Classical holography

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Content

Local, nonlocal and weakly nonlocal

Locality in space(time)

- Local fields and local field equations. Example: $\varphi(t, \mathrm{x})$, Poisson equation.
- Space integrals in the field equations: strong nonlocality.
- Nonlocal fields: Example: $f(t, \mathrm{x}_1, \mathrm{x}_2)$, Liouville equation, entanglement.
- Weak nonlocality: extension of the field equations with higher order space derivatives. Example: gradient fluids, Horndeski gravity.

Locality in time

- Locality in time. No memory. Markov process.
- Memory functionals in the field equations: strong memory. Example: principle of fading memory.
- Multitemporal fields: science fiction?
- Weak 'nonlocality' in time: higher order time derivatives in the field equations. Example: second sound, delay and inertia.

Holography

Holography $\leftarrow$ holos+graphe = complete, whole + drawing, writing.

Optical holography

- Dennis Gabor. Reproduction of 3 dimensional information from 2 dimensional projections.
- Interferometric. Amplitude and phase. For any wavelike propagation. E.g. ambisonic sound.

Holography in quantum field theories

- Generalisation of black hole thermodynamics. Hawking, t'Hooft, Susskind. Entropy is area. (Is that? Biró et al. 2018, 2020)
- Abstracted in string theory. Expected in quantum gravity.
- AdS-CFT correspondence.
- Entanglement. Unruh effect (thermodynamics!). Emergent gravity of Verlinde.

Classical holography I

Balance of momentum. Global form: $\dot{M} = -F + P$.

$$\rho\dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P} = -\rho\nabla\varphi, \qquad \rho\dot{v}^i + \partial_k P^{ik} = -\rho\partial^i\varphi. \tag{1}$$

Bulk and surface forces. Substantial or comoving derivative:

$$\dot{a} = \partial_t a + v^k \partial_k a$$

Convective and conductive current densities:

$$\boldsymbol{P}_{conv} = \boldsymbol{P}_{cond} + \rho\boldsymbol{v}\boldsymbol{v}$$

Newtonian gravity ($\Delta\varphi = 4\pi G\rho$):

$$\rho\nabla\varphi = \nabla \cdot \boldsymbol{P}_{grav}(\nabla\varphi) = \nabla \cdot \left(\frac{1}{4\pi G}\left[\nabla\varphi\nabla\varphi - \frac{1}{2}(\nabla\varphi)^2\boldsymbol{I}\right]\right)$$

Maxwell stress tensor.

Classical holography II

Gravity is holographic

$$\rho\dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P}_{grav} = 0 \quad \Longleftrightarrow \quad \dot{\boldsymbol{v}} = -\nabla\varphi$$

Particle or field?? Test particle and integrating screens. Constant background field or field theory? ($\dot{\rho} + \rho\nabla \cdot \boldsymbol{v} = 0$, $\Delta\varphi = 4\pi G\rho$)

Holographic Euler fluids

Ideal Euler fluids: $\boldsymbol{P}_{Euler} = p(u, \rho)\boldsymbol{I}$. p is the thermostatic pressure, e.g. ideal gas.

$$\nabla \cdot \boldsymbol{P}_{Euler} = \nabla p = \rho\nabla\mu + \rho s\nabla T$$

Follows from the Gibbs-Duhem relation. For isothermal processes of ANY fluid the chemical potential is a mechanical potential. Friedmann equation

Classical holographic property

$$\nabla \cdot \boldsymbol{P}(...) = \rho\nabla\phi(...)$$

Constitutive (...), material property. Thermodynamics or field equation dependent?

How to solve a constrained inequality?

Constitutive state space

Coleman-Noll and Liu procedures. Separation of functions and variables.
The entropy inequality is *conditional*:

$$\rho\dot{e} + \nabla\cdot\boldsymbol{q}(e,\nabla e) = 0,$$

$$\rho\dot{s}(e,\nabla e) + \nabla\cdot\boldsymbol{J}(e,\nabla e) - \Lambda(e,\nabla e)(\rho\dot{e} + \nabla\cdot\boldsymbol{q}(e,\nabla e)) =$$

$$\rho\frac{\partial s}{\partial\nabla e}(\nabla e)^{\cdot} + \rho\left(\frac{1}{T} - \Lambda\right)\dot{e} + \ldots \geq 0$$

Liu-procedure, Lagrange–Farkas-multipliers. It follows that:

$$\frac{\partial s}{\partial\nabla e}(e,\nabla e) = 0, \qquad \Lambda = \frac{1}{T}, \quad \text{and} \quad \boldsymbol{q}(e,\nabla e)\cdot\nabla\left(\frac{1}{T}(e)\right) \geq 0$$

Constitutive state variables: $(e,\nabla e)$
→ thermodynamic state variables: (e)
Process direction variables: $(\dot{e},(\nabla e)^{\cdot},\nabla^2 e)$

Various theories with weakly nonlocal extensions

Classified by constitutive state spaces and constraints

- Heat conduction. Internal energy or temperature. $(e, \nabla e, \nabla^2 e, ...)$.
  Constraint: balance of internal energy.
- Internal variables (e.g. phase fields). $(\varphi, \nabla\varphi, \nabla^2\varphi, ...)$. Tensorial order may be arbitrary.
  Constraint: evolution equation, free or balance (Ginzburg-Landau-Alen-Cahn, Cahn-Hilliard).
- Internal variables and fluid mechanics $(\rho, \nabla\rho, v, \nabla v, e, \nabla e, \varphi, \nabla\varphi, \nabla^2\varphi))$ Constraint: evolution equation, balances of mass momentum and energy ($\rightarrow$ Newtonian gravity and more)
- Fluid mechanics. Mass, velocity and energy. $(\rho, \nabla\rho, \nabla^2\rho, v, \nabla v, e, \nabla e))$
  Constraints: balances of mass, momentum and energy ($\rightarrow$ quantum mechanics and more)
- Solid mechanics. Mass, strain and energy $(\varepsilon, \nabla\varepsilon, \nabla^2\varepsilon, ...)$, and more gradients.
  Constraints: kinematics, balances of mass, momentum and energy.

Newtonian gravity

VP-Abe (Physica A, 2022)
Abe-VP (Symmetry, 2022)
Pszota-VP (arXiv: 2306.01825)

Scalar field and hydrodynamics

$s(e - \varphi - \frac{\nabla\varphi\cdot\nabla\varphi}{8\pi G\rho}, \rho)$. Gibbs relation:

$$du = Tds + \frac{p}{\rho^2}d\rho = de - d\left(\varphi + \frac{\nabla\varphi\cdot\nabla\varphi}{8\pi G\rho}\right).$$

The potential energy, φ, the field energy and internal energy are separated.

Balances of mass, momentum, internal energy + field equation:

$$\dot{\rho} + \rho\nabla\cdot\boldsymbol{v} = 0,$$
$$\rho\dot{\boldsymbol{v}} + \nabla\cdot\boldsymbol{P} = \boldsymbol{0},$$
$$\rho\dot{e} + \nabla\cdot\boldsymbol{q} = -\boldsymbol{P} : \nabla\boldsymbol{v},$$
$$\dot{\varphi} = f.$$

Constraints of the entropy inequality:

$$\boxed{\rho\dot{s} + \nabla\cdot\boldsymbol{J} = \Sigma \geq 0}$$

Gravity

Constitutive state variables: $(e, \nabla e, \rho, \nabla\rho, (\boldsymbol{v}), \nabla\boldsymbol{v}, \varphi, \nabla\varphi, \nabla^2\varphi)$
$\rightarrow$ thermodynamic state variabless: $(e, \rho, \varphi, \nabla\varphi)$

$$\rho\dot{s} + \nabla\cdot\boldsymbol{J} =$$

$$\left(\boldsymbol{q} + \frac{\dot{\varphi}}{4\pi G}\nabla\varphi\right)\cdot\nabla\left(\frac{1}{T}\right)$$

$$+\boxed{\frac{\dot{\varphi}}{4\pi G T}\left(\Delta\varphi - 4\pi G\rho\right)}$$

$$-\left[\boldsymbol{P} - p\boldsymbol{I} - \frac{1}{4\pi G}\left(\nabla\varphi\nabla\varphi - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\boldsymbol{I}\right)\right] : \frac{\nabla\boldsymbol{v}}{T} \geq 0$$

- With Liu procedure as well.
- Holographic property:

$$\nabla\cdot\left(p\boldsymbol{I} + \frac{1}{4\pi G}\left(\nabla\varphi\nabla\varphi - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\boldsymbol{I}\right)\right) = \rho\nabla(\mu + \varphi)$$

Nonlinear extension, static, nondissipative field

Stationary nondissipative field equation:

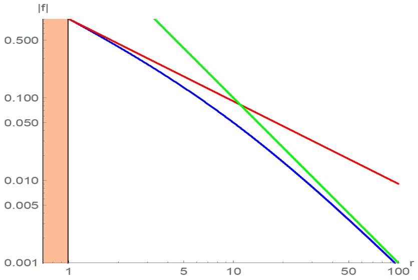
$$\boxed{0 = \Delta\varphi - 4\pi G\rho - K\nabla\varphi\cdot\nabla\varphi.}$$

Vacuum solutions $\rho = 0$:

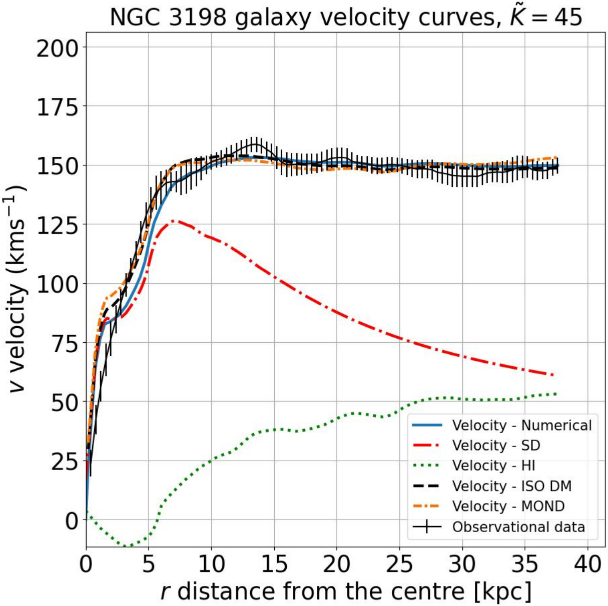
$$\varphi(\boldsymbol{r}) = \frac{1}{K}\ln(r)$$

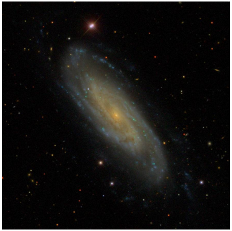
Spherical symmetric force field. Crossover. Apparent and real masses:

$$f(r) = -\frac{r_1}{Kr(r+r_1)} = -\frac{GM_{aa}}{r(r+r_1)}$$



Modified gravity and Dark Matter





NGC 3198	
M_{DM+BM}	M_{aa}
190	110

Unit: $10^9 M_\odot$

Interplay: hidden Galilean covariance

Spacetime aspects - separation of material and motion

$$\frac{\partial(\rho e_{total})}{\partial t} + \nabla\cdot(\boldsymbol{q} + \rho \boldsymbol{v} e_{total} + \boldsymbol{P}\cdot\boldsymbol{v}) = 0, \quad \rightarrow \quad \rho \dot{e}_{total} + \nabla\cdot(\boldsymbol{q} + \boldsymbol{P}\cdot\boldsymbol{v}) = 0$$

It is a change of frame:

- Comoving(substantial) time derivative: $\dot{e} = \frac{\partial e}{\partial t} + \boldsymbol{v}\cdot\nabla e$,
- Galilean four-vector: $(\rho e, \boldsymbol{q})$, convective and conductive current densities.
- constitutive state space: ∇e is spacelike covector,
- total and internal energies: $e = e_{TOT} - v^2/2$.

Consequences

- What is comoving? Mass? Energy? Observer representations. Flow-frame.
- Total energy, kinetic energy and internal energy. Galilean relativistic energy-momentum-mass four-tensor. Consequence: entropy production is objective.
- Temperature is a Galilean relativistic four-vector: thermal reference frames.

Korteweg fluids

VP-Fülöp (Proc. Roy. Soc., 2004)
VP-Kovács (Phil. Trans. Roy. Soc. A, 2020)
VP (Physics of Fluids, 2023)

Korteweg fluids: history

Capillarity.
Van der Waals: gradient of density is a thermodynamic variable.
Korteweg (1905): second gradient of density, pressure expansion.

Balances of mass, momentum and internal energy:

$$\dot{\rho} + \rho \nabla \cdot \boldsymbol{v} = 0,$$
$$\rho \dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P} = \boldsymbol{0},$$
$$(\rho \dot{e} + \nabla \cdot \boldsymbol{q} = -\boldsymbol{P} : \nabla \boldsymbol{v}.)$$

$$\boldsymbol{P} = \left(p - \alpha \Delta \rho - \beta (\nabla \rho)^2\right) \boldsymbol{I} - \delta \nabla \rho \circ \nabla \rho - \gamma \nabla^2 \rho$$

$\alpha, \beta, \gamma, \delta$ are density dependent material parameters.

Instable. Second law? Eckart fluids 1948, Dunn and Serrin 1985.

Korteweg fluids – Liu procedure

Constitutive state variables: $(e, \nabla e, \rho, \nabla\rho, (\boldsymbol{v}), \nabla\boldsymbol{v})$
$\rightarrow$ thermodynamic state variables: $(e, \rho, \nabla\rho)$
Process direction: $(\dot{e}, (\nabla e)^{\cdot}, \nabla^2 e, \dot{\rho}, (\nabla\rho)^{\cdot}, \nabla^2\rho, \dot{\boldsymbol{v}}, (\nabla^2\boldsymbol{v})^{\cdot})$

$$\rho\dot{s} + \nabla\cdot\boldsymbol{J} = \boldsymbol{q}\cdot\nabla\left(\frac{1}{T}\right) -$$
$$-\left[\boldsymbol{P} - p\boldsymbol{I}\right] : \frac{\nabla\boldsymbol{v}}{T} \geq 0$$

- Classical irreversible thermodynamics.
- The pressure of an ideal, non-dissipative Euler fluid:

$$\boxed{\boldsymbol{P} = p(e,\rho)\boldsymbol{I}}$$

Korteweg fluids – Liu procedure

Constitutive state variables: $(e, \nabla e, \rho, \nabla\rho, \nabla^2\rho, (\boldsymbol{v}), \nabla\boldsymbol{v})$

$\rightarrow$ thermodynamic state variables: $(e, \rho, \nabla\rho)$

Process direction: $(\dot{e}, (\nabla e)^{\cdot}, \nabla^2 e, \dot{\rho}, (\nabla\rho)^{\cdot}, (\nabla^2\rho)^{\cdot}, \nabla^3\rho, \dot{\boldsymbol{v}}, (\nabla^2\boldsymbol{v})^{\cdot})$

$$\rho\dot{s} + \nabla\cdot\boldsymbol{J} = \boldsymbol{q}\cdot\nabla\left(\frac{1}{T}\right) -$$
$$-\left[\boldsymbol{P} - p\boldsymbol{I} - \frac{\rho^2}{2}\left(\nabla\cdot\frac{\partial s}{\partial\nabla\rho}\boldsymbol{I} + \nabla\frac{\partial s}{\partial\nabla\rho}\right)\right] : \frac{\nabla\boldsymbol{v}}{T} \geq 0$$

- Rigorous methods are essential.
- The pressure of an ideal, non-dissipative Korteweg fluid:

$$\boxed{\boldsymbol{P} = p(e,\rho)\boldsymbol{I} + \frac{\rho^2}{2}\left(\nabla\cdot\frac{\partial s}{\partial\nabla\rho}\boldsymbol{I} + \nabla\frac{\partial s}{\partial\nabla\rho}\right)}$$

Perfect Korteweg fluids – holography

$$\boldsymbol{P}_K = \frac{\rho^2}{2}\left(\nabla\cdot\frac{\partial s}{\partial\nabla\rho}\boldsymbol{I} + \nabla\frac{\partial s}{\partial\nabla\rho}\right)$$

- Classical holographic property, with internal energy:

$$\boxed{\nabla\cdot\boldsymbol{P}_K = \rho(\nabla\phi + T\nabla s)}, \quad \text{where} \quad \phi = \frac{\partial\rho u}{\partial\rho} - \nabla\cdot\frac{\partial(\rho u)}{\partial\nabla\rho} = \delta_\rho(\rho u)|_{\rho s}$$

Functional derivative. Equivalently: chemical potential, homotherm.

- Momentum balance: continuum AND point mass

$$\rho\dot{\boldsymbol{v}} + \nabla\cdot\boldsymbol{P}_K = \rho(\dot{\boldsymbol{v}} + \nabla\phi) = 0 \quad \rightarrow \quad \dot{\boldsymbol{v}} = -\nabla\phi$$

- Conserved vorticity follows.
- Bohm potential?

Probabilistic Korteweg fluids – additivity

Zeroth Law of thermodynamics: separability of independent physical systems.
Multicomponent normal fluids. Notation: $\rho_1 = \rho_1(\boldsymbol{x}_1)$.

$$u(\rho_1 + \rho_2) = u(\rho_1) + u(\rho_2).$$

Multicomponent probabilistic fluids:

$$u(\rho_1 \rho_2) = u(\rho_1) + u(\rho_2).$$

Functional condition, $\rho_{tot} = \rho_1 \rho_1$:

$$u(\rho_{tot}, (\nabla \rho_{tot})^2) = u\left(\rho_1\rho_2, (\rho_2 \nabla_1 \rho_1)^2 + (\rho_1 \nabla_2 \rho_2)^2\right) =$$
$$u(\rho_1, (\nabla_1 \rho_1)^2) + u(\rho_2, (\nabla_2 \rho_2)^2).$$

Unique solution:

$$\boxed{u(\rho, (\nabla\rho)^2) = k \ln \rho + \frac{K}{2}\frac{(\nabla\rho)^2}{\rho^2}}$$

Independent Schrödinger equations for independent particles/components.

Summary

There is a thermodynamic road to quantum physics:

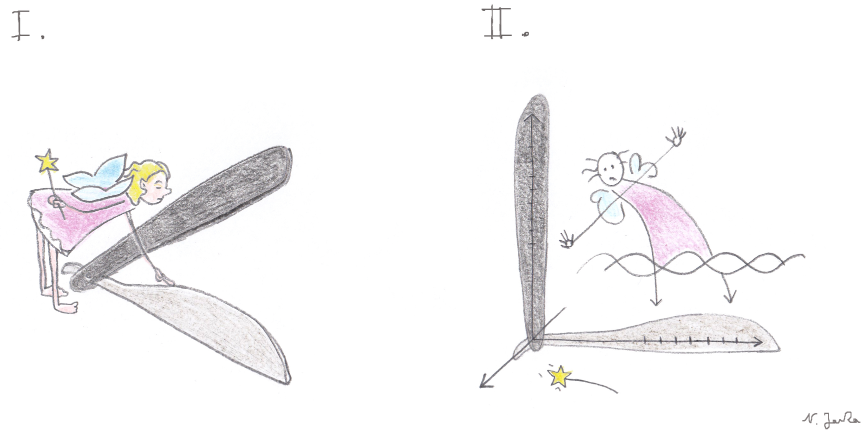
Korteweg fluids $\Longrightarrow$ superfluids $\Longrightarrow$ quantum mechanics

- Second law and ideality conditions:
  classical holography = wave-particle duality.
- Additivity $\Longrightarrow$ superfluids
- Density independent potential, $\hbar$: $\Longrightarrow$ Schrödinger equation

Classical holography

- The Second Law of Thermodynamics is fundamental.
- It is applicable for fields and informative in the marginal case of zero dissipation.
- There are real physical systems all along the road.
- It is quantisation method.

Thank you for the attention!



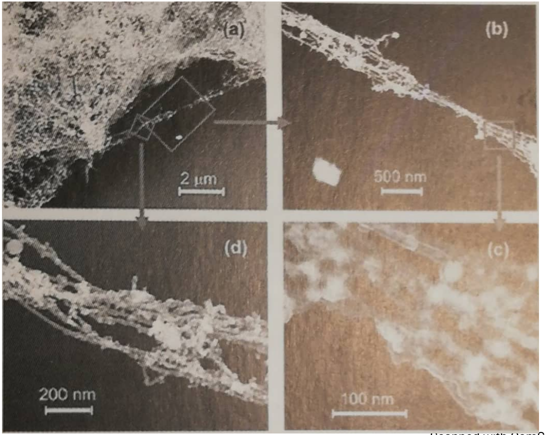
"This may be true, because it is mathematically trivial."
(somebody from Princeton, according to R. Pisarski)

What is quantum mechanics?

Early history

- Schrödinger, *Quantisierung als Eigenwertproblem I-II.*, AdP (1926), Heft 4, closed 18 March 1926;
- Madelung, *Quantentheorie in hydrodynamischer Form*, ZfP, (1926), submitted 25 Oktober 1926;
- Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables I-II.*, PhysRev (1952);
- de Broglie, Fényes, Nelson, Jánossy, Takabayashi, Bell, Vigier, Holland, etc...
- Jackiw et al. (2004). Every non-Abelian gauge theory can be reformulated as hydrodynamics. QCD, QGP. (See G. Torrieri I guess).
- Cosmology. K. Huang, Khoury, Hossenfelder, etc... Dark matter vs MOND: superfluidity.
- Biró-Ván: *Splitting the source term for the Einstein equation to classical and quantum parts*, FoP (2015). Transformed Klein-Gordon and Einstein equations.

Quantum fluids exist:



Vortice lines in He II, from boundary to boundary. Donelly, 1991.

Superfluidity

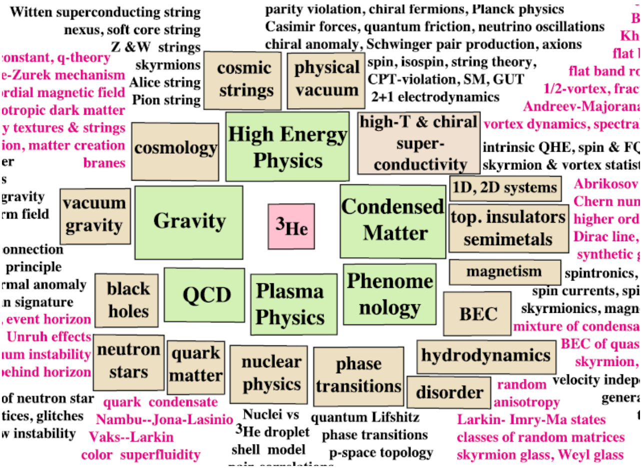
Histocical remarks and more

- Landau, ZhETP (1941).
- Bohm and Pines (1952), Gross and Pitaevskii (1961).
- Dense plasmas. Critical survey of Bonitz et al. (2019).
- QGP, T. Kodama et al., stochastic qm., Jackiw et al (2004).
- G.E. Volovik: The Universe in a Helium Droplet (2003).

Theory notes: there is no quantisation

- There is a Hamiltonian but there is no Lagrangian.
- Hydro is an effective theory and cannot be quantised.
- (Note: Like gravity. Therefore gravity cannot be quantized, q-gravity does not exist.)

A recent view with trimmed boundaries.



From Volovik JLTP (2021). Inflational expansion.

Absolute temperature

VP (PTRSA, 2023)

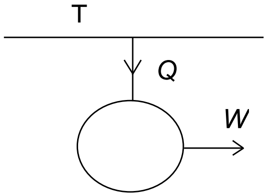
Equilibrium thermodynamics or thermostatics

Absolute temperature: does not depend on the material and the method of the measurement. It is not the scale, not the zero point, it is the *concept*. (Lord Kelvin, 1848. See e.g. Kardar, 2007).

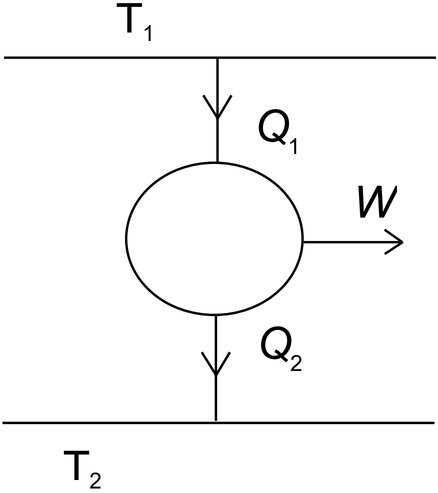
Kelvin-Planck form of the second law

It is impossible to devise a cyclically operating device, the sole effect of which is to absorb energy in the form of heat from a single thermal reservoir and to deliver an equivalent amount of work.

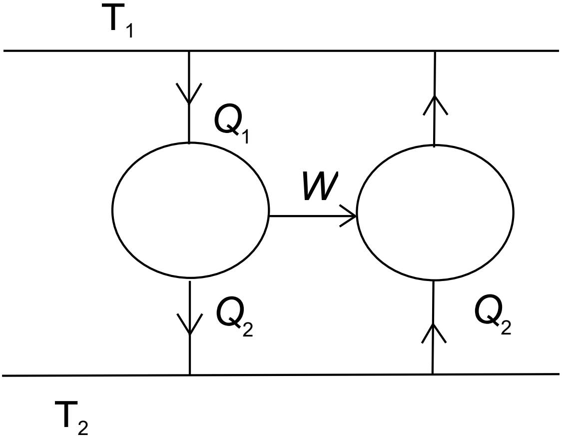
A perpetuum mobile of second kind:



A heat engine. Absorbs and emits heat and produces work.

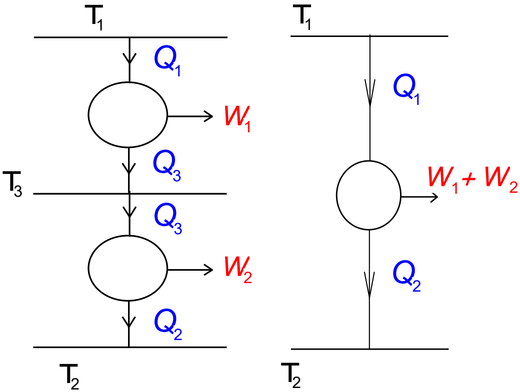


The reversible device is the most effective:



Efficiency depends only on the reservoir temperatures: $\eta(T_H, T_L)$

More heat engines:



Properties of efficiency:

$$\eta_1(T_1, T_3)\eta_2(T_3, T_2) = \eta_3(T_1, T_2) \quad \rightarrow \quad \eta(T_1, T_2) = \frac{\phi(T_1)}{\phi(T_2)} \quad \rightarrow \quad T = \frac{1}{\phi}.$$

Conditions: reversible process and the Kelvin-Planck form.

Variational principles for dissipative processes

Variational principles for dissipative processes

Condition: symmetry

$$\boxed{\hat{\Theta}(\varphi) = 0}, \qquad \exists F : Dom(\hat{\Theta}) \to \mathbb{R}, \quad \delta F(\varphi) = \hat{\Theta}(\varphi)$$

δ derivation in a Banach (or Frechet) spaces, boundary conditions, ...

Necessary condition: $\hat{\Theta}$ is symmetric.

Many different variational principles

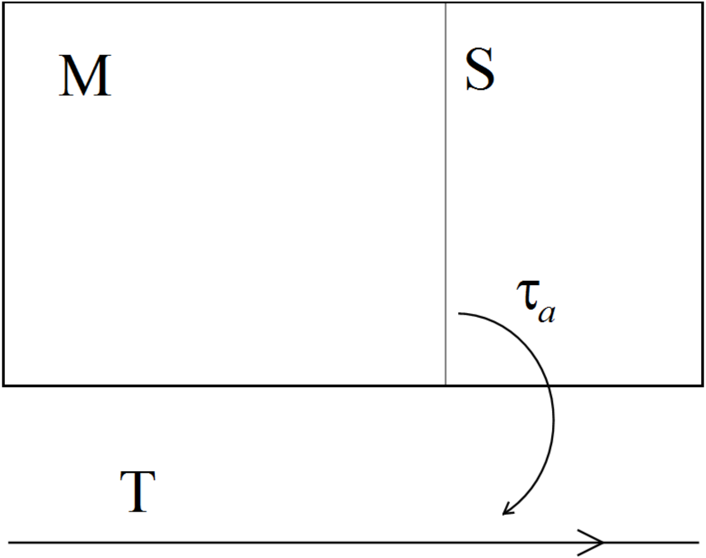
- Potentials: $\hat{\Theta} \circ \hat{\varphi}(\varphi) = 0$, where $\hat{\Theta} \circ \hat{\varphi}$ is symmetric
- Integrating multipliers: $\hat{T} \circ \hat{\Theta} = 0$, where $\hat{T} \circ \hat{\Theta}$ is symmetric
- Change the operator: $(\hat{\Theta}(\varphi))^2 = 0$, and neglect parts
- Change the function space: Gyarmati principle, ...,

All of them are right, which one is the true?

Galilean relativistic spacetime

VP (Continuum Mechanics and Thermodynamics, 20219)

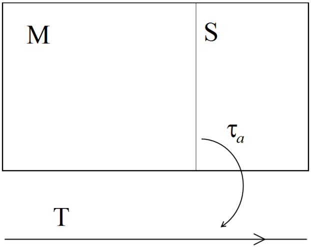
The four dimensions of Galilean relativistic space-time



Mathematical structure of Galilean relativistic space-time

1. The *space-time* $\mathbb{M}$ is an oriented four dimensional vector space of the $x^a \in \mathbb{M}$ *world points or events*. There are no Euclidean or pseudoeuclidean structures on $\mathbb{M}$: the length of a space-time vector does not exist.
2. The *time* $\mathbb{I}$ is a one dimensional oriented vector space of $t \in \mathbb{I}$ *instants*.
3. $\tau_a : \mathbb{M} \to \mathbb{I}$ is the *timing* or *time evaluation*, a linear surjection.
4. $\delta_{\bar{a}\bar{b}} : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \otimes \mathbb{R}$ Euclidean structure is a symmetric bilinear mapping, where $\mathbb{E} := Ker(\tau) \subset \mathbb{M}$ is the three dimensional vector space of *space vectors*.

- Simplification: space-time and time are affine spaces
- Simplification: measure lines.
- Abstract indexes: $a, b, c, ...$ for $\mathbb{M}$, $\bar{a}, \bar{b}, \bar{c}, ...$ for S



$$\begin{pmatrix} t' \\ x'^i \end{pmatrix} = \begin{pmatrix} t \\ x^i + v^i t \end{pmatrix}$$

Vector transformations (extensives):

$$\begin{pmatrix} A' \\ A'^i \end{pmatrix} = \begin{pmatrix} A \\ A^i + v^i A \end{pmatrix}$$

$$\boxed{A'^\alpha B'_\beta = A^\alpha B_\beta = AB + A^i B_i}$$

Covector transformations (derivatives):

$$\begin{pmatrix} B' & B'_i \end{pmatrix} = \begin{pmatrix} B - B_k v^k & B_i \end{pmatrix}$$

Balances: absolute, local and substantial

$$\boxed{\partial_a A^a = 0} \qquad \longrightarrow \qquad \begin{aligned} u^a : \quad D_u A + \partial_i A^i &= d_t A + \partial_i A^i = 0, \\ u'^a : \quad D_{u'} A + \partial_i A'^i &= \partial_t A + \partial_i A'^i = 0. \end{aligned}$$

$(a,b,c \in \{0,1,2,3\})$

Transformed: $(d_t - v^i \partial_i) A + \partial_i (A^i + A v^i) = d_t A + A \partial_i v^i + \partial_i A^i = 0$

Mass, energy and momentum

What kind of quantity is the energy?

- Square of the relative velocity: 2nd order tensor
- Kinetic theory: trace of a contravariant second order tensor.
- Energy density and flux: additional order

Basic field:

$$\boxed{Z^{abc} = z^{bc}u^a + z^{abc} : \qquad \text{mass-energy-momentum density-flux tensor}}$$

$a, b, c \in \{0,1,2,3\}, \quad \bar{a}, b, c \in \{1,2,3\}$

$$z^{bc} \to \begin{pmatrix} \rho & p^b \\ p^c & e^{bc} \end{pmatrix}, \qquad z^{\bar{a}bc} \to \begin{pmatrix} j^{\bar{a}} & P^{ab} \\ P^{ac} & q^{\bar{a}\bar{b}\bar{c}} \end{pmatrix}, \qquad e = \frac{e^b{}_b}{2}$$

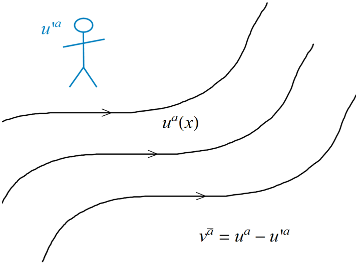
Galilean transformation

$$\boxed{Z'^{\alpha\beta\gamma} = G^{\alpha}_{\mu} G^{\beta}_{\nu} G^{\gamma}_{\kappa} Z^{\mu\nu\kappa}}$$

$$Z^{\alpha\beta\gamma} = \left(\begin{pmatrix} \rho & p^i \\ p^j & e^{ji} \end{pmatrix} \quad \begin{pmatrix} j^k & P^{ki} \\ P^{kj} & q^{kij} \end{pmatrix} \right), \quad G^{\alpha}_{\nu} = \begin{pmatrix} 1 & 0^i \\ v^j & \delta^{ji} \end{pmatrix}, \quad e = \frac{e^i{}_i}{2}$$

Transformation rules follow:

$$\begin{aligned} \rho' &= \rho, \\ p'^i &= p^i + \rho v^i, \\ e' &= e + p^i v_i + \rho \frac{v^2}{2}, \\ j'^i &= j^i + \rho v^i, \end{aligned}$$



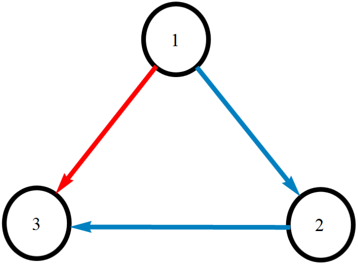
$$\begin{aligned} P'^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\ q'^i &= q^i + e v^i + P^{ij} v_j + p^j v_j v^i + (j^i + \rho v^i) \frac{v^2}{2}. \end{aligned}$$

Galiean transformation of energy

Transitivity:

$$\left.\begin{aligned} e_2 &= e_1 + p_1 v_{12} + \rho\frac{v_{12}^2}{2} \\ e_3 &= e_2 + p_2 v_{23} + \rho\frac{v_{23}^2}{2} \end{aligned}\right\} \rightarrow e_3 = e_1 + p_1 v_{13} + \rho\frac{v_{13}^2}{2}$$

$$p_2 = p_1 + \rho v_{12}, \qquad v_{13} = v_{12} + v_{23}$$



Balance transformations

Absolute

$$\partial_a Z^{abc} = \dot{z}^{bc} + z^{bc}\partial_a u^a + \partial_a z^{\bar{a}bc} = 0$$

Rest frame

$$\begin{aligned}
\dot{\rho} + \partial_i j^i &= 0, \\
\dot{p}^i + \partial_k P^{ik} &= 0^i, \\
\dot{e} + \partial_i q^i &= 0.
\end{aligned}$$

Inertial reference frame

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\
\dot{p}^i + p^i\partial_k v^k + \partial_k P^{ik} + \rho\dot{v}^i + j^k\partial_k v^i &= 0^i, \\
\dot{e} + e\partial_i v^i + \partial_i q^i + p^i\dot{v}_i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$