Limiting acceleration, radiation reaction, and nonlinear electromagnetism

Will Price, Martin Formanek, and Johann Rafelski

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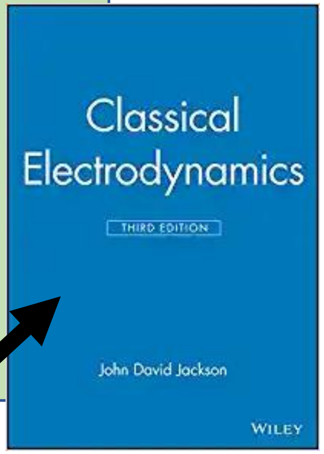
"A completely satisfactory treatment of the reactive effects of radiation does not exist. The difficulties presented by this problem touch on one of the most fundamental aspects of physics, the nature of an elementary particle. Although partial solutions, workable within limited area, can be given, the basic problem remains unsolved."

- J.D. Jackson, "Classical Electrodynamics," 3rd Edition, page 579



Most widely used EM textbook in US

$$\tau_0 = \frac{2}{3}\frac{e^2}{4\pi\epsilon_0 mc^3}$$

radiation reaction timescale $= 6.27 \cdot 10^{-24}$ seconds for electrons

Radiation Reaction

Acceleration = Radiation

Larmor-Abraham formula for instantaneous power radiated:

$$P_{rad} = -\frac{2}{3}\frac{e^2}{4\pi\epsilon_0 c^3} a_\mu a^\mu \equiv -m\tau_0 a_\mu a^\mu$$

Lorentz force on a charge needs to be supplemented with the backreaction force of the radiation field: the radiation reaction (RR) force

$$ma^\mu = eF^{\mu\nu}_{ext} u_\nu + \mathcal{F}^\mu_{RR}$$

← RR force is the dominant term in strong fields with strong acceleration

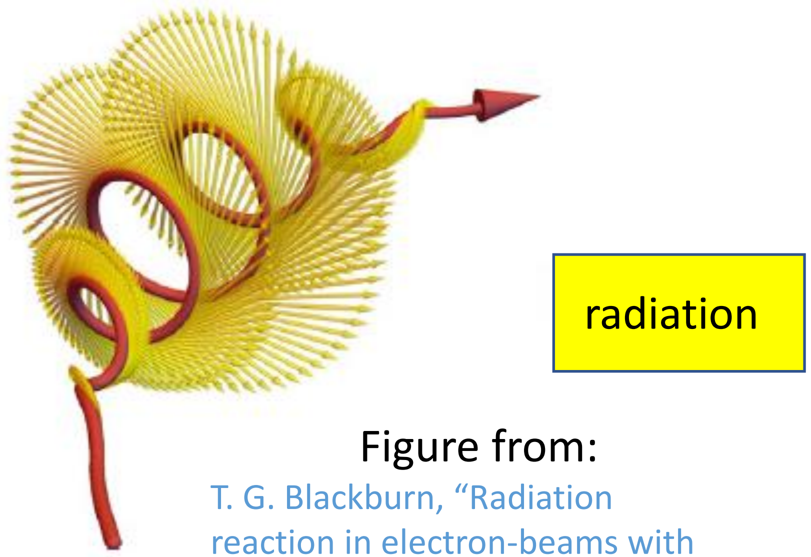


Figure from:
T. G. Blackburn, "Radiation reaction in electron-beams with high intensity lasers," Reviews of Modern Plasma Physics (2020).

Lorentz-Abraham-Dirac (LAD) Equation

$$ma^{\mu} = eF^{\mu\nu}u_{\nu} + m\tau_0\left(\frac{da^{\mu}}{d\tau} + \frac{a_{\nu}a^{\nu}}{c^2}u^{\mu}\right)$$

Renormalized mass:

$$m = \mathrm{m}_{\mathrm{bare}} + \mathrm{m}_{\mathrm{EM}}$$

- Derived by computing the action of the self-field on a point particle
- Produces unphysical runaway solutions: charge accelerates exponentially even in absence of external EM field
- Classical EM theory inconsistent, quantum EM theory (built upon the classical theory)…also inconsistent?

Classical theory of radiating electrons

By P. A. M. Dirac, F.R.S., *St John's College, Cambridge*

(*Received* 15 *March* 1938)

Introduction

The Lorentz model of the electron as a small sphere charged with electricity, possessing mass on account of the energy of the electric field around it, has proved very valuable in accounting for the motion and radiation of electrons in a certain domain of problems, in which the electromagnetic field does not vary too rapidly and the accelerations of the electrons are not too great. Beyond this domain it will not go unless supplemented by further assumptions about the forces that hold the charge on an electron together. No natural way of introducing such further assumptions has been discovered, and it seems that the Lorentz model has reached the limit of its usefulness and must be abandoned before we can make further progress.

Dirac clarified earlier work by Lorentz and Abraham, deriving a covariant equation of motion from energy conservation + Maxwell's equations

P. A. M. Dirac, "Classical theory of radiating electrons," Proc. R. Soc. A 167, 148 (1938).

Landau-Lifshitz Approximation

RR is a perturbative effect for small acceleration,

Approximate: $a^{\mu} \rightarrow \frac{e}{m} F^{\mu\nu} u_{\nu} + \cdots$ on the RHS of LAD,

$$P^{\mu\nu} \equiv g^{\mu\nu} - \frac{u^{\mu} u^{\nu}}{c^2}$$

Landau-Lifshitz (LL) Equation:

$$m a^{\mu} = e F^{\mu\nu} u_{\nu} + e \tau_0 \left(u^{\alpha} \partial_{\alpha} F^{\mu\nu} u_{\nu} + \frac{e}{m} P^{\mu\nu} F_{\nu\alpha} F^{\alpha\beta} u_{\beta} \right)$$

L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Second Edition, London, England: Pergamon(1962).

PHYSICAL REVIEW D **80**, 024031 (2009)

Rigorous derivation of electromagnetic self-force

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(Received 14 May 2009; published 23 July 2009)

During the past century, there has been considerable discussion and analysis of the motion of a point charge in an external electromagnetic field in special relativity, taking into account "self-force" effects due to the particle's own electromagnetic field. We analyze the issue of "particle motion" in classical electromagnetism in a rigorous and systematic way by considering a one-parameter family of solutions to

S. E. Gralla, A. I. Harte, R. M. Wald, "A rigorous derivation of electromagnetic self-force" Phys. Rev. D 80, 024031 (2009).

LL equation can be derived rigorously from Maxwell's equations as a first order perturbation to Lorentz force motion

Current state of EM theory

- Classical theory = LAD equation is inconsistent and predicts unphysical particle motion
- For weak enough external fields, the LL equation gives reliable results, as long as radiation reaction is a small perturbation to Lorentz force motion
- How do we treat particle motion in strong fields?

Limiting Acceleration

Quantum particles emit a limited amount of energy in a single photon:

$\hbar\omega \leq \gamma mc^2$ ➔ limited radiation rate

Radiation reaction has a natural timescale of τ_0 which gives us a natural acceleration scale:

$$a_{RR} \equiv \frac{c}{\tau_0} = \frac{3}{2}\frac{4\pi\epsilon_0 mc^4}{e^2} = 4.8 \cdot 10^{31}\ m/s^2$$

(value for electrons)

Limiting rate of radiation emission corresponding to a_{RR}:

$$P_{RR} = \frac{mc^2}{\tau_0}$$

Eliezer-Ford-O’Connell (EFO) Equation

$$P^{\mu\nu} \equiv g^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}$$

$$ma^\mu = eF^{\mu\nu}u_\nu + \tau_0 P^\mu_\nu \frac{d}{d\tau}(eF^{\nu\alpha}u_\alpha)$$

← RR force is derivative of external force

- Eliezer (1948) invented the equation by considering a charge of radius $c\tau_0$
- Ford, O’Connell (1991) re-invented it later by treating the radiation field as a thermodynamic system
- We (2021) independently re-re-invented it by looking for a RR force mixing acceleration and field dependence

Eliezer: particles with structure

On the classical theory of particles

By C. Jayaratnam Eliezer, *Christ's College, University of Cambridge*

(*Communicated by R. E. Peierls, F.R.S.—Received* 7 *February* 1948)

A set of classical relativistic equations of motion of an electron in an electromagnetic field is postulated. These equations are free from 'run-away' solutions, and give the same results as the Maxwell-Lorentz theory for non-relativistic motions when the external electromagnetic field does not vary too rapidly. For the scattering of light by an electron, the scattering cross-section is independent of the frequency and is a universal constant. This brings out a point of difference from the Lorentz-Dirac equations according to which the scattering cross-section varies inversely as the square of the frequency of the incident light, for large frequencies. For the motion of an electron towards a fixed proton, the equations allow a collision, unlike the Lorentz-Dirac equations according to which the electron is brought to rest before it reaches the proton.

(C. J. Eliezer, "On the classical theory of particles," Proc.R. Soc. Lond. A194: 543-555 (1948))

Derived by expanding RR force in power series of acceleration derivatives and charge radius

$$m\vec{a} = \left(1 + \frac{r_0}{c}\frac{d}{dt}\right)\vec{F}_{ext}$$

$$r_0 \equiv c\tau_0 = \text{charge radius}$$

Ford and O'Connell – RR from thermodynamics

Radiation reaction in electrodynamics and the elimination of runaway solutions

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Received 20 February 1991; accepted for publication 18 March 1991
Communicated by J.P. Vigier

The familiar Abraham–Lorentz theory of radiation reaction in classical non-relativistic electrodynamics exhibits many problems such as "runaway solutions" and violation of causality. As shown by many authors, such problems can be alleviated by dropping the assumption of a point electron. We also drop this assumption (by introducing a form-factor with a large cutoff frequency Ω) but we present a new approach based on the use of the generalized *quantum* Langevin equation. For an electric dipole interaction, an exact treatment is possible and we obtain a new equation of motion which, in spite of being third order, does not lead to runaway solutions or solutions which violate causality (the sole proviso being that Ω cannot exceed an upper limit of $3Mc^3/2e^2 = 1.60\times10^{23}\ \mathrm{s}^{-1}$). Furthermore, Ω appears in the third-derivative term but we show that, to a very good approximation, this term may be dropped so that we end up with a simple second-order equation which does not contain Ω and whose solutions are well-behaved.

G. W. Ford and R. F. O'Connell, Radiation reaction in electrodynamics and the elimination of runaway solutions, Phys. Lett. A157, 217 (1991).

Derived equation from considering the classical limit of a quantum particle interacting with a quantized blackbody radiation field

$$\left(\frac{1}{\Omega} - \tau_0\right)\frac{d\vec{a}}{dt} + m\vec{a} = \left(1 + \frac{1}{\Omega}\frac{d}{dt}\right)\vec{F}_{ext}$$

Ω is the inverse size of the charge. Choosing $\Omega \to \infty$ corresponds to a point charge and the nonrelativistic LAD equation, choosing $\Omega = \frac{1}{\tau_0}$ corresponds to a structured charge and the nonrelativistic EFO equation:

EFO - limiting acceleration

- Acceleration is limited by $a_{RR} = \frac{c}{\tau_0}$ for certain field configurations
- LL equation = EFO equation expanded to first order in τ_0

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Radiation reaction and limiting acceleration

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We investigate the strong acceleration properties of the radiation reaction force and identify a new and promising limiting acceleration feature in the Eliezer-Ford-O'Connell model; in the strong field regime, for many field configurations, we find an upper limit to acceleration resulting in a bound to the rate of radiation emission. If this model applies, strongly accelerated particles are losing energy at a much slower pace than predicted by the usual radiation reaction benchmark, the Landau-Lifshitz equation, which certainly cannot be used in this regime. We explore examples involving various "constant" electromagnetic field configurations and study particle motion in a light plane wave as well as in a material medium.

DOI: 10.1103/PhysRevD.105.016024

W. Price, M. Formanek and J. Rafelski, "Radiation reaction and limiting acceleration," Phys. Rev. D 105 (2022) no.1, 016024 doi:10.1103/PhysRevD.105.016024 [arXiv:2112.04444 [hep-ph]].

Limiting Acceleration from EFO

Re-arranged EFO equation:

$$\left(\delta^{\mu}_{\nu} - \frac{e\tau_0}{m} P^{\mu\alpha} F_{\alpha\nu}\right) a^{\nu} = \frac{e}{m}\left(F^{\mu\nu} + \tau_0 \frac{d}{d\tau} F^{\mu\nu}\right) u_{\nu}$$

Schematic form of solution for a^{μ}:

$$a = \frac{e}{m} \frac{F + \tau_0 \dot{F}}{1 - \frac{e\tau_0}{m} PF} u$$

→ If $F \to \infty$ and $u \to c$,
then $a \to \frac{c}{\tau_0}$

EFO equation for constant fields

Solve for $a_\mu a^\mu$:

$$\left(\delta^\mu_\nu - \frac{e\tau_0}{m} P^{\mu\alpha} F_{\alpha\nu}\right) a^\nu = \frac{e}{m} F^{\mu\nu} u_\nu$$

$$a_\mu a^\mu = a_{LF}^2 \frac{1 + \left(\frac{c\tau_0 e^2}{m^2}\right)^2 \frac{\mathcal{P}}{|a_{LF}^2|}}{1 + \tau_0^2 \left(\frac{e^2}{m^2} 2\mathcal{S} + \frac{|a_{LF}^2|}{c^2}\right)}$$

For $|a_{LF}| \gg \mathcal{S}, \mathcal{P}$, the acceleration is limited:

$$|a_\mu a^\mu| \leq \frac{c^2}{\tau_0^2}$$

$a_{LF}^\mu = \frac{e}{m} F^{\mu\nu} u_\nu$ (LF = Lorentz Force)

$\mathcal{S} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}\left(B^2 - \frac{E^2}{c^2}\right)$

$\mathcal{P} = \frac{1}{4} F^{\mu\nu} \tilde{F}_{\mu\nu} = \vec{\mathrm{E}} \cdot \vec{B}/c$ $\quad \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$

Limiting acceleration will still hold in time-dependent fields will still hold if

$$\omega \ll 1/\gamma\tau_0$$

ω = frequency of external field

$$\Omega_B \equiv \frac{eB}{m}$$

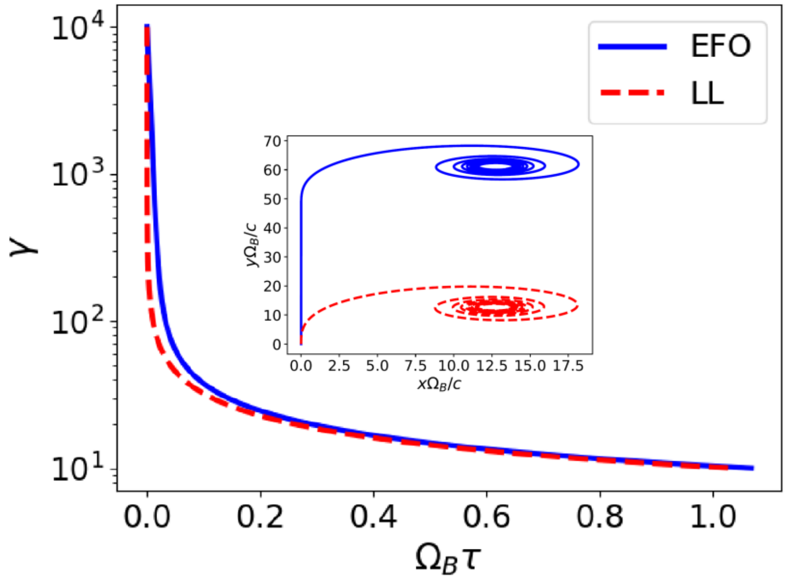
Example: Constant Magnetic Field

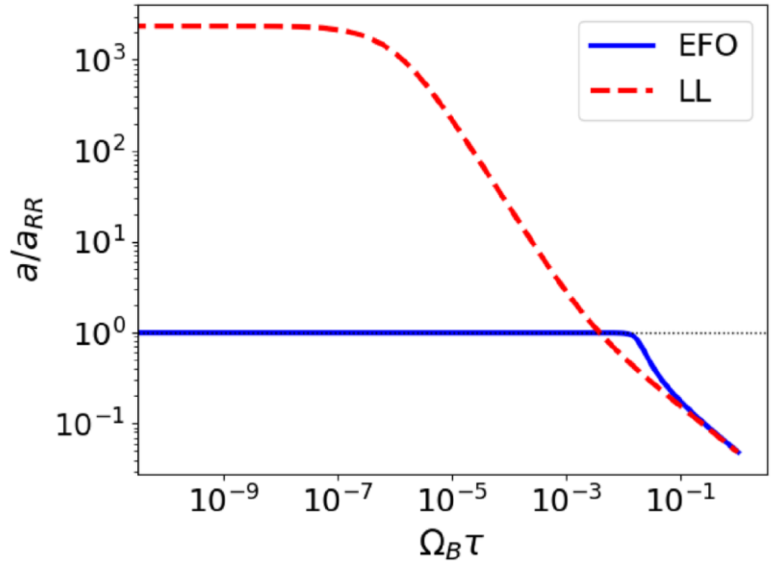
<u>EFO</u>: $$a_\mu a^\mu = -\frac{\Omega_B^2 c^2(\gamma^2-1)}{1+(\Omega_B\tau_0\gamma)^2}$$

In the limit of $B \to \infty, \gamma \to \infty$: $a_\mu a^\mu \to -\left(\frac{c}{\tau_0}\right)^2$

<u>LL</u>: $$a_\mu a^\mu = -\Omega_B^2 c^2(\gamma^2-1)(1-(\Omega_B\tau_0\gamma)^2)$$

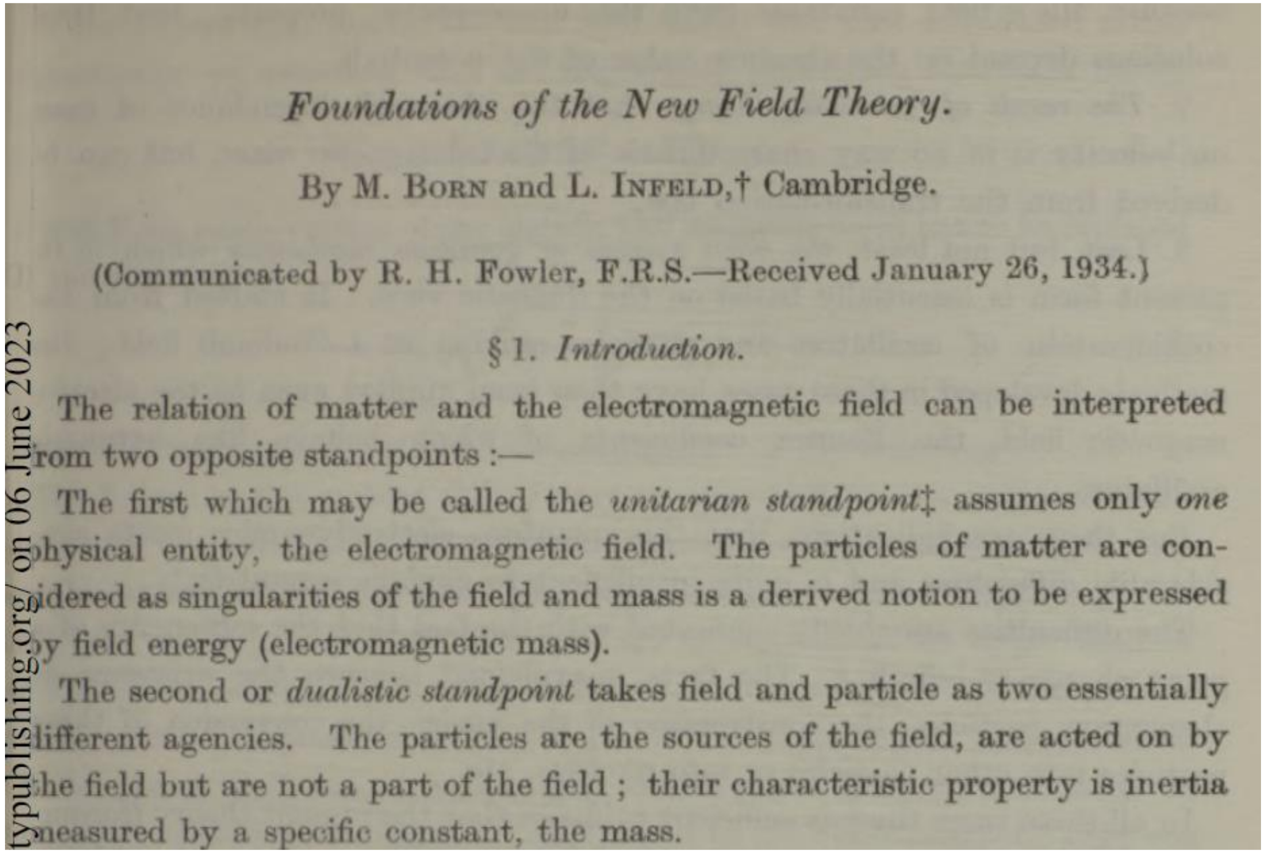
In the limit of $B \to \infty, \gamma \to \infty$: $a_\mu a^\mu \to -\infty$





Numerical solutions of EFO and LL equations for $B = 4.41 \cdot 10^9$ Tesla

Born-Infeld (BI): limiting field theory



Foundations of the New Field Theory.

By M. Born and L. Infeld,† Cambridge.

(Communicated by R. H. Fowler, F.R.S.—Received January 26, 1934.)

§ 1. *Introduction.*

The relation of matter and the electromagnetic field can be interpreted from two opposite standpoints :—

The first which may be called the *unitarian standpoint*‡ assumes only *one* physical entity, the electromagnetic field. The particles of matter are considered as singularities of the field and mass is a derived notion to be expressed by field energy (electromagnetic mass).

The second or *dualistic standpoint* takes field and particle as two essentially different agencies. The particles are the sources of the field, are acted on by the field but are not a part of the field ; their characteristic property is inertia measured by a specific constant, the mass.

- Nonlinear EM theory with a limiting field strength E_0 to solve the divergence problems of EM interactions

- E_0 determined by assuming all mass of an electron is electromagnetic:

$$m_e c^2 = \frac{\epsilon_0}{2} \int d^3x \, \vec{E}^2(E_0)$$

$$\rightarrow E_0 = 1.2 \cdot 10^{20} \, \frac{V}{m}$$

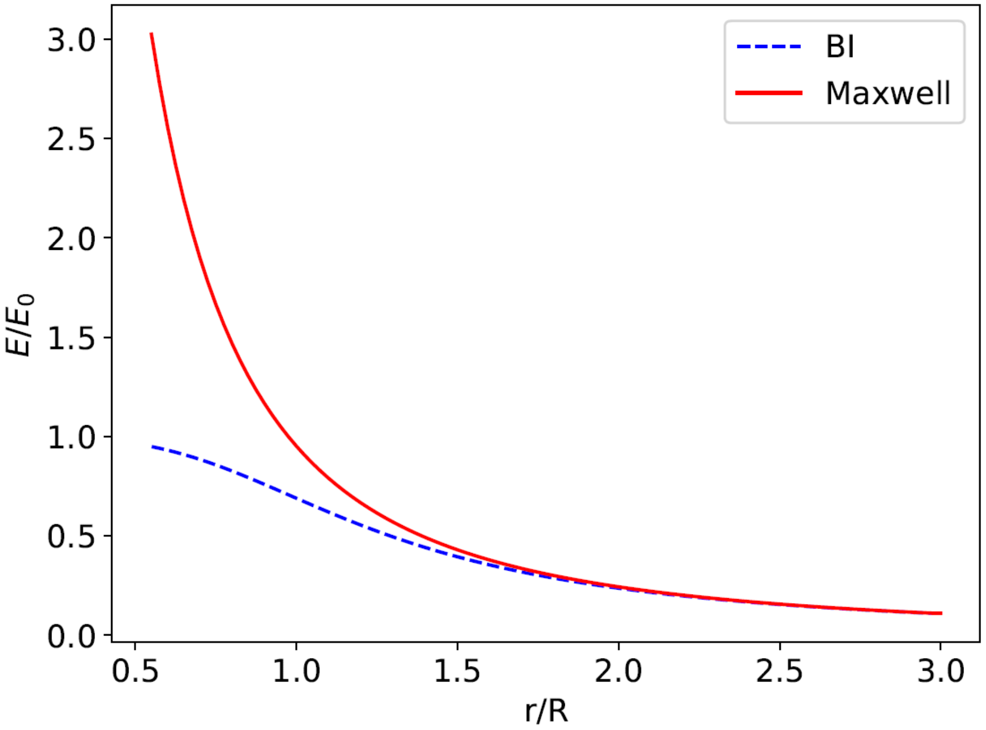
Born-Infeld equations

$$\mathcal{L} = \epsilon_0 E_0^2 \left(1 - \sqrt{det\left(g_{\mu\nu} + \frac{F_{\mu\nu}}{c}\right)} \right) = \epsilon_0 E_0^2 \left(1 - \sqrt{1 + \frac{2\mathcal{S}c^2}{E_0^2} - \frac{\mathcal{P}^4 c^4}{E_0^4}} \right)$$

Field Equations

$$\partial_\mu \frac{F^{\mu\nu} - \frac{\mathcal{P}c^2}{E_0^2}\tilde{F}^{\mu\nu}}{\sqrt{1 + \frac{2\mathcal{S}c^2}{E_0^2} - \frac{\mathcal{P}^4 c^4}{E_0^4}}} = \mu_0 j^\nu$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$



Coulomb solution:

$$\vec{E}_{BI} = \frac{E_0 \hat{r}}{\sqrt{1 + \frac{r^4}{R^4}}}$$

$$R^2 \equiv \frac{e}{4\pi\epsilon_0 E_0}$$

Born-Infeld as a limiting field theory

Born–Infeld Nonlinear Electromagnetism in Relativistic Heavy Ion Collisions

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Doi: 10.12693/APhysPolA.XX.TEMP--9502 *e-mail: wprice@arizona.edu

We study the effect of the limiting field strength of Born–Infeld electromagnetism on the dynamics of charged particle scattering. We formulate the Born–Infeld limiting field in an invariant manner, showing that it is the electric field-dominated eigenvalue "a" of the field tensor $F^{\mu\nu}$ which is limited rather than the individual field vectors. Heavy ion collisions in particular provide uniquely large values of the field invariants that appear in the Born–Infeld action, amplifying nonlinear effects. Thus "a" is the dominant input into the force between heavy ions that we use to compute the scattering angle as a function of the impact parameter. We evaluate the Born–Infeld effects, showing relevance at small impact parameters and exhibiting their dependence on the value of the limiting field strength.

topics: Born–Infeld electromagnetism, heavy-ion collisions, strong fields

***paper under embargo until June 14**

$$\lambda_a = \sqrt{-S + \sqrt{S^2 + P^2}}$$

$$\lambda_b = \sqrt{S + \sqrt{S^2 + P^2}}$$

BI Lagrangian can be rewritten in terms of field tensor eigenvalues λ_a, λ_b:

$$\mathcal{L} = \epsilon_0 E_0^2 \left(1 - \sqrt{\left(1 - \frac{c^2 \lambda_a^2}{E_0^2}\right)\left(1 + \frac{c^2 \lambda_b^2}{E_0^2}\right)} \right)$$

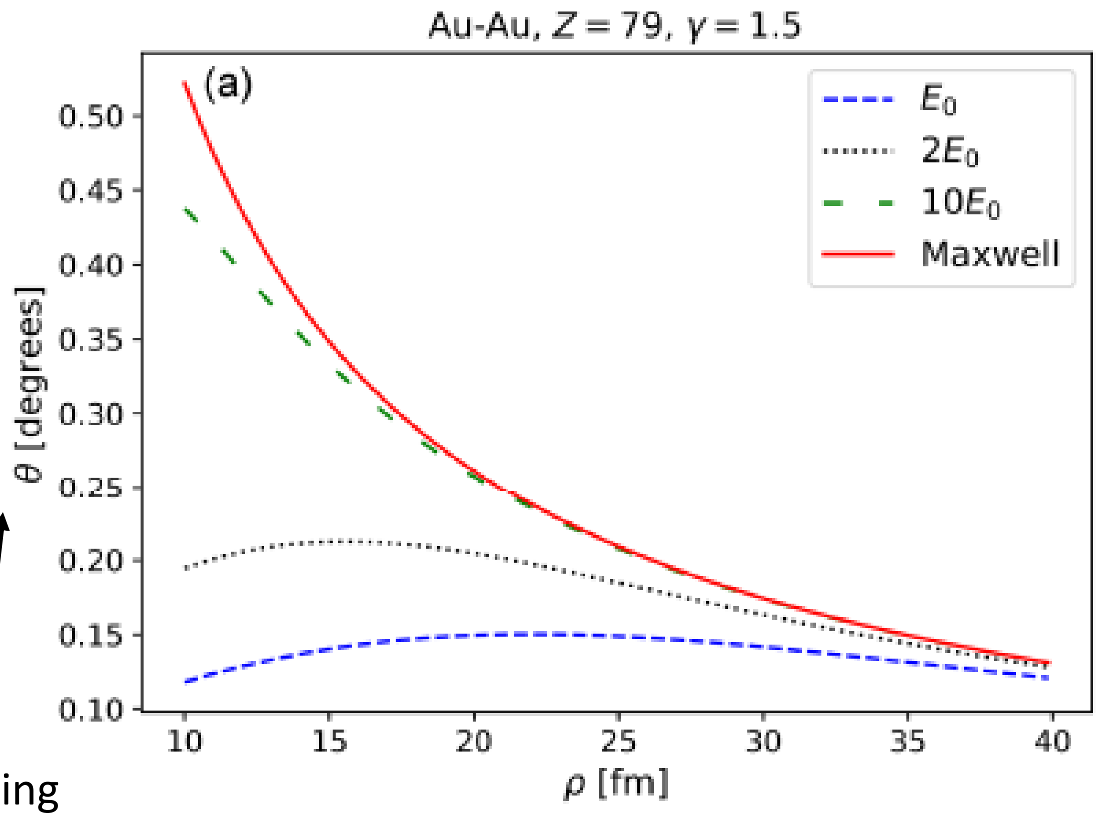
Invariant eigenvalue λ_a must be limited: → $\lambda_a < \dfrac{E_0}{c}$

Limiting field strength implies limiting acceleration:

$$a_{BI} = \frac{eE_0}{m} = 2.1 \cdot 10^{31} \frac{m}{s^2}$$

(value for electrons)

Limiting field effects in heavy ion collisions



Scattering angle

Impact parameter

- Scattering of two identical heavy ions calculated numerically using the Lorentz force with BI fields of each ion
- BI limiting field strength decreases the force on the particles at distance of closest approach, causing less scattering

$$\vec{E} = \frac{Ze\gamma}{4\pi\epsilon_0\left(x_\perp^2 + \gamma^2 x_\parallel^2\right)^{3/2}} \frac{\vec{x}}{\sqrt{1 + \frac{\left(\frac{Ze}{4\pi\epsilon_0\left(x_\perp^2+\gamma^2 x_\parallel^2\right)}\right)^2}{E_0^2}}}$$

$$\vec{B} = \vec{v} \times \vec{E} \qquad m a^\mu = Z e F^{\mu\nu} u_\nu$$

Conclusions

- Radiation = acceleration; limiting acceleration limits the rate a particle emits radiation
- Acceleration can be limited through balance between radiation reaction force and external force (EFO)
- Acceleration can also be limited by a limiting field strength through of nonlinear EM theory (BI)
- A solution to the problem of radiation reaction may be found by using limiting acceleration as a guiding principle and building from foundations of EFO and BI as models