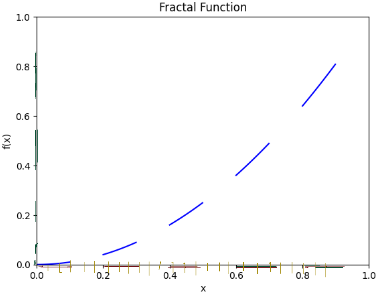
K. Falconer: Fractal Geometry: Mathematical Foundations and Applications

δ-measure - $\mathcal{H}^{\alpha}_{\delta}(\mathbb{F}) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^{\alpha} : \{U_i\} \text{is a cover of } \mathbb{F}\right\}$

Haussdorff measure - $\mathcal{H}^{\alpha}(\mathbb{F}) = \lim_{\delta\to 0} \mathcal{H}^{\alpha}_{\delta}(\mathbb{F})$

If $\mathbb{F}$ is a Borel set in $\mathbb{R}^{\alpha}$ $\Rightarrow \mathcal{H}^{\alpha}(\mathbb{F}) = c_{\alpha}^{-1} vol^{\alpha}(\mathbb{F})$

Mass distribution $\gamma^{\alpha}_{\mathbb{F}}(a,b) = \mathcal{H}^{\alpha}([a,b])$ if $0 < \mathcal{H}^{\alpha} < \infty$



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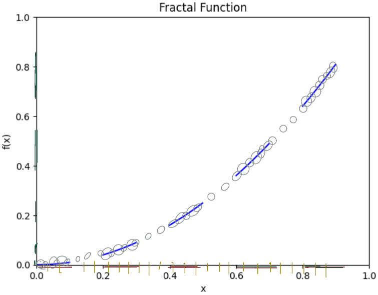

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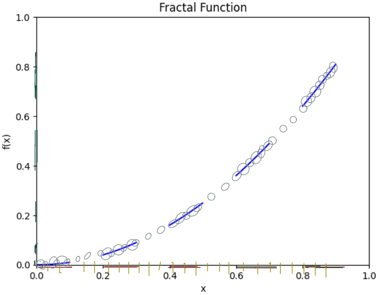

FractalDerivative

Parvate & Gangal, Fractals 17 (2009) 53

$$D^\alpha_{\mathbb{F},a} f(x_o) = \begin{cases} F\lim_{x\to x_o} \frac{f(x)-f(x_o)}{S^\alpha_{F,a_o}(x) - S^\alpha_{F,a_o}(x_o)} \\ 0 \quad \text{otherwise} \end{cases}$$

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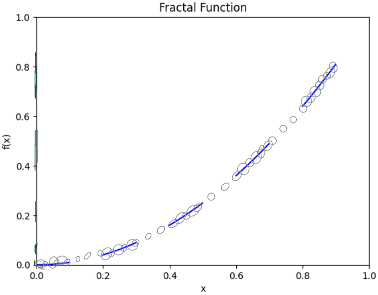

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$$D^{\alpha}_{\mathbb{F},\varphi} f(x_o) = \begin{cases} F\lim_{x\to x_o} \frac{S^{\alpha}_{F,\varphi}(f_x)-S^{\alpha}_{F,\varphi}(f_{x_o})}{x-x_o} \\ 0 \quad \text{otherwise} \end{cases}$$

Continuous Approximations and Fractional Derivatives

Continuous Approximation:

$dS^{\alpha}_{F,\varphi}(x) = S^{\alpha}_{F,\varphi}(x+dx) - S^{\alpha}_{F,\varphi}(x)$

$dS^{\alpha}_{F,\varphi}(x)$ is the volume of the ball at $x+dx$.

$$dS^{\alpha}_{F,\varphi}(x) = \begin{cases} A(\alpha)x^{\alpha-1}dx \text{ if } x, x+dx \in \mathbb{F} \\ 0 \text{ otherwise} \end{cases} .$$

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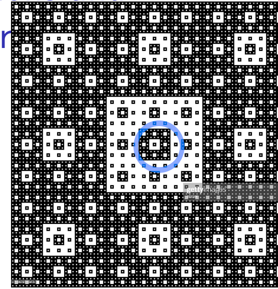

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Fractal Derivatives

In domain space

$D_{conf}f(x) = x^{1-\alpha}\frac{df}{dx}(x)$

$D_C f(x) = \int_{x_o}^{x}(x-t)^{1-\alpha}\frac{df}{dt}(t)dt$

In image space

$D_q f(x) = f^{1-q}\frac{df}{dx}(x)\,, \quad \alpha = 2-q$

$D_{CI}f(x) = \int_{x_o}^{x}[f(x)-f(t)]^{1-q}\frac{df}{dt}dt$

AD, E. Megías & R. Pasechnik, arXiv:2305.04633

Continuous Approximations and

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$dS^{\alpha}_{F,\varphi}(x)$ is the volume of t

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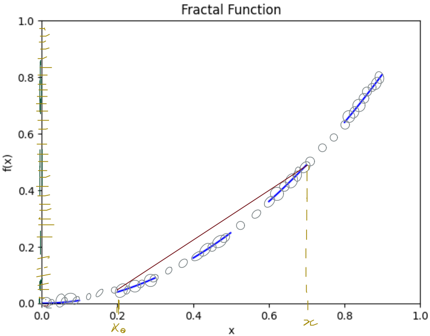

Fractal

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Conclusions:

Thermofractal in QCD (and any YMF theory)

Thermofractals and Tsallis Statistics

Emergence of Tsallis Statistics in QGP

Determination of q in terms of N_c and N_f.

Agreement with experimental data

Possibility of thermofractal structures in hadrons.

Dynamics of non-additive systems: heavy-qyark in the medium

Fractal derivative and fractional derivatives.

q-Deformed derivative as continuous approximation of fractal derivative.

Thank you