The QCD phase diagram and Complex Langevin simulations

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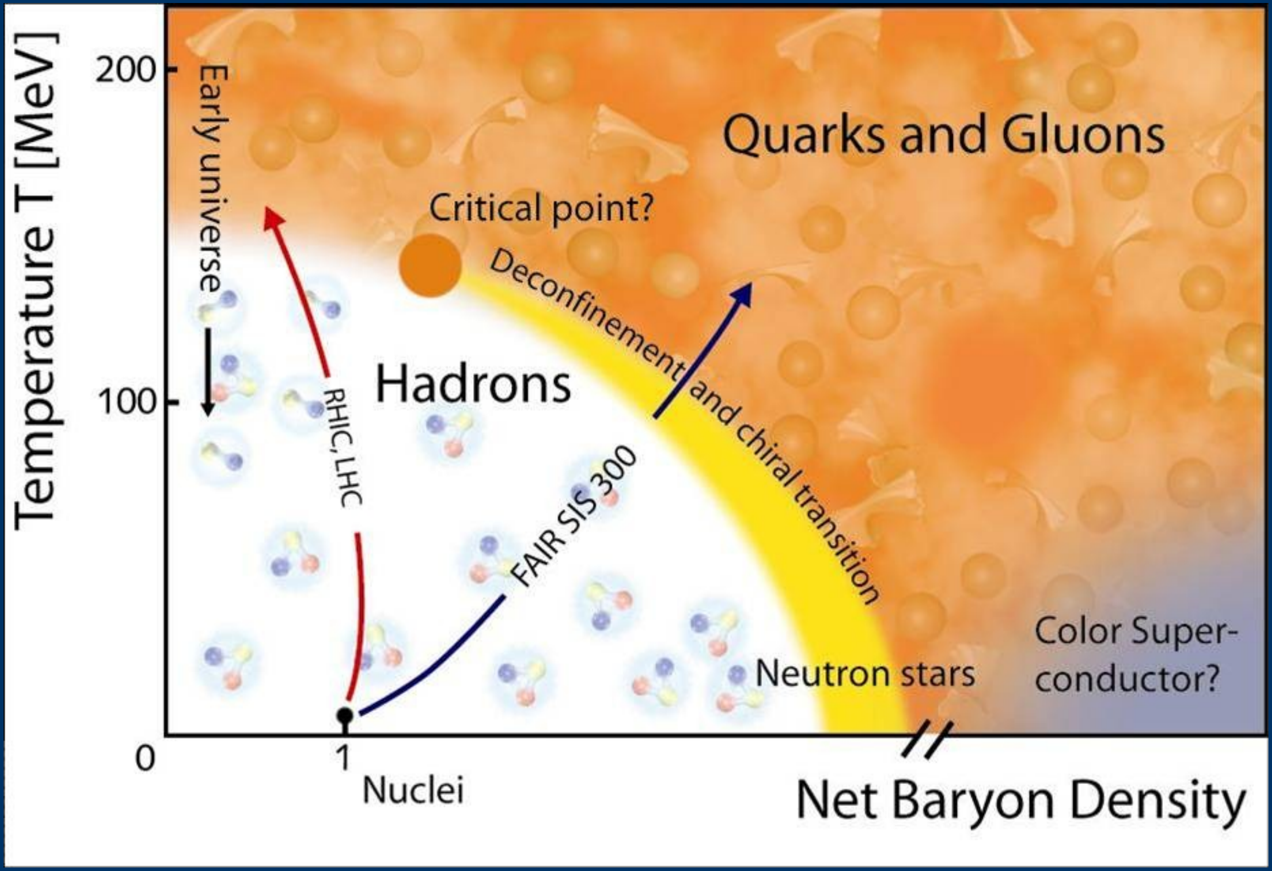
1. Introduction to Complex Langevin
2. Boundary terms; test in full QCD; correction
3. Kernels in the CLE – How to get rid of boundary terms
4. Results for full QCD: EoS and phase diagram, dynamic stabilization

Phase diagram of QCD

Zero density axis well known

transition temperature

zero temperature:
hadron masses
scattering amplitudes, etc.



At nonzero density much less solid knowledge

What phases are present?
Is there a critical point?
compressibility of nuclear matter?

Why is non-zero density so hard?

QCD sign problem

Euclidean SU(3) gauge theory with fermions:

$$Z = \int DU \exp(-S_E[U]) det(M[U])$$

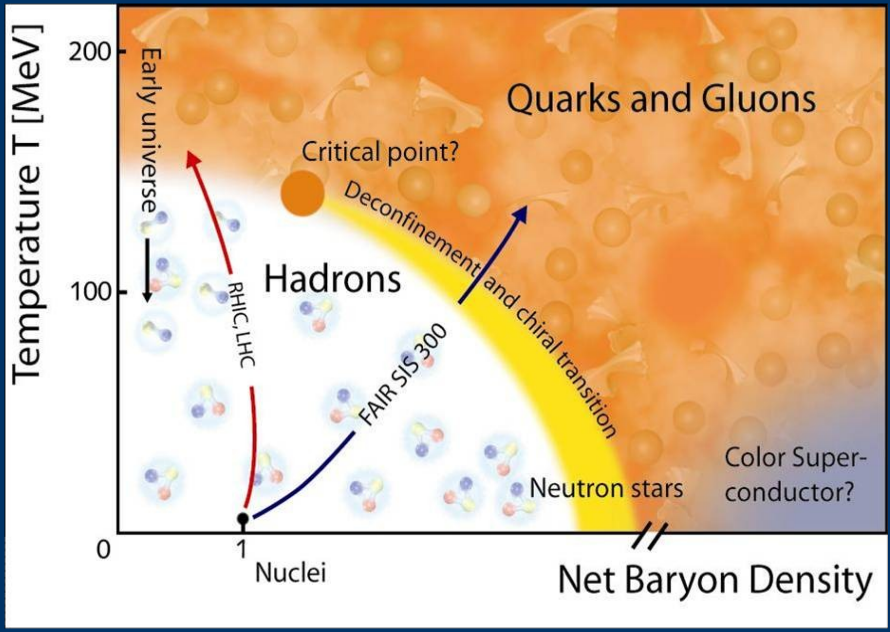
for $\det(M[U]) > 0$ Importance sampling is possible

Non-zero chemical potential

For nonzero chemical potential, the fermion determinant is complex

$$\det(M(U, -\mu^*)) = (\det(M(U), \mu))^*$$

Sign problem $\longrightarrow$ Naive Monte-Carlo breaks down



How to solve the sign problem (of QCD)?

Extrapolation from a positive ensemble

Reweighting $$\langle X\rangle_W = \frac{\sum_c W_c X_c}{\sum_c W_c} = \frac{\sum_c W'_c (W_c / W'_c) X_c}{\sum_c W'_c (W_c / W'_c)} = \frac{\langle (W/W') X\rangle_{W'}}{\langle W/W'\rangle_{W'}}$$

Taylor expansion $$Z(\mu) = Z(\mu=0) + \frac{1}{2}\mu^2 \partial_\mu^2 Z(\mu=0) + \ldots$$

Analytic continuation from imaginary sources
(chemical potentials, theta angle,..)

Using analyticity

Complex Langevin
Complexified variables – enlarged manifolds

Other ideas (not yet for QCD)

Lefschetz thimbles , sign improved manifolds,
Dual variables, worldlines,
Density of States, Subsets, ...

In QCD direct simulation only possible at $\mu = 0$

Taylor extrapolation, Reweighting, continuation from imaginary μ, canonical ens.
all break down around

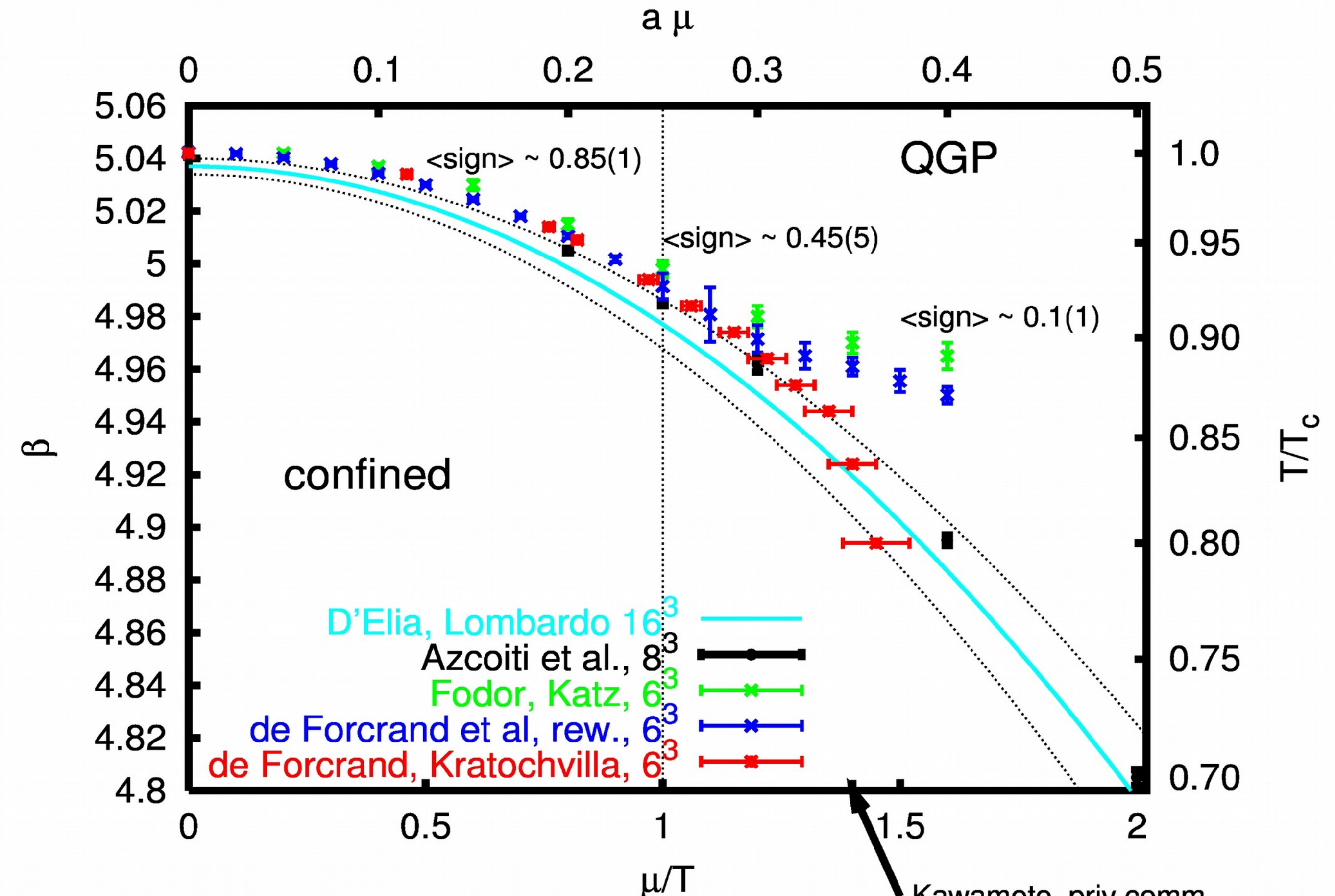
$$\frac{\mu_q}{T} \approx 1 \qquad \frac{\mu_B}{T} \approx 3$$

Around the transition temperature
Breakdown at $\mu_q \approx 150-200\,\mathrm{MeV}$ $\mu_B \approx 450-600\,\mathrm{MeV}$

Results on

$N_T = 4, N_F = 4, ma = 0.05$

using
Imaginary mu,
Reweighting,
Canonical ensemble



Agreement only at $\mu/T < 1$

Langevin Equation (aka. stochatic quantisation)

Given an action $S(x)$

Stochastic process for x: $$\frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta(\tau)$$

Gaussian noise
$$\langle \eta(\tau) \rangle = 0$$
$$\langle \eta(\tau)\eta(\tau') \rangle = \delta(\tau - \tau')$$

Random walk in configuration space

Averages are calculated along the trajectories:

$$\langle O \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T O(x(\tau))\, d\tau = \frac{\int e^{-S(x)} O(x)\, dx}{\int e^{-S(x)}\, dx}$$

Numerically,
results are extrapolated to $\Delta\tau \to 0$

Complex Langevin Equation

Given an action $S(x)$

Stochastic process for x: $$\frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta(\tau)$$

Gaussian noise

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$$\frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta(\tau)$$

The field is complexified

real scalar $\longrightarrow$ complex scalar

link variables: SU(N) $\longrightarrow$ SL(N,C)

compact non-compact

$$det(U) = 1, \quad U^{+} \neq U^{-1}$$

Analytically continued observables

$$\frac{1}{Z} \int P_{comp}(x) O(x)\, dx = \frac{1}{Z} \int P_{real}(x, y) O(x + iy)\, dx\, dy$$

$$\langle x^2 \rangle_{real} \rightarrow \langle x^2 - y^2 \rangle_{complexified}$$

Gaussian Example

$$S[x]=\sigma x^2+i\lambda x$$

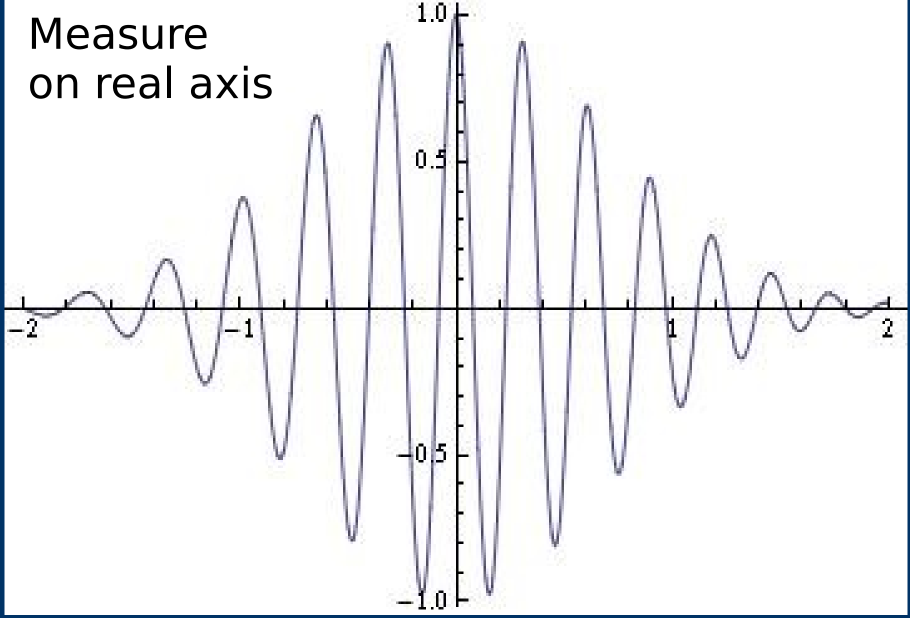
CLE

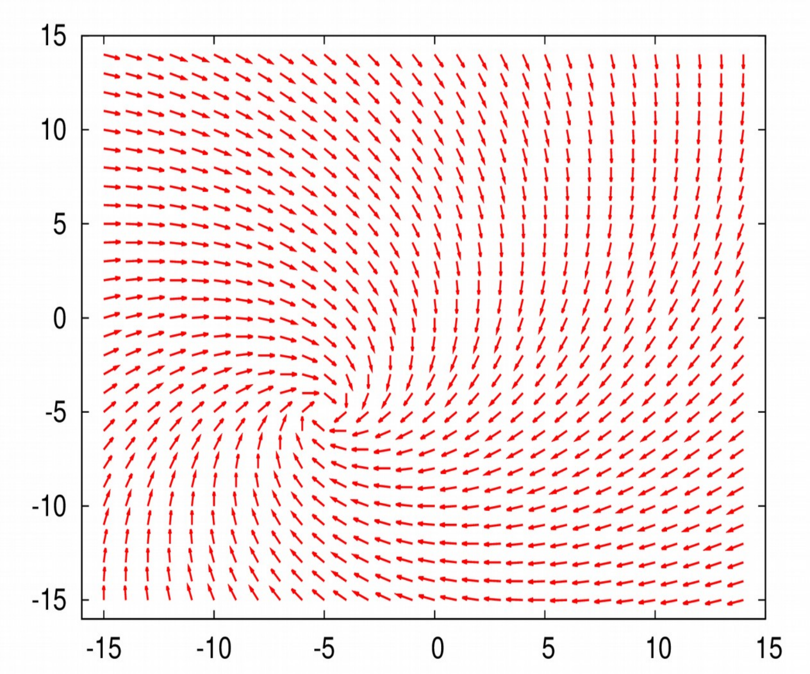
$$\frac{d}{d\tau}(x+iy)=-2\sigma(x+iy)-i\lambda+\eta$$

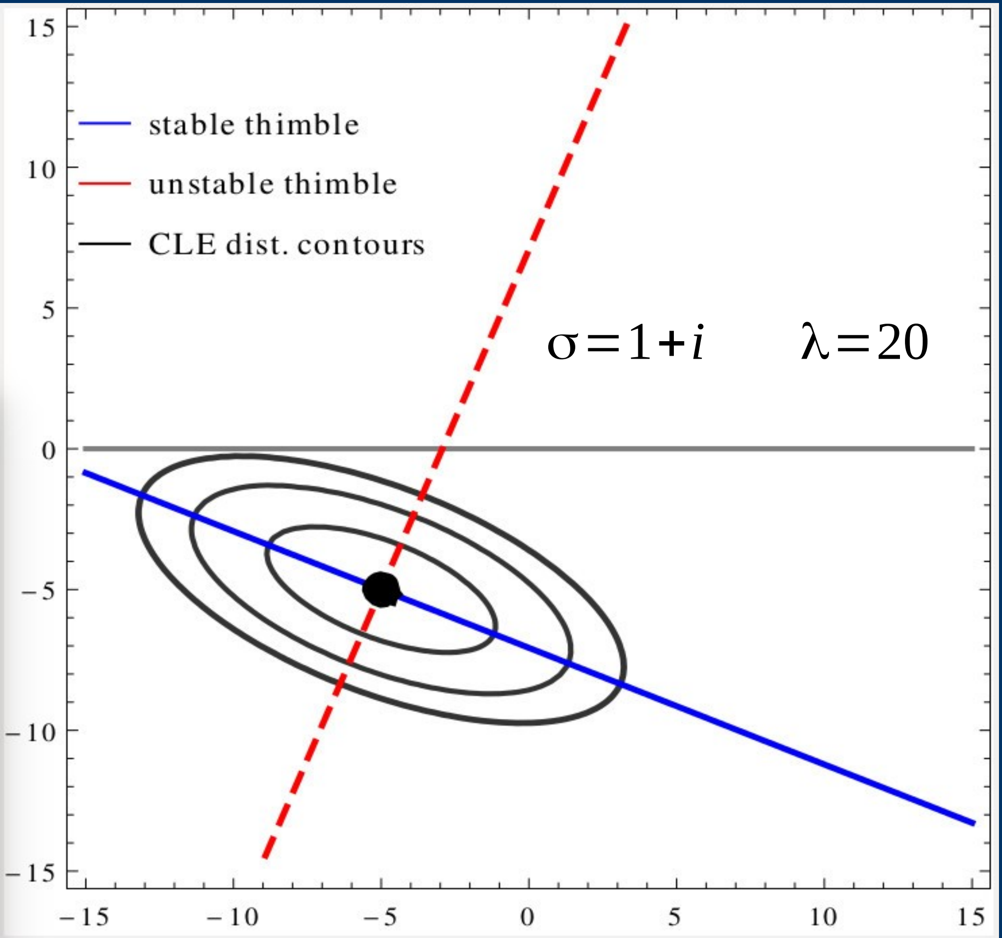
$$P(x,y)=e^{-a(x-x_0)^2-b(y-y_0)^2-c(x-x_0)(y-y_0)}$$

Gaussian distribution
around critical point

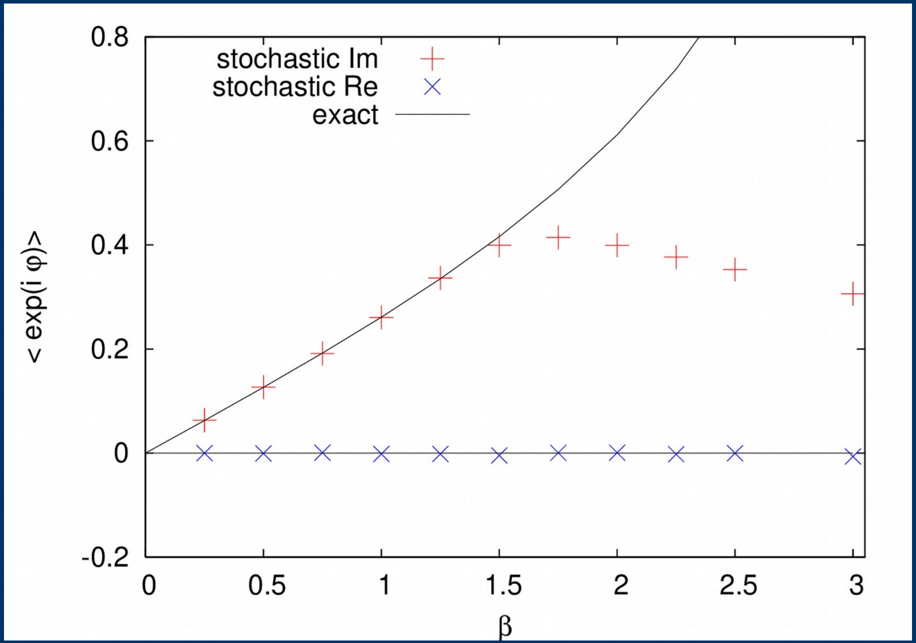
$$\left.\frac{\partial S(z)}{\partial z}\right|_{z_0}=0$$







“troubled past”: Convergence to wrong results
Lack of theoretical understanding
Runaway trajectories



$$S(\varphi)=i\beta\cos\varphi+i\varphi$$

Correct in one parameter region
Incorrect in an other

Convergent in both

Klauder '83, Parisi '83, Hueffel, Rumpf '83, Karsch. Wyld '84, Gausterer, Klauder '86. Matsui, Nakamura '86, ...
Interest went down as difficulties appeared
Renewed interest in connection of otherwise unsolvable problems
applied to nonequilibrium: Berges, Stamatescu '05, ...
aimed at nonzero density QCD: Aarts, Stamatescu '08 ... many important results since revival

Argument for correctness of CLE results

If there is fast decay $P(x, y) \to 0$ as $x, y \to \infty$

and a holomorphic action $S(x)$

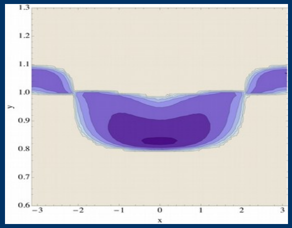
then CLE converges to the correct result

[Aarts, Seiler, Stamatescu (2009)
Aarts, James, Seiler, Stamatescu (2011)]

Loophole 1: Non-holomorphic action for nonzero density

$$S = S_W[U_\mu] + \ln \mathrm{Det}\, M(\mu)$$

measure has zeros $(Det\, M = 0)$
complex logarithm has a branch cut
→ meromorphic drift



No problems if poles are not 'touched' by distribution

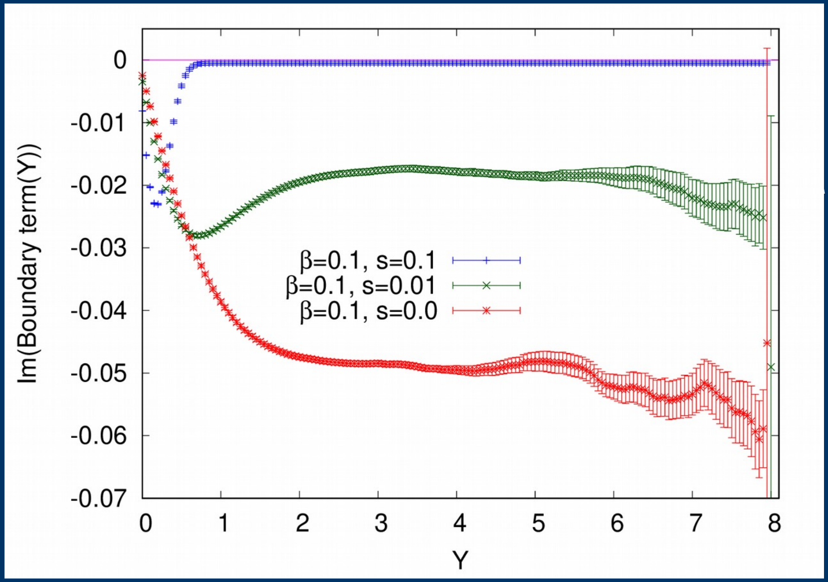
satisfied for: HDQCD, full QCD at high temperatures

[Aarts, Seiler, Sexty, Stamatescu '17]

Loophole 2: decay not fast enough

boundary terms can be nonzero
explicit calculation of boundary terms possible

[Scherzer, Seiler, Sexty, Stamatescu (2018)+(2019)]



Unambigous detection of boundary terms

given by plateau as 'cutoff' $Y \to \infty$

Observable cheap also for lattice systems

Measuring "corrected observable"

in case boundary term nonzero

Using an interpolation function $F(t,\tau)$

CLE works, if

What we want		What we get with CLE
$\int dx\, \rho(x) O(x)$	$=$	$\int dx\, dy\, P(x,y) O(x+iy)$
$F(t,t)$		$F(t,0)$

Boundary terms as a volume integral

[Scherzer, Seiler, Sexty, Stamatescu (2018+2019)]

Calculating an observable defined on a compact boundary in many dimensions can be inconvenient

$$\partial_\tau F_O(Y,t,\tau=0) = B_O(Y,t,\tau=0) =$$

$$\int_{-Y}^{Y} P(x,y,t)\, L_c O(x+iy) - \int_{-Y}^{Y} (L^T P)\, O(x+iy,0)$$

Observable with a cutoff
easy to do in many dimensions

Vanishes as process equilibrates

$L_c O(x+iy)$ consistency conditions $\approx$ Schwinger-Dyson eqs.

Order of limits crucial

$$\lim_{t\to\infty} \lim_{Y\to\infty} \int_{-Y}^{Y} P(x,y,t)\, L_c O(x+iy) \text{ can be undefined}$$

Measuring boundary terms

$$\int_{-Y}^{Y} P(x,y,t) L_c O(x+iy) = \int P(x,y,t) L_c O(x+iy) \theta(Y-y)$$

$$L_c = \sum \partial_i^2 + K_i \partial_i$$

Many variables: define cutoff to extend SU(N) manifold
to compact submanifold of SL(N,C)

$$\text{e.g.} \quad \mathrm{Im}\, z\,; \quad \max_i Tr\left(U_i^{+} U_i - 1\right)^2$$

Measure “unitarity norm” and observable

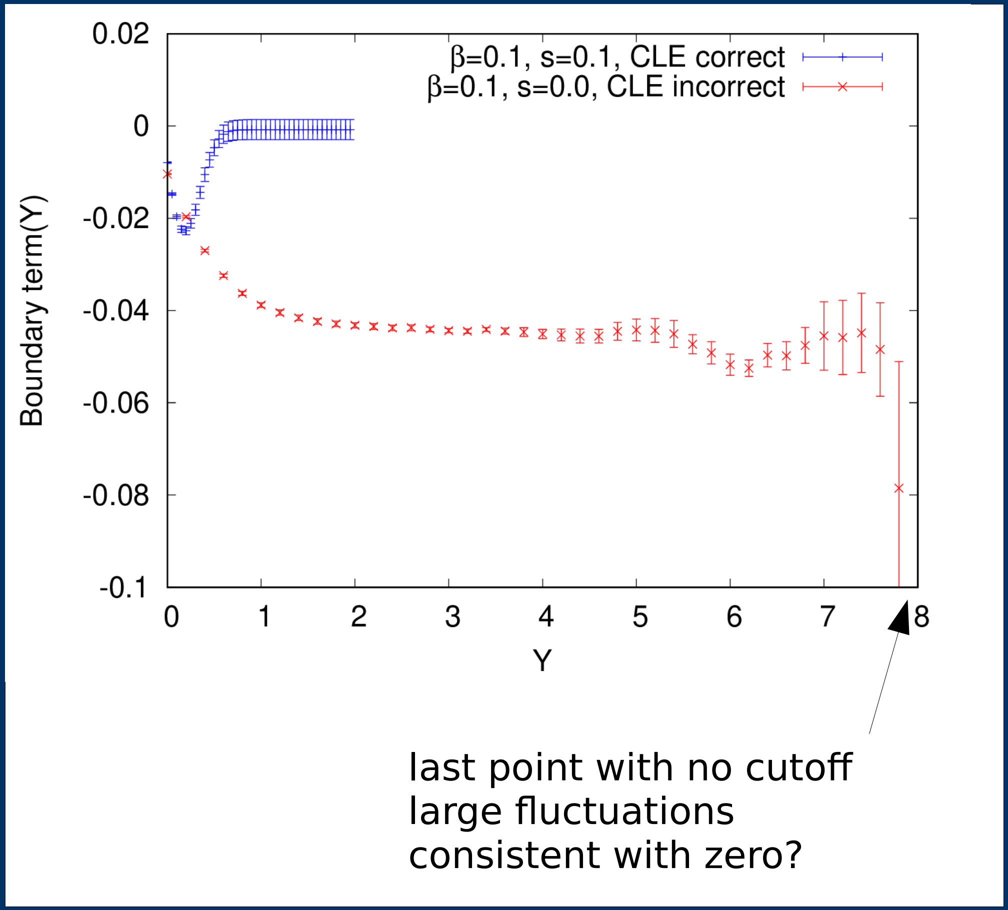
Analyze for any cutoff

Trick for second term:

$$\sum K_i \partial_i O = \frac{1}{\epsilon}\left[O(z(\tau+\epsilon, \eta=0)) - O(z(\tau))\right]$$

Measure observable after doing a noiseless update step with stepsize ϵ

One plaquette model with regulator $S(x) = i\beta\cos(x) + \frac{s}{2}x^2$



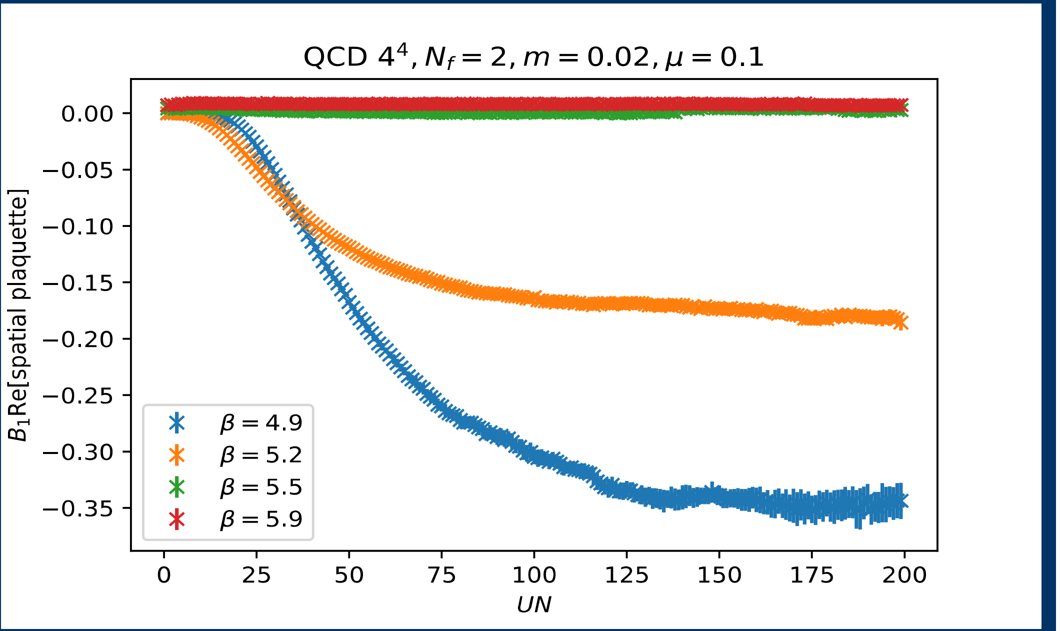
Unambigous detection of boundary terms

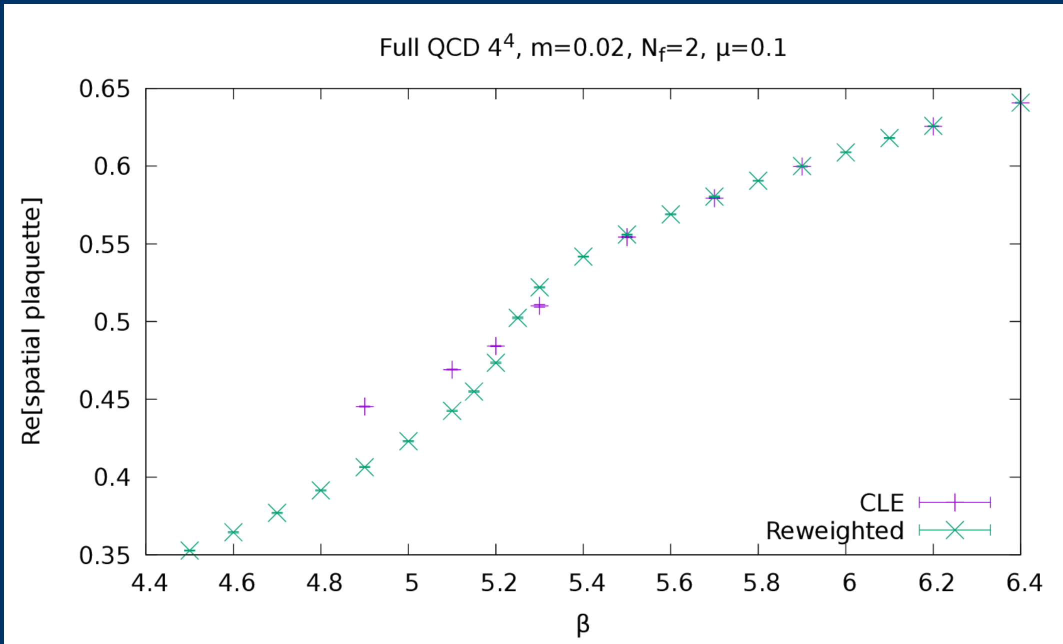
Observable cheap also for lattice systems

In full QCD this confirms already known signals
Quantifies error

Faster than exponential decay of histograms of observables
Drift criterion = same for drift term observable

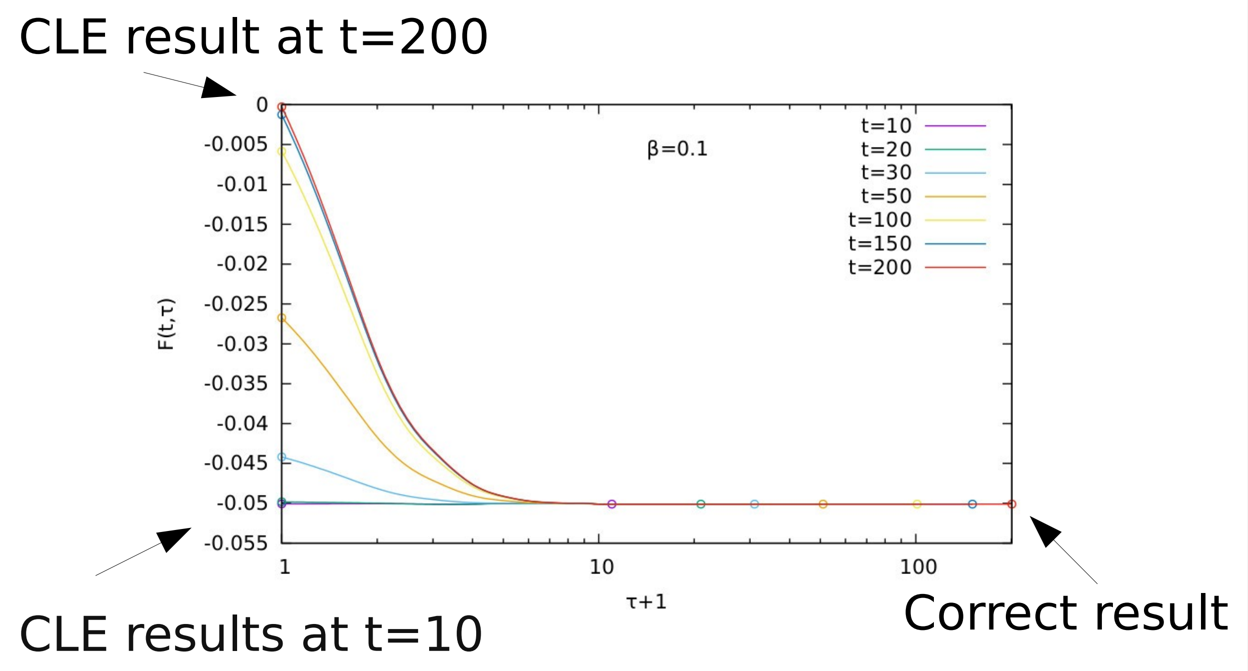
Boundary terms appear at small β = large lattice spacing





[Hansen, Sexty in. prep]

Correcting CLE using boundary terms



Interpolation function

$$F(t,\tau)=\sum A_n \exp(-\omega_n \tau)$$

Higher order boundary terms

$$\frac{\partial^n F(t,\tau)}{\partial \tau^n}=B_n=\langle L_c^n O\rangle$$

Ansatz

$$F(t,\tau)=A_0+A_1 \exp(-\omega_1 \tau)$$

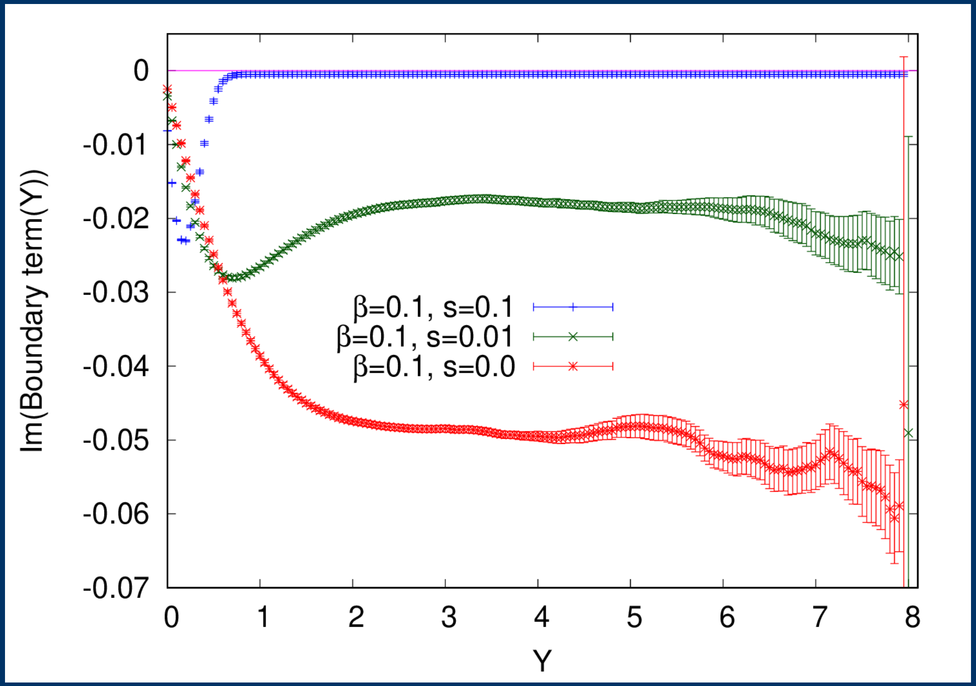
Systematic error of CLE

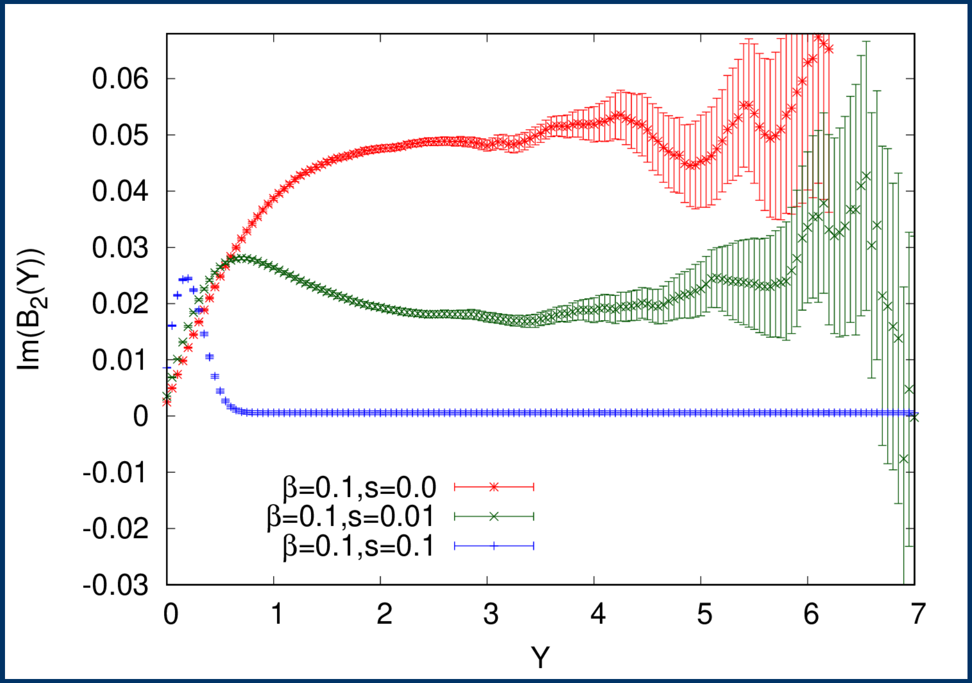
$$F(t,0)-F(t,t)=B_1^2/B_2$$

Correction using Boundary terms in U(1) toy model

$$S(x) = i\beta \cos(x) + \frac{s}{2} x^2$$

Measuring B_1, B_2 allows correction of results when CLE fails





β, s	B_1	B_2	B_1^2/B_2	CL error	CL	correct	corrected CL
0.1, 0	-0.04859(45)	0.0493(11)	0.04786(79)	0.04891(45)	-0.00115(45)	-0.05006	-0.04901(62)
0.1, 0.01	-0.01795(49)	0.01801(80)	0.01789(60)	0.01689(50)	-0.03318(50)	-0.05006	-0.05106(40)
0.1, 0.1	-0.00048(30)	0.00057(35)	0.00039(28)	0.00049(31)	-0.04957(31)	-0.05006	-0.04997(6)
0.5, 0	-0.2474(11)	0.237(11)	0.258(11)	0.25818(23)	0.00003(23)	−0.25815	-0.258(11)
0.5, 0.3	−0.05309(86)	0.0552(51)	0.0507(41)	0.04183(70)	-0.19658(70)	−0.23841	−0.2473(37)

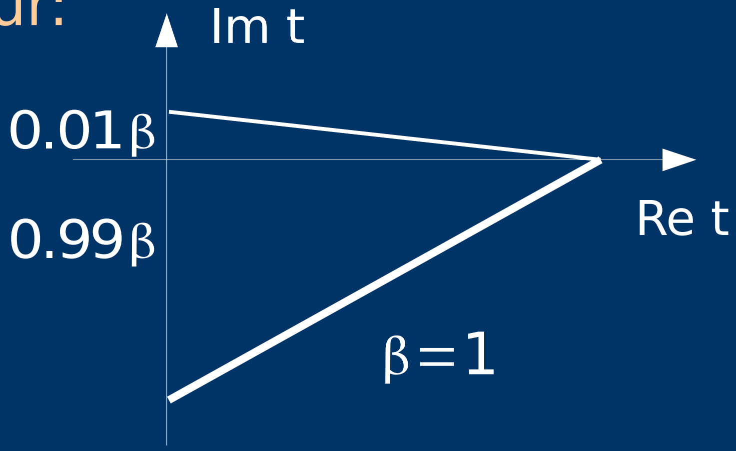
Real-time two point function of quantum oscillator

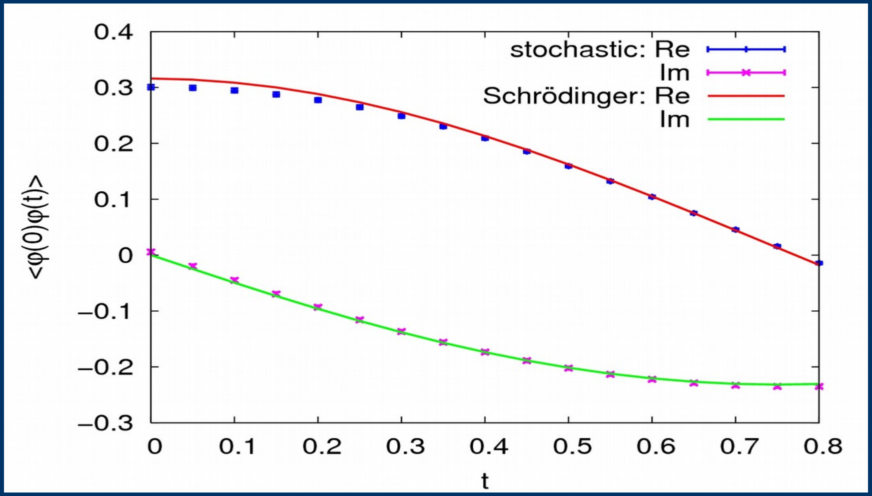
Thermal equilibrium:
periodic boundary cond.

Imaginary extent gives $\beta = \frac{1}{T}$

short real-time extent

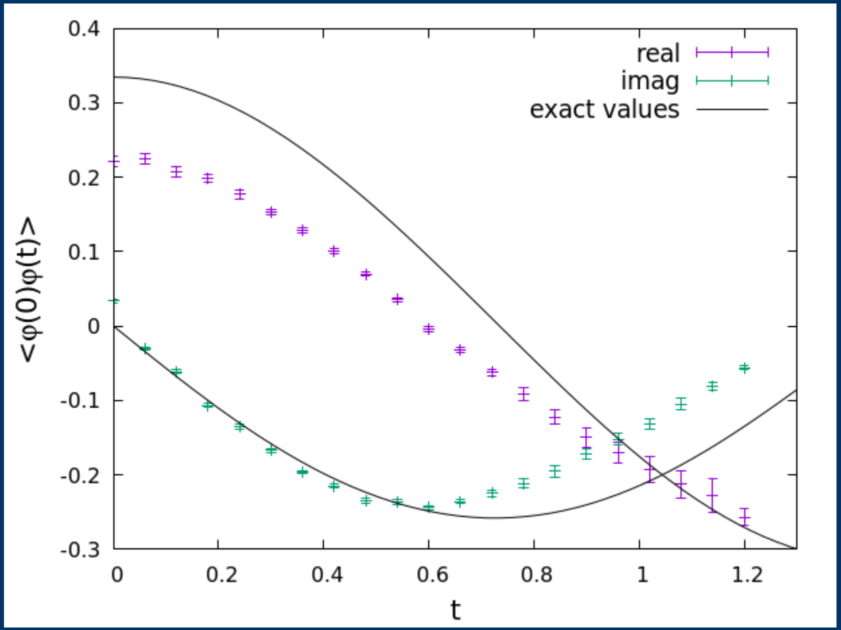
Asymmetric contour:





large real-time extent

Boundary terms appear



Kernels in the Langevin equation

Introducing a Kernel

$$\dot{z} = -\frac{\partial S}{\partial z} + \eta \quad \rightarrow \quad \dot{z} = -K(z)\frac{\partial S}{\partial z} + \frac{\partial K(z)}{\partial z} + \sqrt{K(z)}\,\eta$$

Leaves the stationary distribution unchanged

Many variables – matrix Kernel

$$\frac{d\phi_i}{d\tau} = -H_{ij}(\phi) H^T_{jk}(\phi) \frac{\partial S}{\partial \phi_k} + \partial_k \left(H_{ij}(\phi) H^T_{jk}(\phi) \right) + H_{ij}(\phi)\,\eta_j$$

Can one use a Kernel to decrease boundary terms in the CLE?

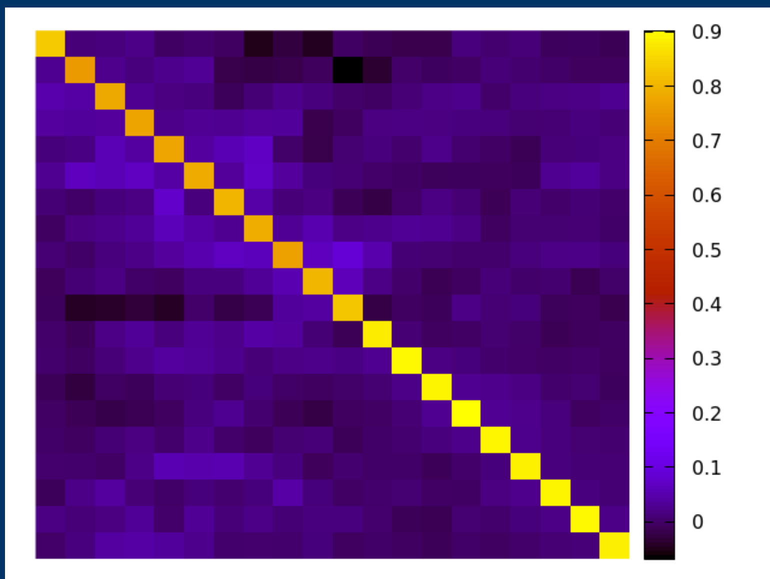
Yes! search for a kernel using stochastic gradient descent
Loss function: Size of the distribution in imaginary directions

[Lampl, Sexty (in prep.)]

First step: Field independent matrix kernel

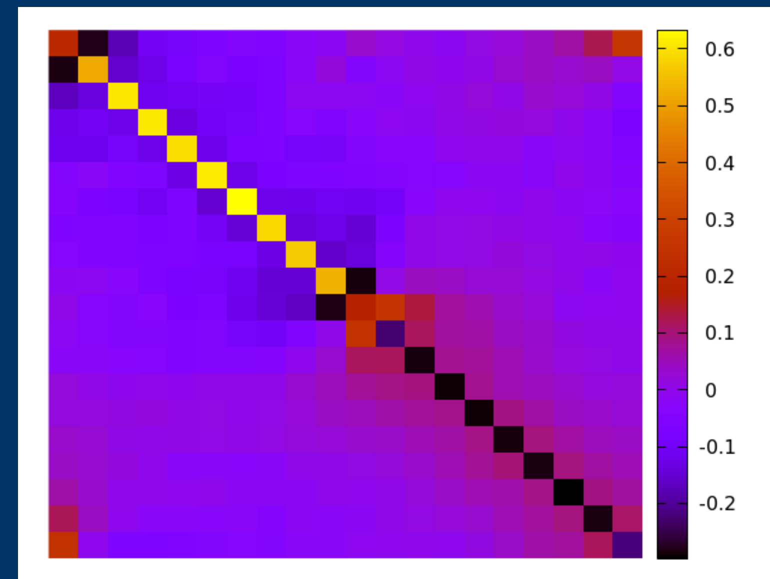
real part

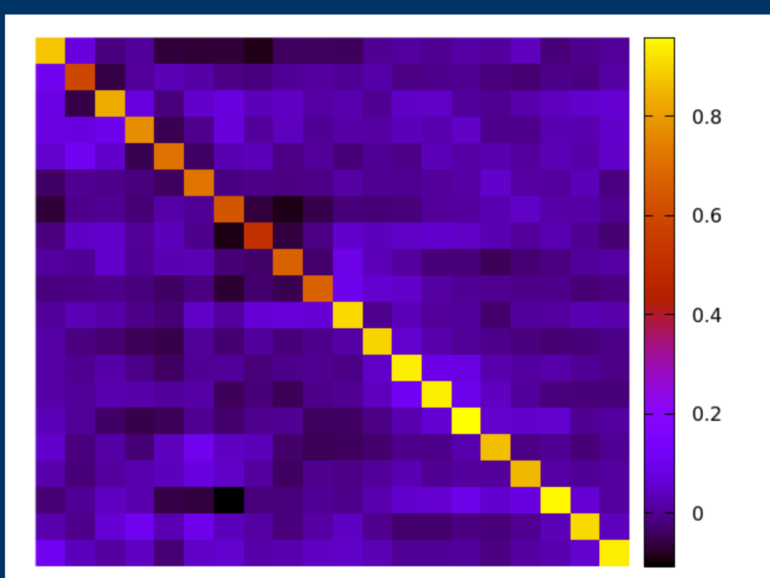
imaginary part



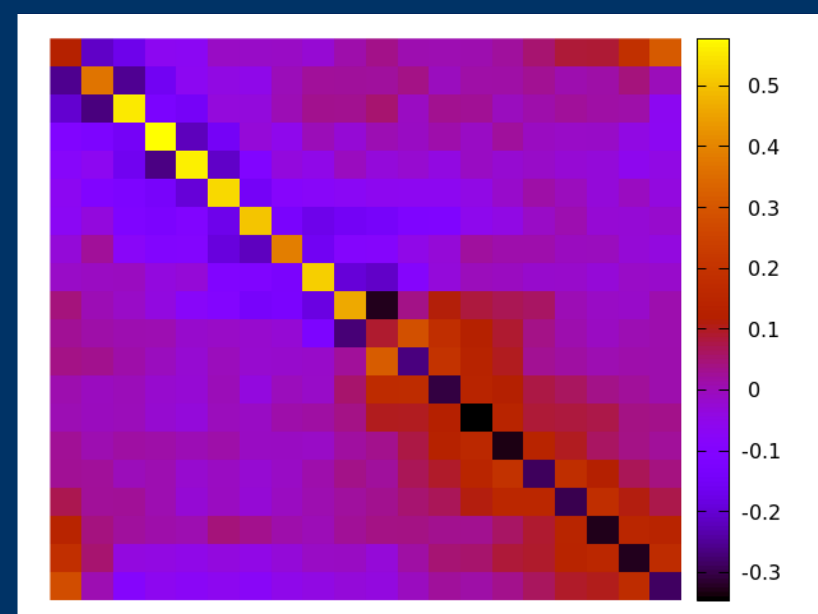
Real-time extent

$t = 1.2$



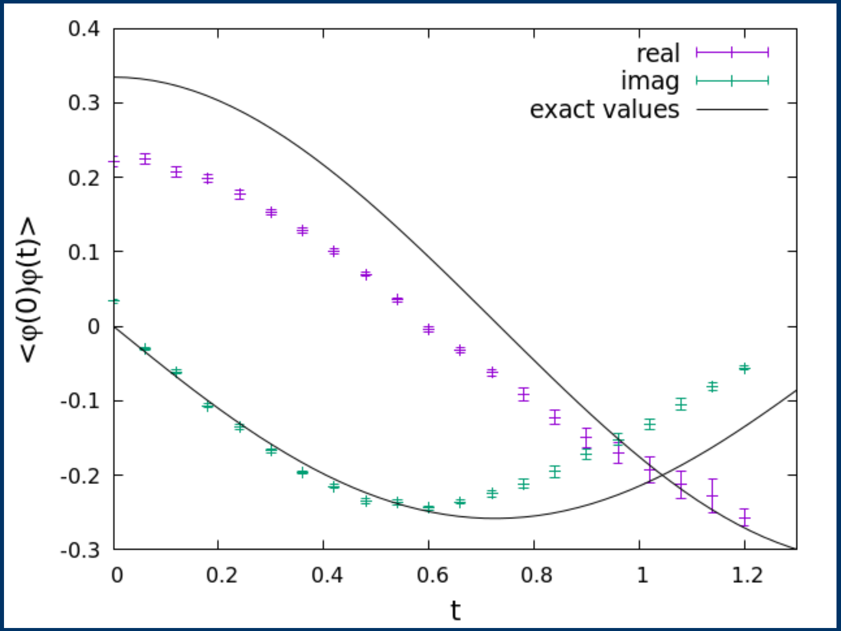


$t = 2.0$

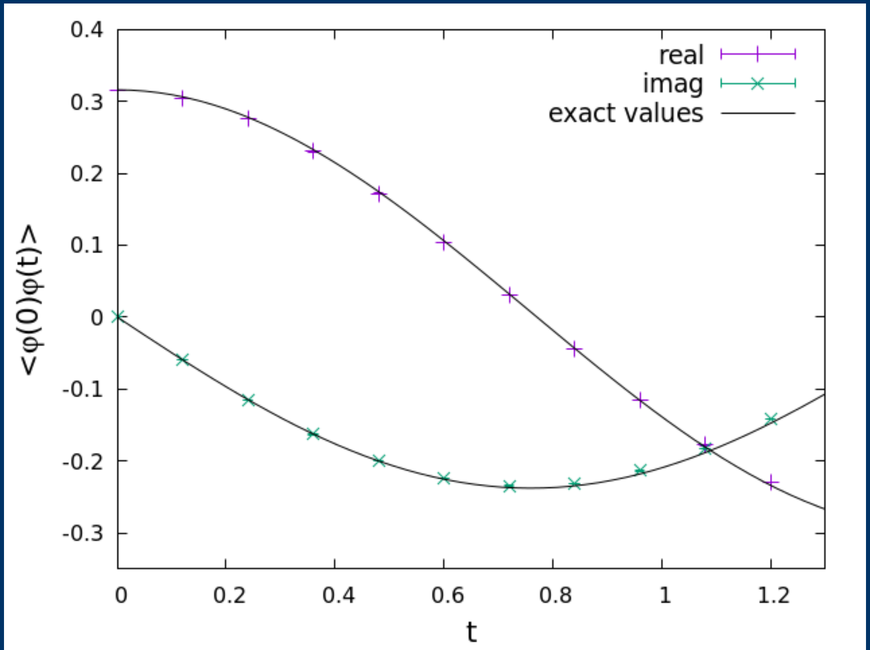


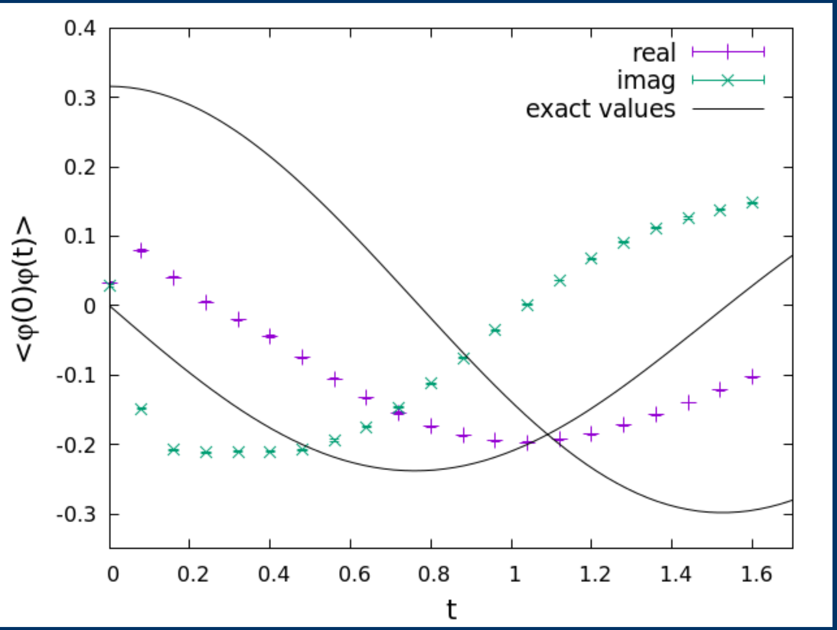
Without kernel

With learned kernel

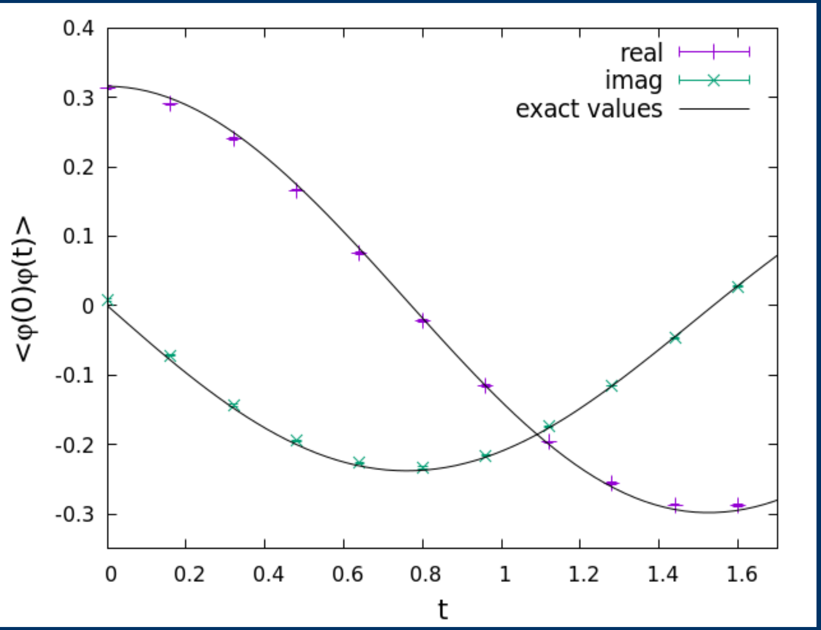


$t = 1.2$





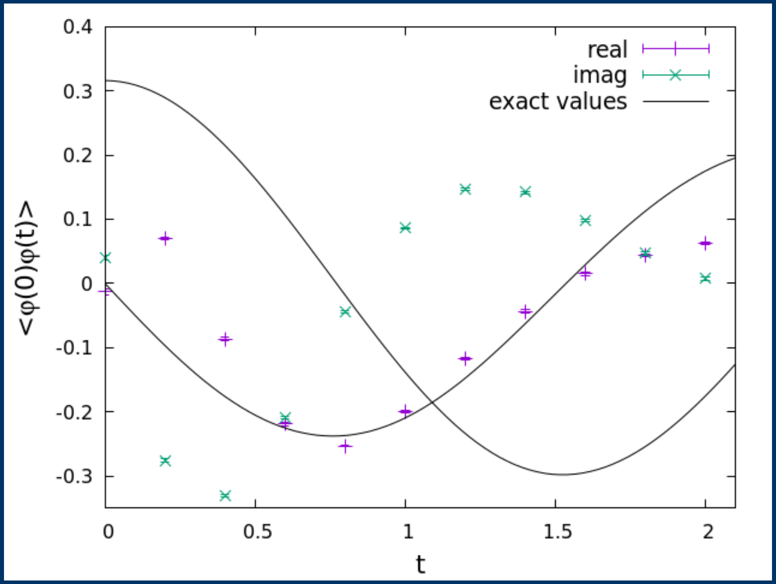
$t = 1.6$



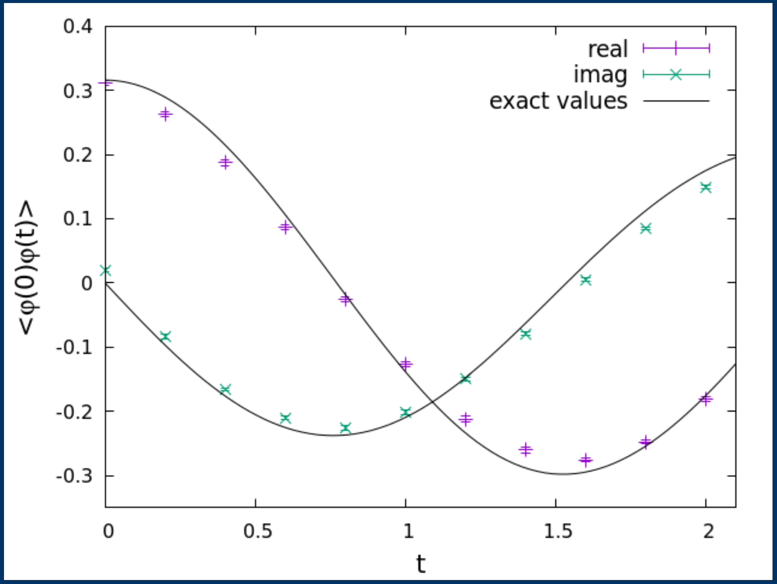
Increasing real time extent, boundary terms appear again

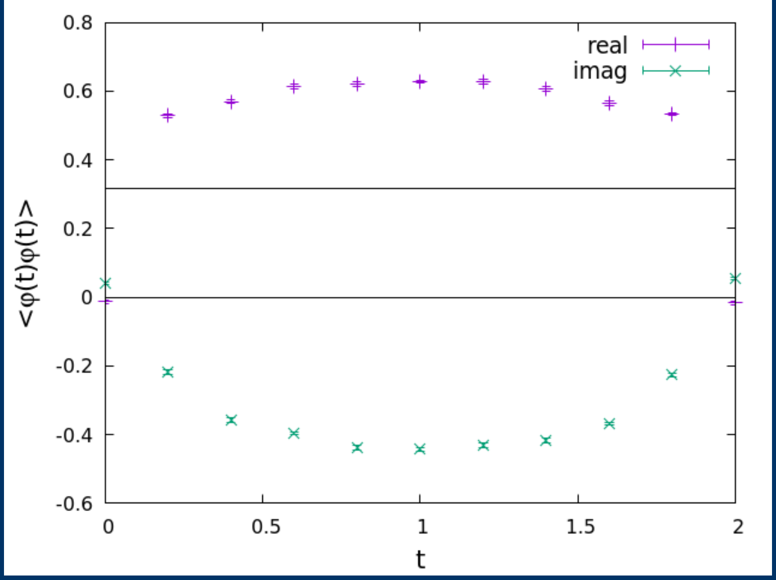
Without kernel

With learned kernel

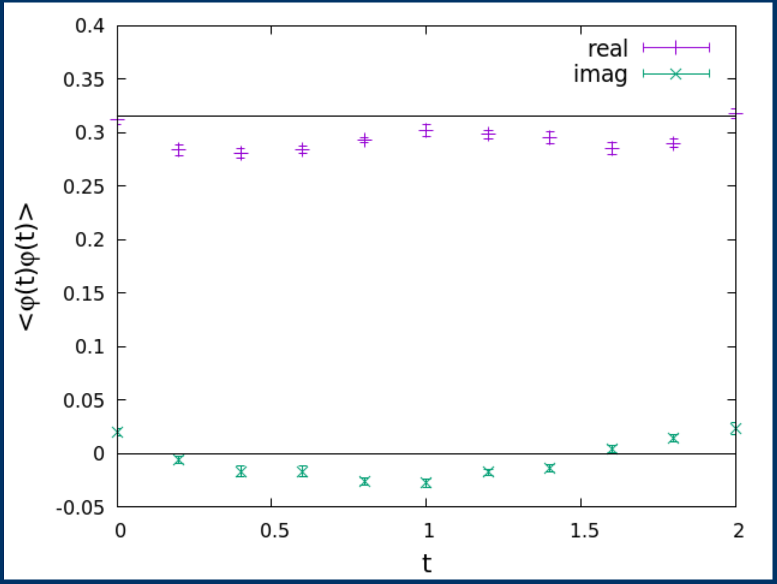


$t = 2.0$





$t = 2.0$



Pressure of the QCD Plasma at non-zero density

$$\frac{p}{T^4} = \frac{\ln Z}{V\,T^3}$$ → Derivatives of the pressure are directly measureable
→ Integrate from T=0

[Engels et. al. (1990)]

Other strategies:

Measure the Stress-momentum tensor using gradient flow

[Suzuki, Makino (2013-)]

Shifted boundary conditions

[Giusti, Pepe, Meyer (2011-)]

Non-equilibrium quench

[Caselle, Nada, Panero (2018)]

First integrate along the temperature axis, then explore $\mu > 0$

Taylor expansion [Bielefeld-Swansea (2002-)]

Simulating at imaginary μ to calculate susceptibilities

[de Forcrand, Philipsen (2002-)]

Pressure of the QCD Plasma at non-zero density

$$\Delta\left(\frac{p}{T^4}\right)=\frac{p}{T^4}(\mu=\mu_q)-\frac{p}{T^4}(\mu=0)$$

If we want to stay at $\mu=0$

$$\Delta\left(\frac{p}{T^4}\right)=\sum_{n>0,even} c_n(T)\left(\frac{\mu}{T}\right)^n$$

$$c_2=\frac{1}{2}\frac{N_T}{N_s^3}\frac{\partial^2 \ln Z}{\partial\mu^2}$$

$$c_4=\frac{1}{24}\frac{1}{N_s^3 N_T}\frac{\partial^4 \ln Z}{\partial\mu^4}$$

Measuring the coefficients of the Taylor expansion

$$\frac{\partial^2 \ln Z}{\partial\mu^2}=N_F^2\langle T_1^2\rangle+N_F\langle T_2\rangle$$

$$\frac{\partial^4 \ln Z}{\partial\mu^4}=-3\left(\langle T_2\rangle+\langle T_1^2\rangle\right)^2+3\langle T_2^2\rangle+\langle T_4\rangle$$

$$+\langle T_1^4\rangle+4\langle T_3 T_1\rangle+6\langle T_1^2 T_2\rangle$$

$$T_1/N_F=\mathrm{Tr}(M^{-1}\partial_\mu M)$$

$$T_{i+1}=\partial_\mu T_i$$

$$T_2/N_F=\mathrm{Tr}(M^{-1}\partial_\mu^2 M)-\mathrm{Tr}\left((M^{-1}\partial_\mu M)^2\right)$$

$$T_3/N_F=\mathrm{Tr}(M^{-1}\partial_\mu^3 M)-3\,\mathrm{Tr}(M^{-1}\partial_\mu M\, M^{-1}\partial_\mu^2 M)$$
$$+2\,\mathrm{Tr}\left((M^{-1}\partial_\mu M)^3\right)$$

$$T_4/N_F=\mathrm{Tr}(M^{-1}\partial_\mu^2 M)-4\,\mathrm{Tr}(M^{-1}\partial_\mu M\, M^{-1}\partial_\mu^3 M)$$
$$-3\,\mathrm{Tr}(M^{-1}\partial_\mu^2 M\, M^{-1}\partial_\mu^2 M)-6\,\mathrm{Tr}\left((M^{-1}\partial\mu\, M)^4\right)$$
$$+12\,\mathrm{Tr}\left((M^{-1}\partial_\mu M)^2 M^{-1}\partial_\mu^2 M\right)$$

Pressure of the QCD Plasma using CLE

[Sexty (2019)]

If we can simulate at $\mu > 0$

$$\Delta\left(\frac{p}{T^4}\right) = \frac{p}{T^4}(\mu = \mu_q) - \frac{p}{T^4}(\mu = 0) = \frac{1}{VT^3}\left(\ln Z(\mu) - \ln Z(0)\right)$$

$$\ln Z(\mu) - \ln Z(0) = \int_0^\mu d\mu \frac{\partial \ln Z(\mu)}{\partial \mu} = \int_0^\mu d\mu\, n(\mu)$$

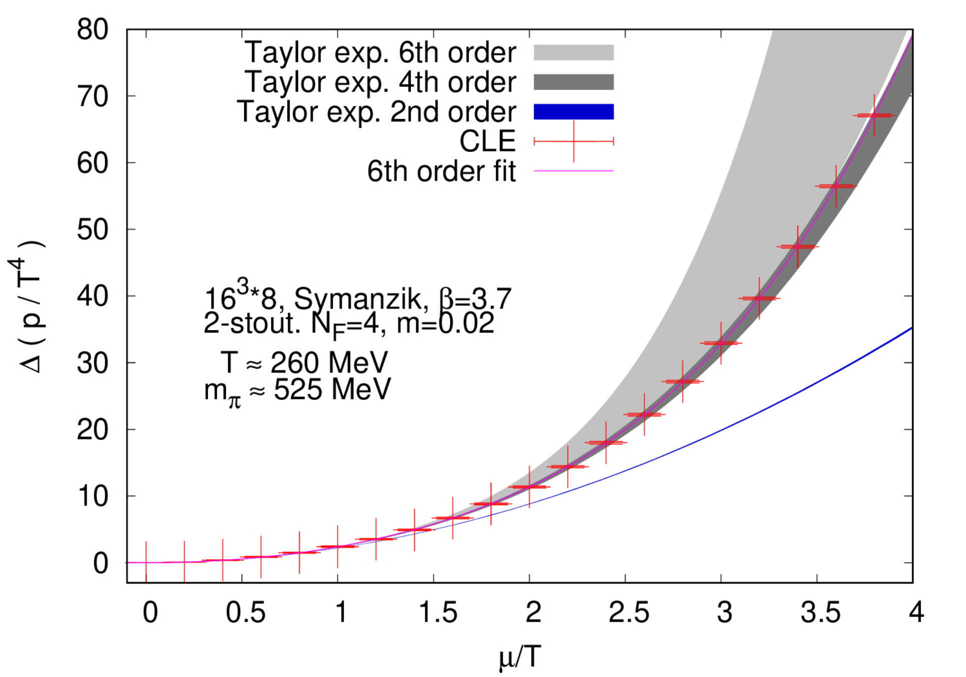
$$n(\mu) = \langle \mathrm{Tr}(M^{-1}(\mu)\partial_\mu M(\mu))\rangle$$

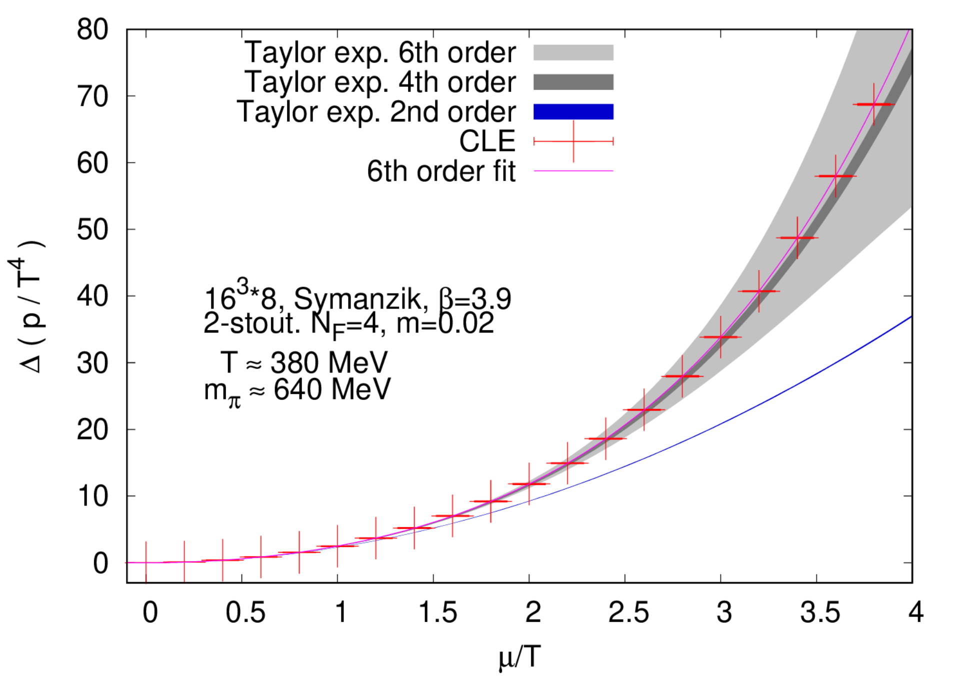
Using CLE it’s enough to measure the density – much cheaper

Pressure with improved action

[Sexty (2019)]

In deconfined phase
Symanzik gauge action
stout smeared staggered fermions





β	c_2 Taylor exp.	c_4 Taylor exp.	c_6 Taylor exp.	c_2 CLE	c_4 CLE	c_6 CLE
3.7	2.206 ± 0.009	0.156 ± 0.016	0.016 ± 0.013	2.33 ± 0.1	0.13 ± 0.02	0.002 ± 0.001
3.9	2.312 ± 0.007	0.150 ± 0.007	0.001 ± 0.005	2.36 ± 0.04	0.14 ± 0.01	0.002 ± 0.001

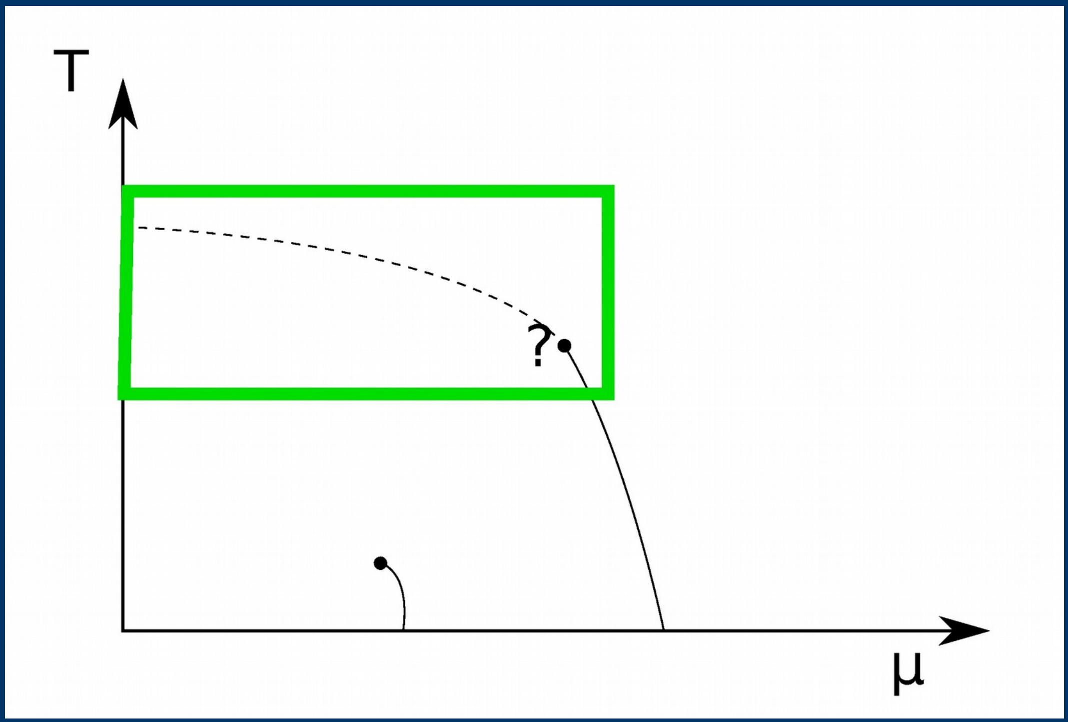
Good agreement at small μ
CLE calculation is much cheaper

further interesting quantities: Energy density, quark number susceptibility, ...

See also Benjamin Jägers's talk using CLE and Dyn. Stab. (after lunch today)

Mapping out the phase transition line

[Scherzer, Sexty, Stamatescu (2020)]



Follow the phase transition line starting from $\mu = 0$

Using Wilson fermions

Fixed lattice spacing and spatial vol.
N_t scan

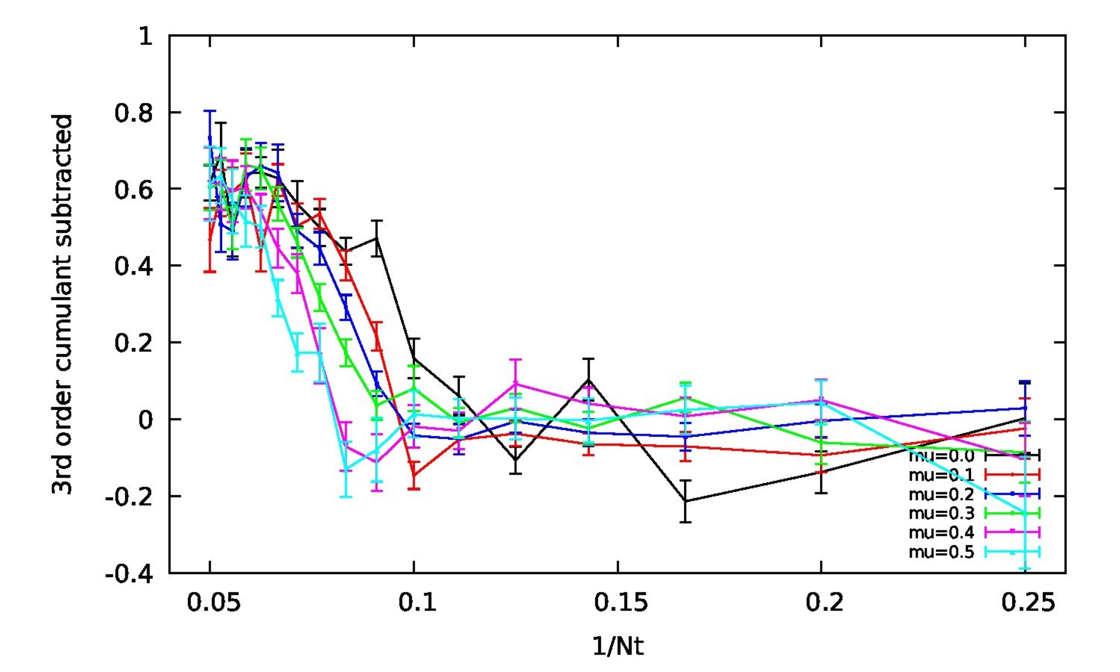
Detection of the phase transition line

Binder cumulant

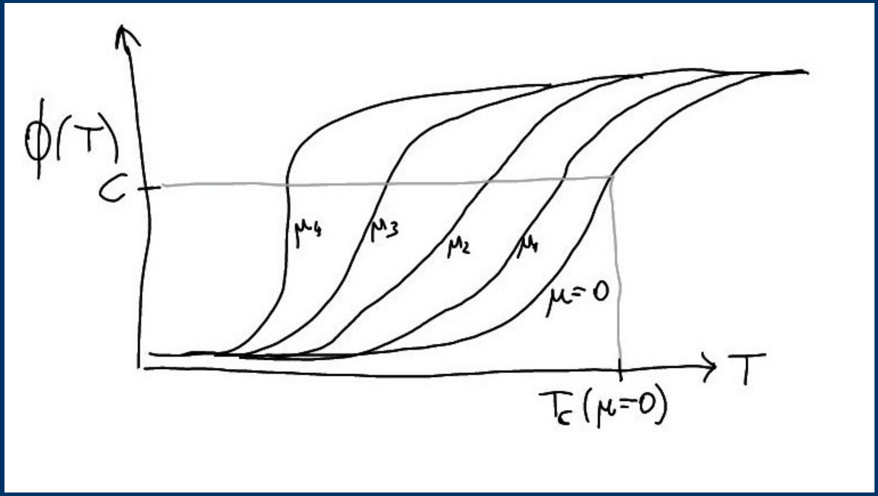
$$B_3 = \frac{\langle O^3 \rangle}{\langle O^2 \rangle^{3/2}}$$

$$O = P - \langle P \rangle \quad \text{with} \quad P = \sqrt{P_{bare} P_{bare}^{-1}}$$

no renormalization
zero crossing defines transition



Shift method



Define $T_c(\mu)$ as $\phi(T_c(\mu),\mu) = C$

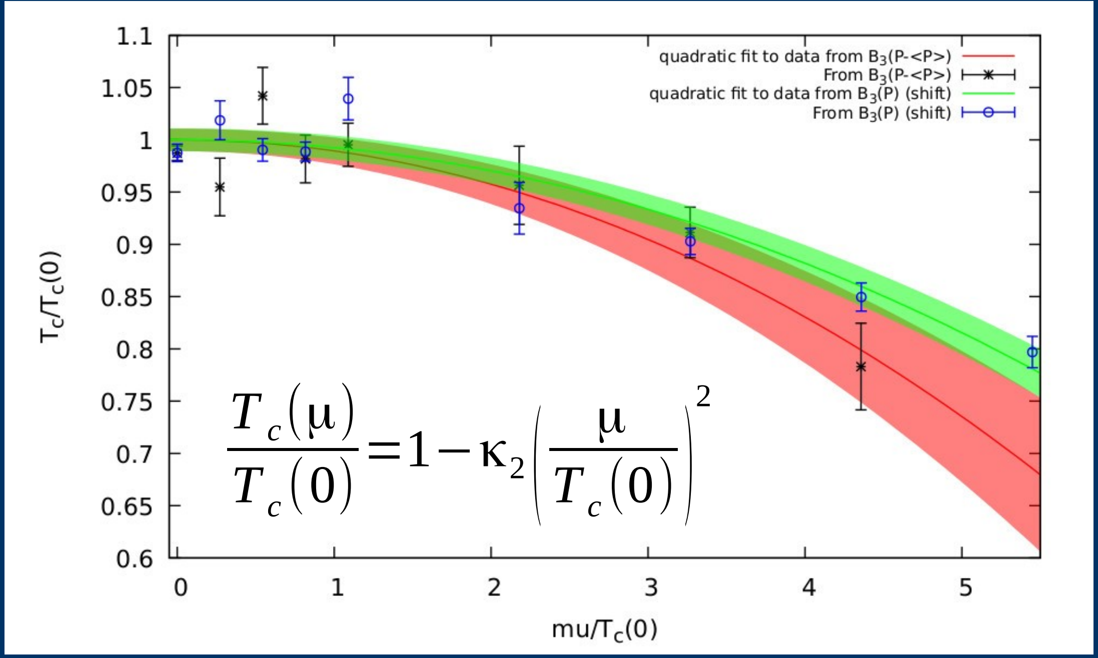
e.g. $B_{3,}$ chiral condensate,
baryon number susceptibility

Works well for small μ
Critial point at μ_4

Lattice spacing: $a = 0.065$ fm

Pion mass: $m_\pi = 1.3$ GeV

Volumes: $8^3, 12^3, 16^3$



Finite size effects large

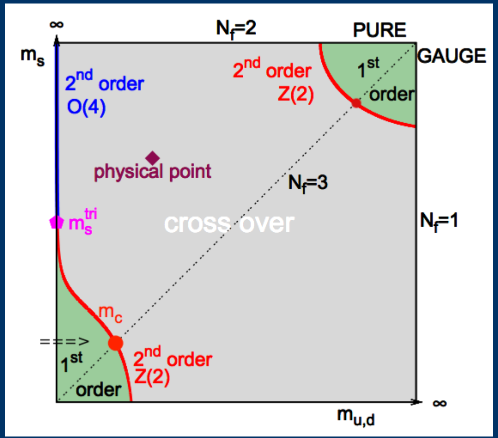
$\kappa_2 \approx 0.0012$

Consistent results

In literature
For physical pion mass
$\kappa_2 = 0.015$

Can follow the line to
quite high μ / T

Open questions
Possible for lighter quarks?
Finite size scaling?
Where is the upper right corner of Columbia plot?
Critical point nearby?



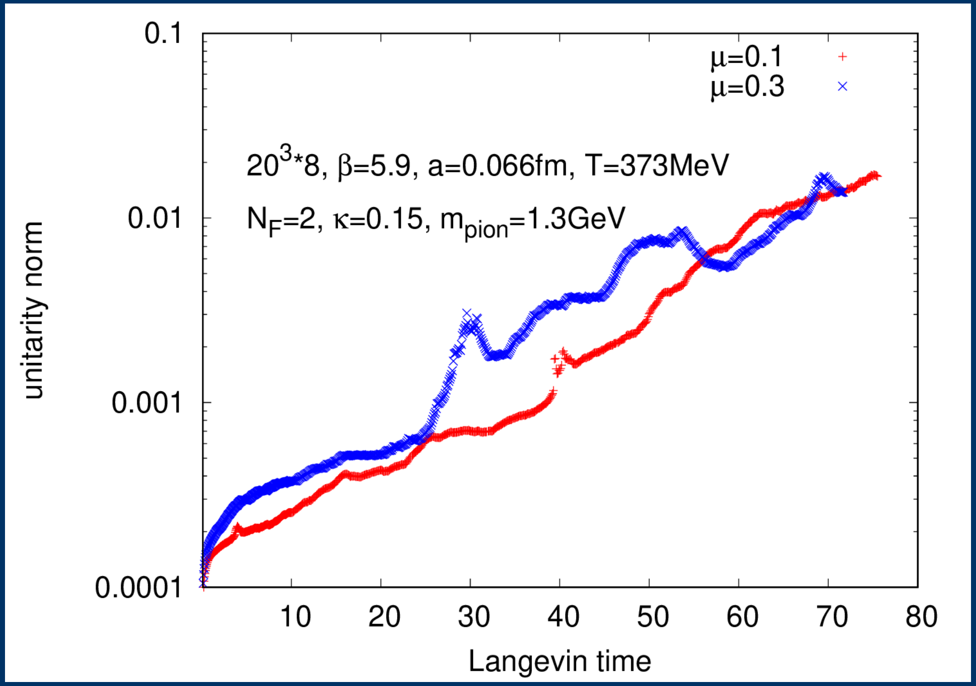
Long runs with CLE

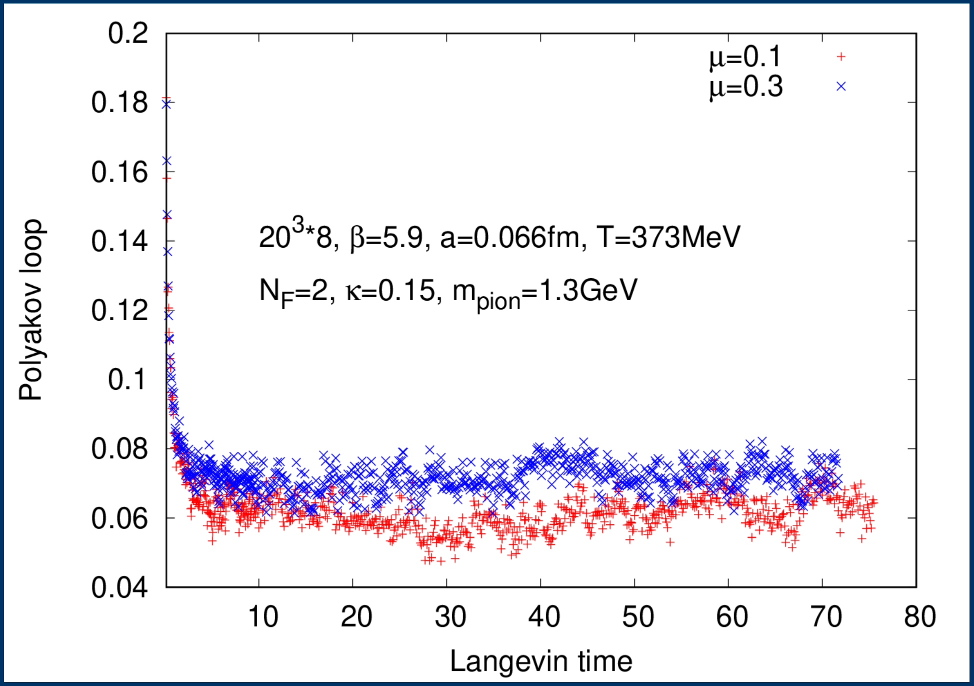
Unitarity norm has a tendency to grow slowly (even with gauge cooling)

$$UN = \sum_{x,\nu} Tr\left(U_{x\nu} U^{+}_{x\nu} - 1\right)$$

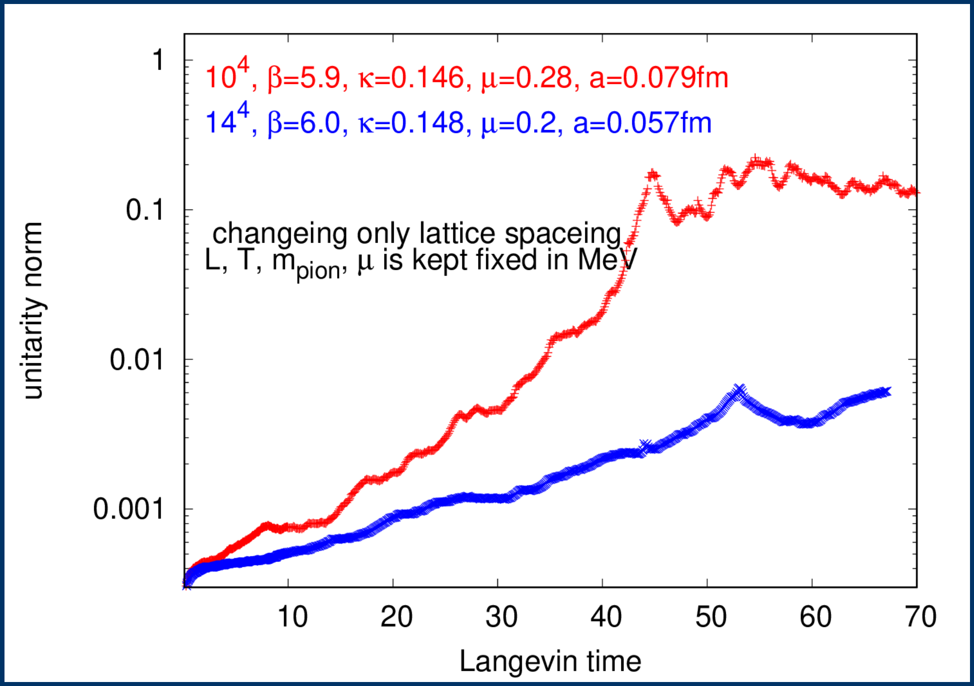
Runs are cut if it reaches ~ 0.1

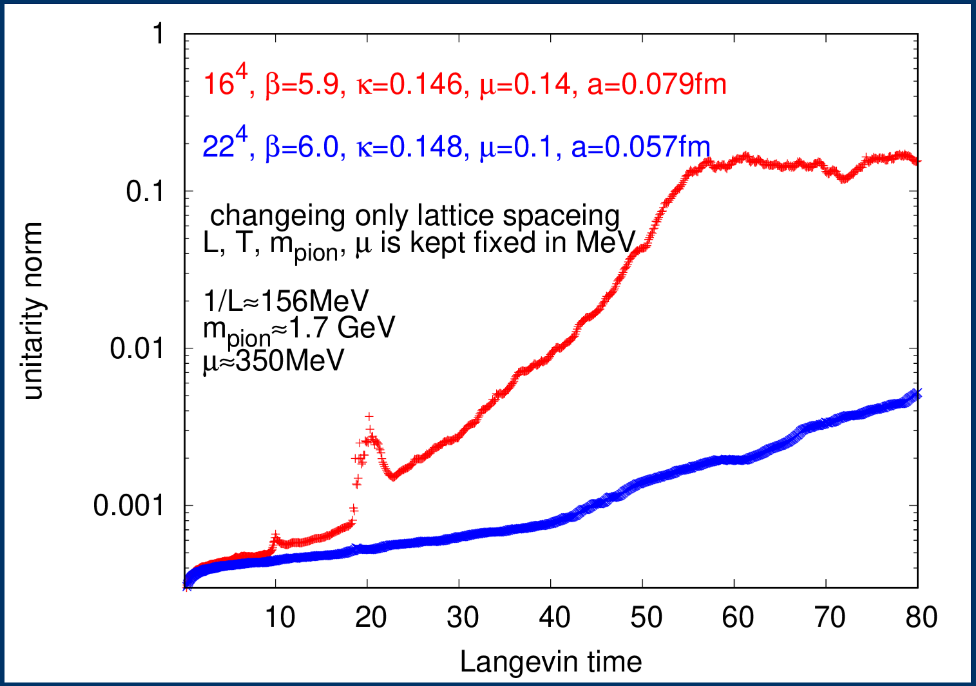
Thermalization usually fast
– might be problematic close to critical point or at low T





Getting closer to continuum limit





Test with Wilson fermions
Increase β by 0.1 – reduces lattice spacing by 30%
change everything else to stay on LCP

behavior of Unitarity norm improves
autocorrelation time decreases as lattice gets finer

Dynamical Stabilization [Attanasio, Jäger (2018)]

Prevent growth of Unitarity norm

“Soft cutoff” in the imaginary directions of SL(3,C)

New term in drift

$$K_{x,\nu}^{a} \rightarrow K_{x,\nu}^{a} + i\,\alpha_{\mathrm{DS}}\,M_{x}^{a}$$

$$M_x^a = i\,b_x^a \left(\sum_c b_x^c b_x^c\right)^3 \qquad b_x^a = Tr\left[\lambda^a \sum_\nu U_{x,\nu} U_{x,\nu}^+\right]$$

New term is SU(3) gauge invariant (not SL(3,C))
Not a derivative of an action
Not holomorphic
Gauge cooling is still used with DS on top
α_{DS} controls strength of attraction to SU(3)

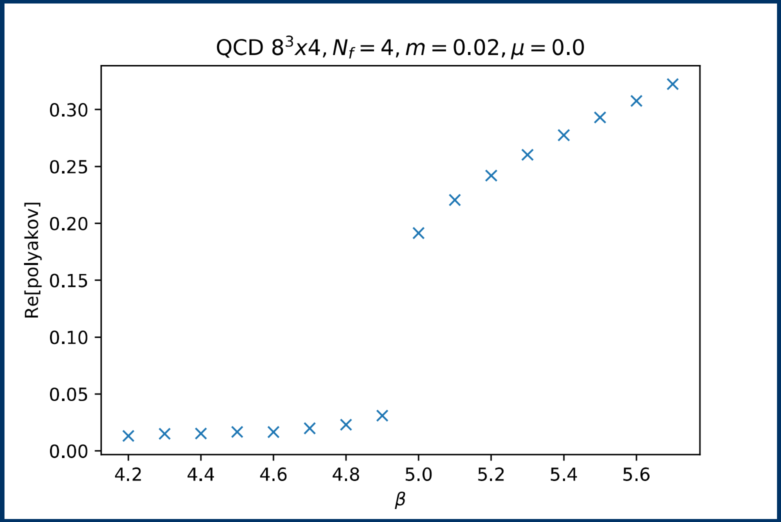
Comparison of Dyn.Stab. And reweighting for full QCD

[Hansen, Sexty (in prep.)]

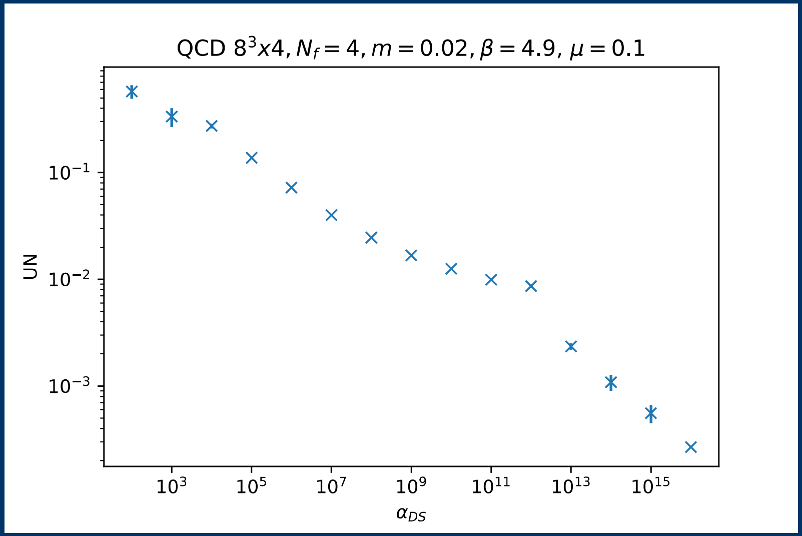
Naive action with staggered quarks

$$8^4 * 4, N_f = 4, m = 0.02, \text{ various } a$$

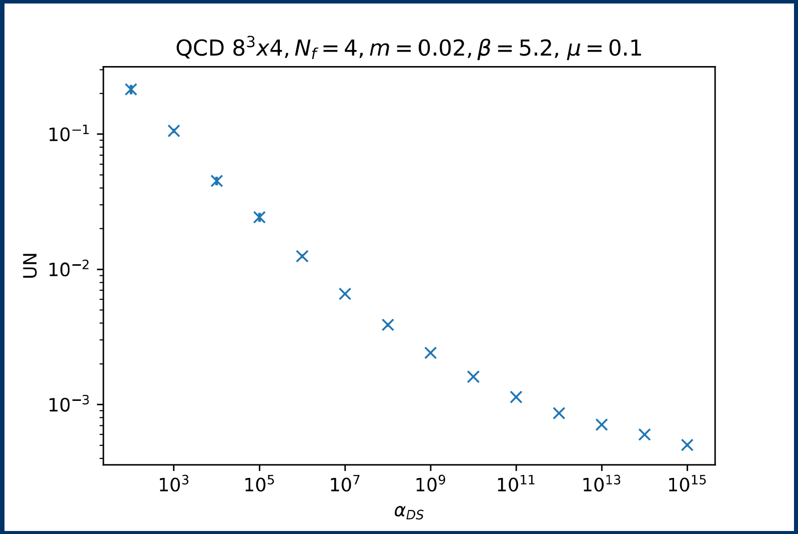
Phase transition for $4.9 < \beta < 5.0$



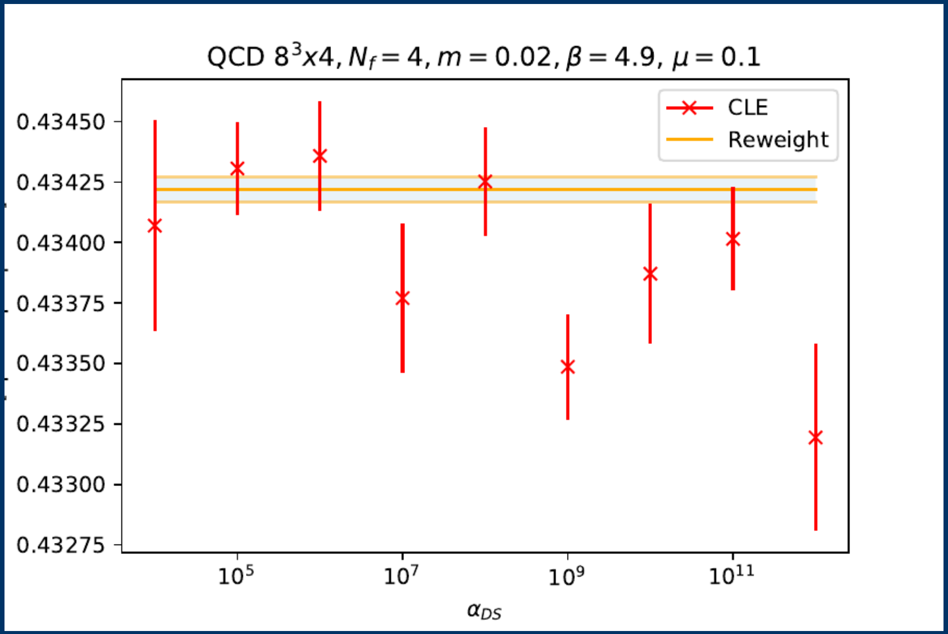
Average UN
at low temperature

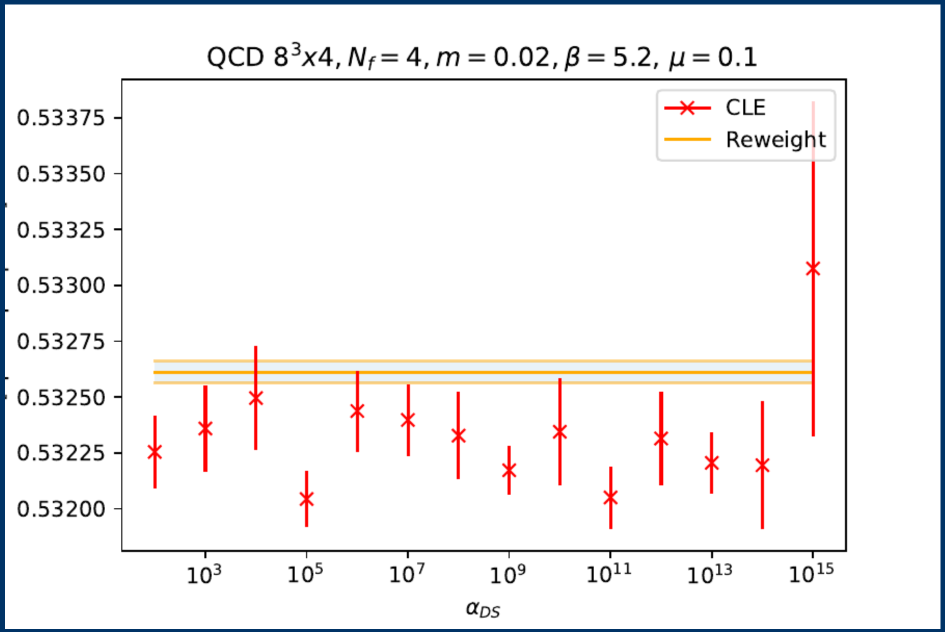


Average UN
at high temperature



Spatial plaquettes





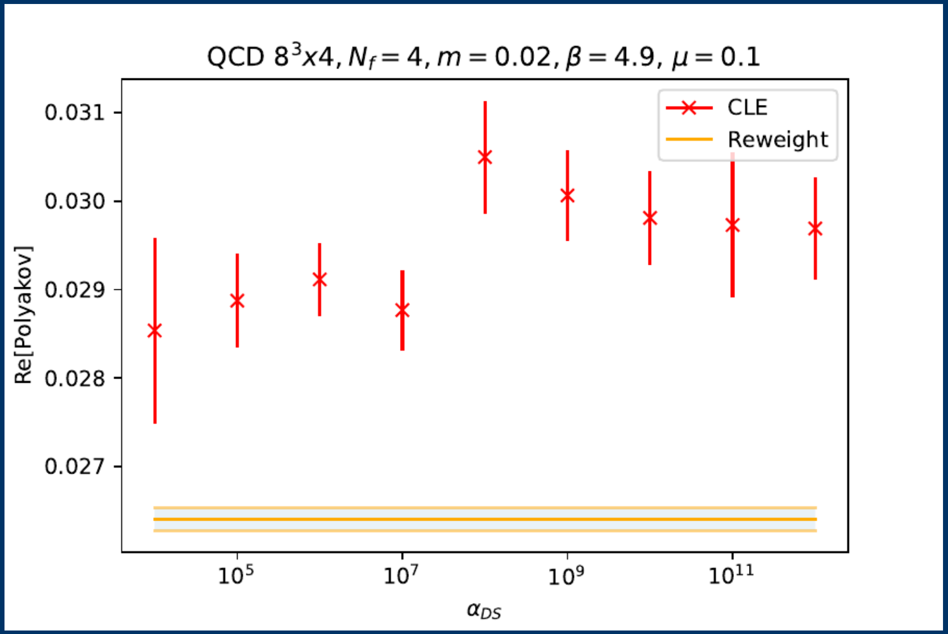
At high temperatures, DS is not needed

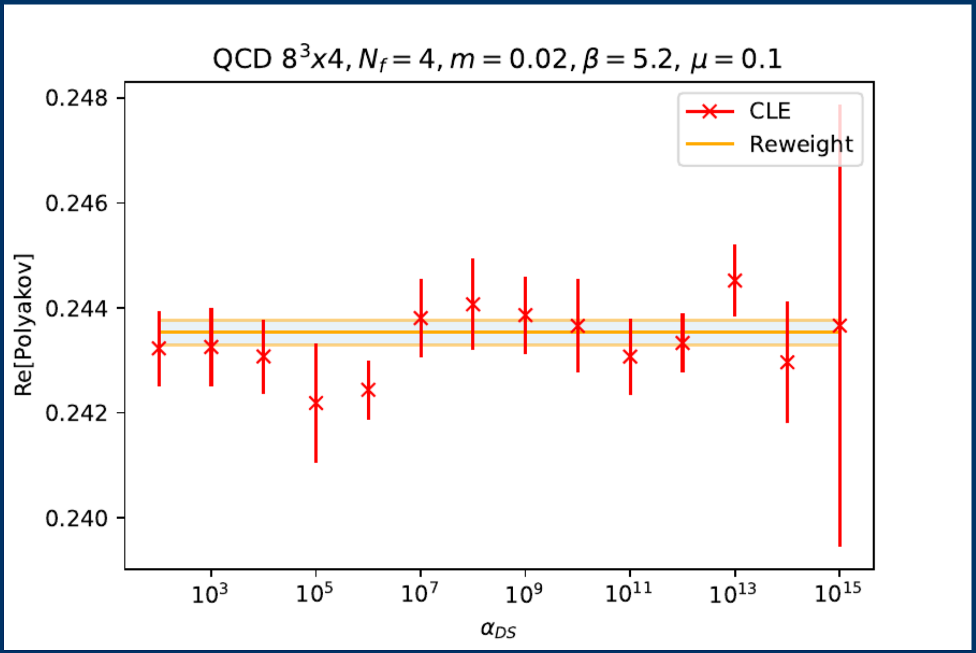
At low temperatures, plaquette is consistent with exact whith a large plateau

low alpha – UN large boundary terms present – simulation instable

High alpha – system restricted to SU(3): phasequenched result

Polyakov loop





At high temperatures, DS is not needed

At low temperatures, value slightly off
bias is introduced by cutoff term. Can we extrapolate to $\alpha_{DS} \rightarrow 0$?

Summary

CLE has potential problems with boundary terms and poles

Monitoring of the process is required:
measuring Boundary terms

lattice models with cheap observable
Correction with higher order boundary terms

Kernels help elimnating boundary terms (if they are present)

Results for the EoS and Phase diag. of QCD

Dynamical stabilization might help at low temperatures of QCD

Sketch of the proof

$P(x, y, t)$: probability density on the complex plane at Langevin time t

Real Fokker-Planck equation

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} - K_x P\right) - \frac{\partial}{\partial y}(K_y P) \quad \text{with} \quad K_i = -\partial_i S$$

Real action $\rightarrow$ $K_y = 0$, positive eigenvalues of H_{FP}

$$P(x, y, \infty) = \delta(y) \exp(-S(x))$$

$\rho(x, t)$: complex measure evolving with the complex Fokker-Planck equation (not associated to a stochastic process)

$$\partial_t \rho(x, t) = \partial_x(\partial_x - K_x)\rho(x, t) = L_c^T \rho(x, t)$$

Stationary solution: $\rho(x, \infty) = \exp(-S(x))$

Assuming spectrum of L_c is fine

CLE works, if

What we want		What we get with CLE
$\int dx\, \rho(x) O(x)$	$=$	$\int dx\, dy\, P(x, y) O(x + iy)$

$$O(z,t)=e^{L_c t}O(z,0)$$
with $L_c=(\partial_z+K(z))\partial_z$

Interpolating function:

$$F(t,\tau)=\int P(x,y,t-\tau)O(x+iy,\tau)\,dx\,dy$$

$$F(t,t)=\langle O(x)\rangle_{\rho(t)} \qquad F(t,0)=\langle O(x+iy)\rangle_{P(t)}$$

$$\int dx\,\rho(x)O(x) \qquad \int dx\,dy\,P(x,y)O(x+iy)$$

$\partial_\tau F(t,\tau)=0$ can be seen with partial integrations

using Cauchy-Riemann eqs. for $\partial_x O(x+iy,\tau)$

QED

Boundary term defined on a surface

$$\partial_\tau F_O(t,\tau)=B_O(Y,t,\tau)=\int K_y(x,Y)P(x,Y,t-\tau)O(x+iY,\tau)\,dx$$
$$-\int K_y(x,-Y)P(x,-Y,t-\tau)O(x-iY,\tau)\,dx$$

Loophole 2: Spectrum on the wrong side

[Seiler, Sexty, Stamatescu (2023)]

Fokker Planck operator

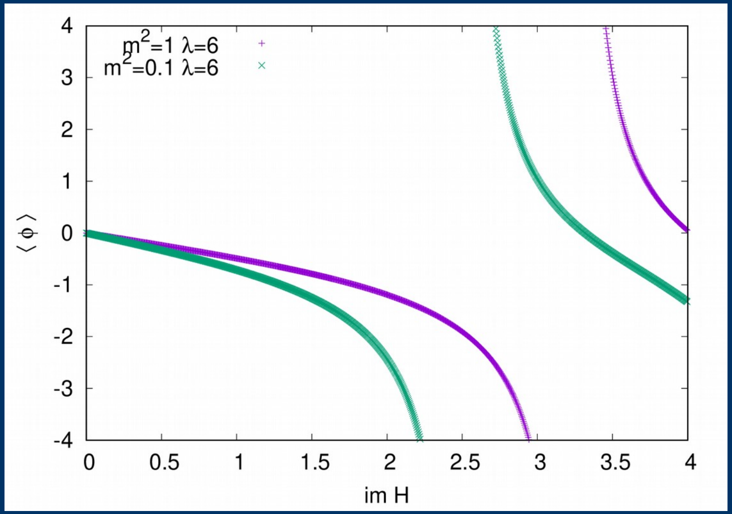
$$L_c = \sum_i \partial_i^2 + K_i \partial_i$$

Determines $\rho(x,t) = e^{t L_c^T} \rho(x,0)$

Toy model:

$$S = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4 + H\phi \quad \rightarrow \quad L_c = \partial_z^2 + \left(-m^2 z - \frac{\lambda}{6} z^3 - H \right) \partial_z$$

At imaginary magnetic field
Lee-Yang zeroes appear



At each Lee-Yang zero
an eigenvalue appears with $\mathrm{Re}(\lambda) > 0$

Slow decay is also present: → Boundary terms signal also this problem

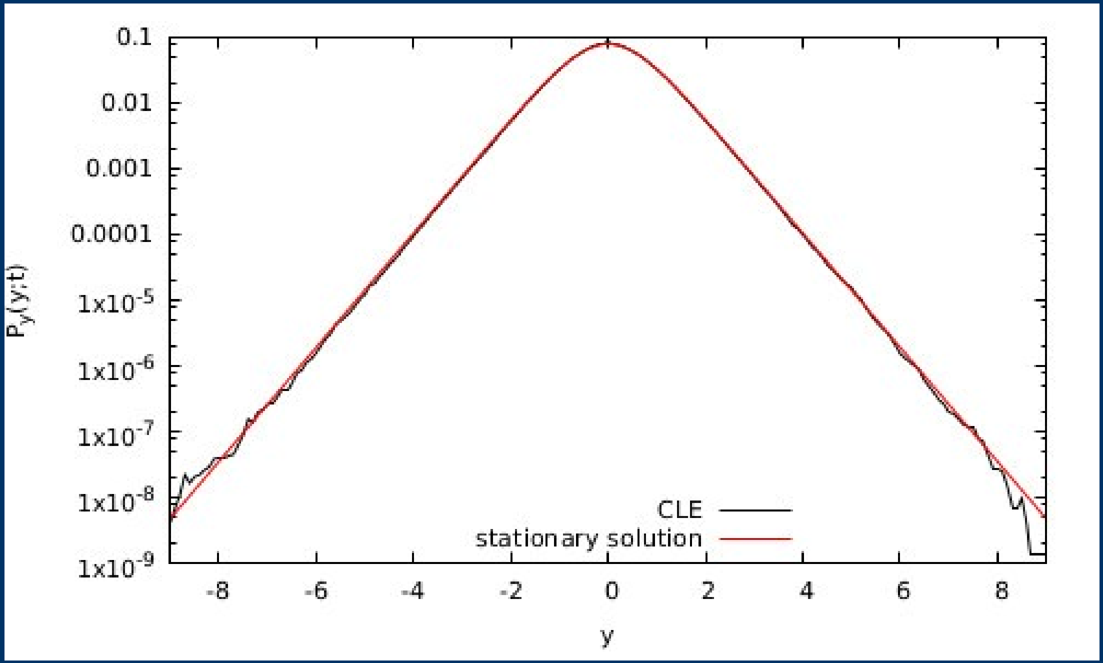
One plaquette model

$$S(\varphi)=i\beta\cos(\varphi) \qquad \langle e^{ikx}\rangle=(-i)^k\frac{J_k(\beta)}{J_0(\beta)}$$

Exact stationary solution of Fokker-Planck eq. [Salcedo, 2017]

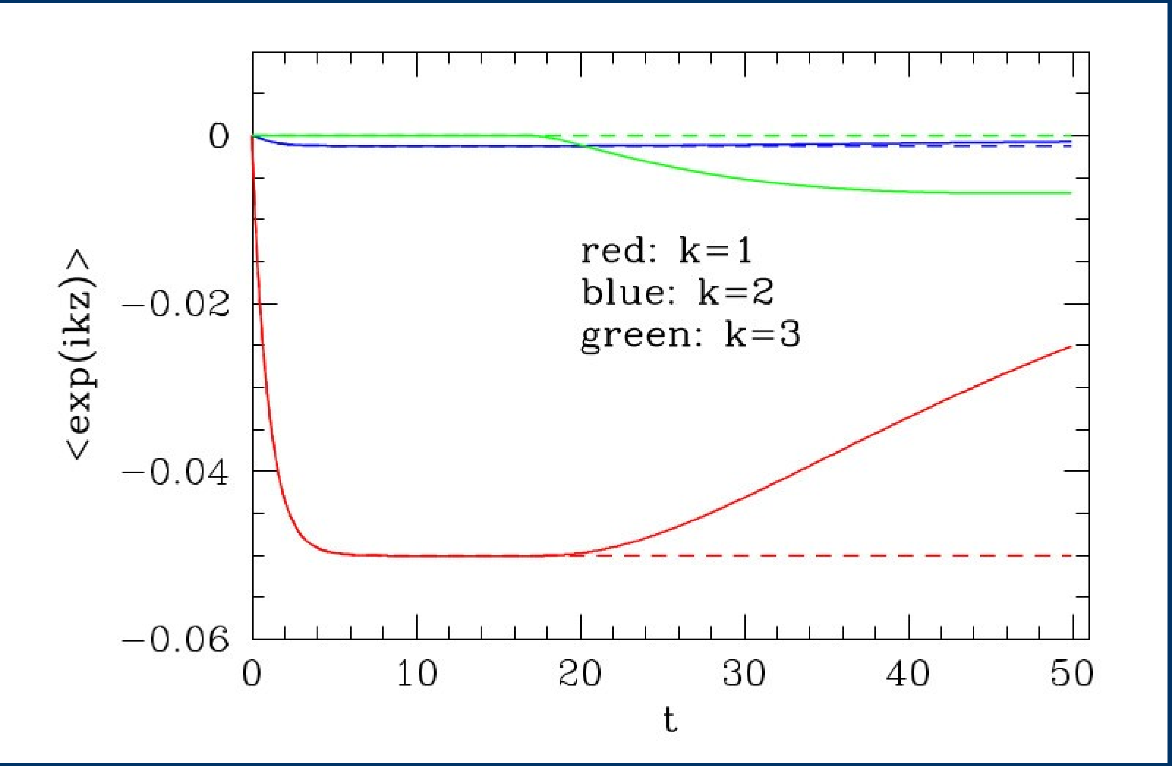
$$P_a(x,y)=\frac{1}{4\pi\cosh^2 y} \qquad \text{independent of } x \text{ and } \beta$$

$$\langle e^{ix}\rangle_{P_a}=0, \quad \langle e^{ikx}\rangle_{P_a} \text{ for } k\geq 2 \text{ is undefined or divergent}$$



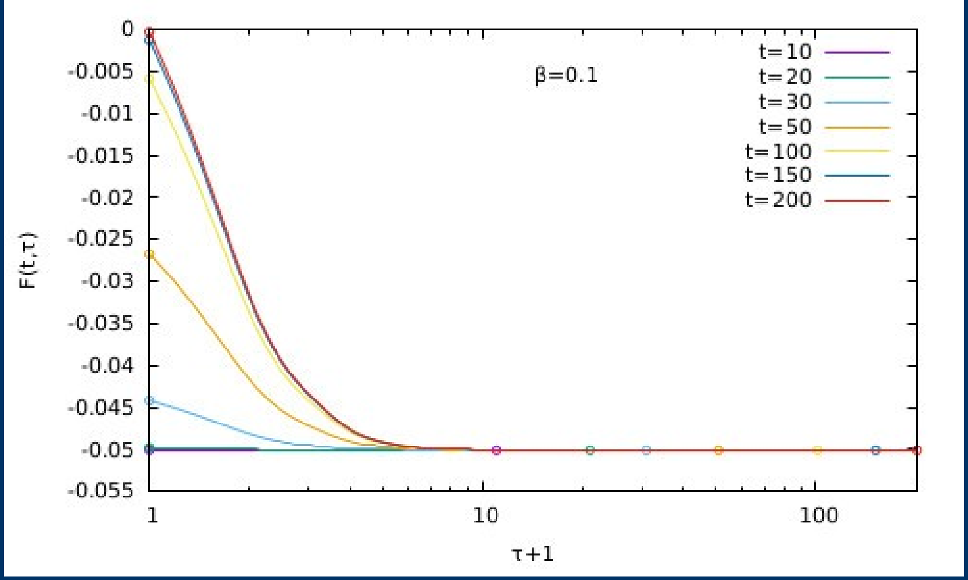
CLE reproduces this (incorrect) solution

Langevin time evolution



For short times
plateau at the correct value

asymptotic result incorrect



$F(t,0)-F(t,t)$ gets >0 above $t=20$

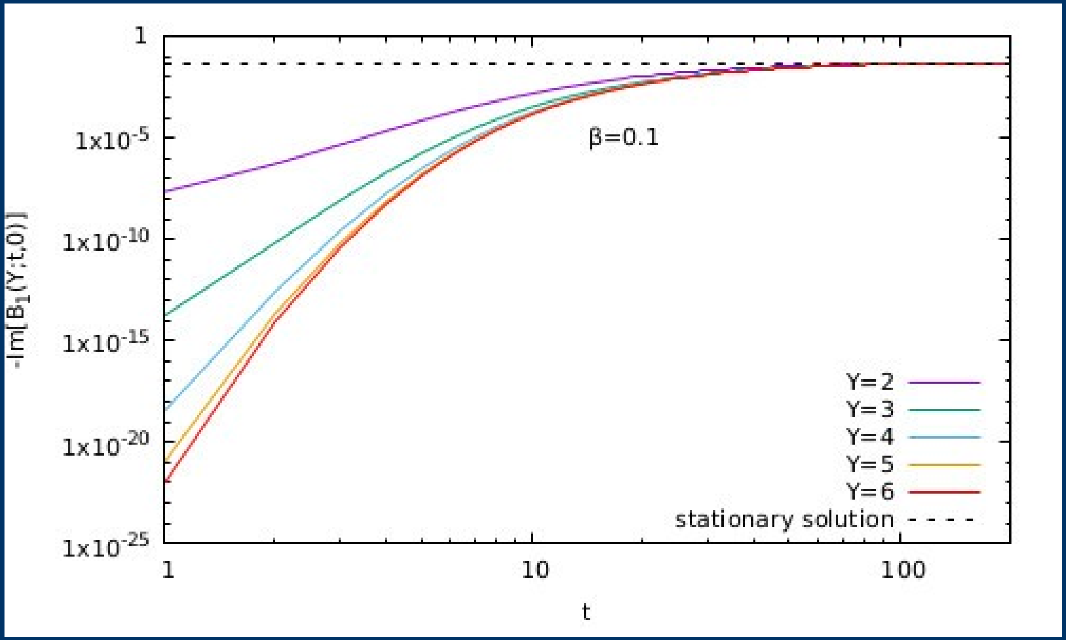
Largest slope at $\tau=0$

$$\partial_\tau F_{(t,\tau=0)} = B(t,0)$$

seems like a good proxy for

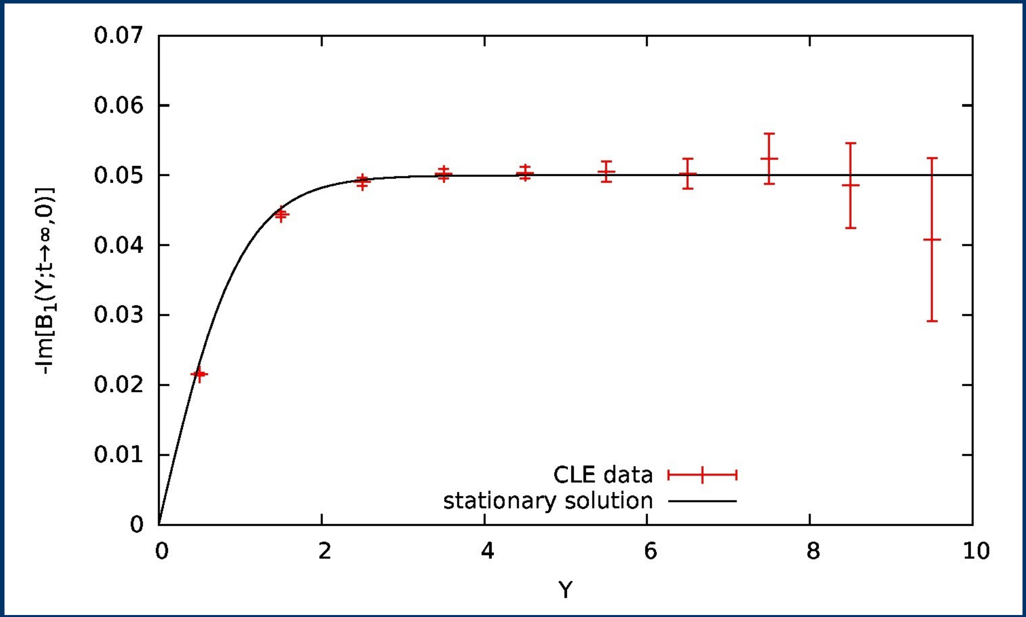
$$F(t,0)-F(t,t)$$

Boundary term



Boundary term

Calculated using Fokker-Planck discretised on a 2d grid



Using Complex Langevin only

Plateau clearly visible

At high cutoff statistics is worse

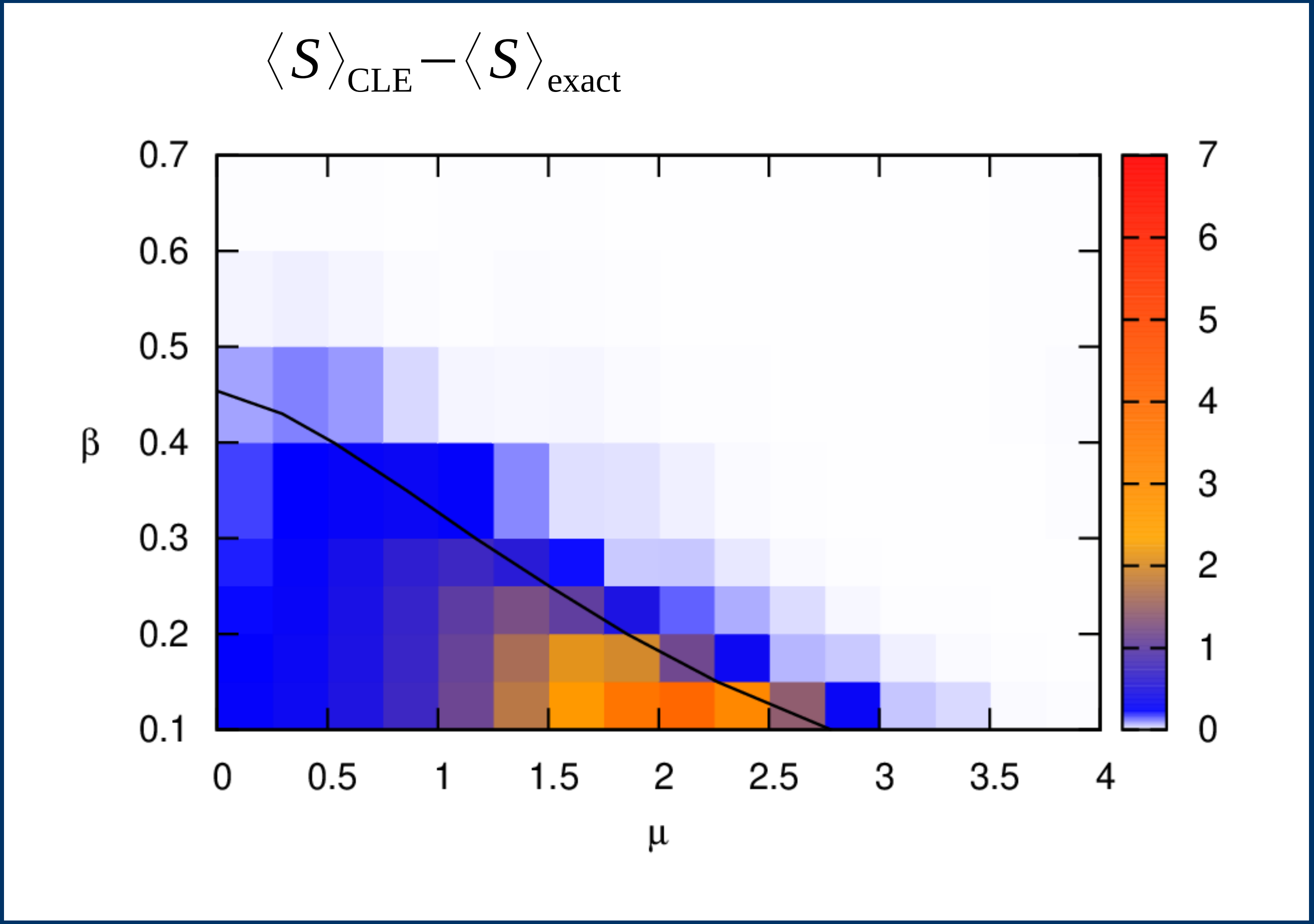
Need to measure on some surface inconvenient in many dimensions

XY model in d=3

$$S = -\beta \sum_x \sum_{\nu=1}^{3} \cos\left(\phi_x - \phi_{x+\nu} - i\mu\,\delta_{\nu 0}\right)$$

Can be solved exactly using dual variables (worldlines)

CLE fails in one of the phases

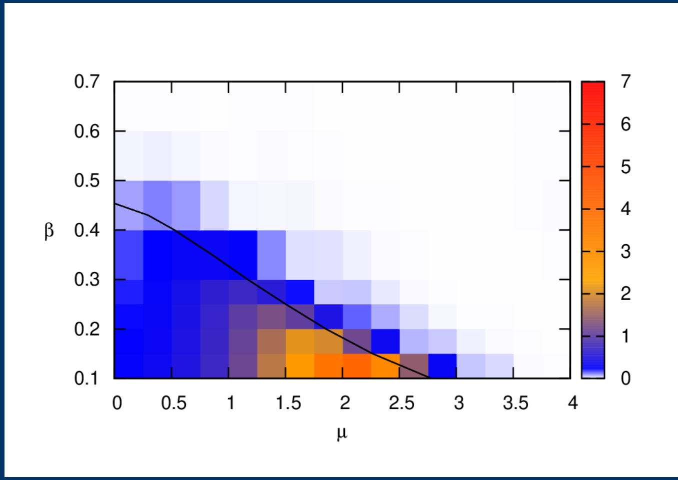


[plot from: Aarts and James (2010)]

Boundary terms in 3d XY model

CLE is actually wrong in the whole phase diag.

Boundary term is very small in one of the phases



Correction

B_2 is very noisy, hard to measure

Step in the right direction

$\mathcal{O}$	β, μ^2	B_1	B_2	B_1^2/B_2	CL error	CL	worldline	corrected CL
S	$0.2, 10^{-6}$	$0.02567(21)$	$-0.0730(47)$	$-0.00902(46)$	$-0.013029(65)$	$-0.075316(65)$	$-0.062288(17)$	$-0.06630(53)$
	$0.2, 0.1$	$0.03309(25)$	$-0.0903(79)$	$-0.01213(89)$	$-0.0169974(91)$	$-0.0792922(91)$	$-0.062295(18)$	$-0.06716(90)$
	$0.2, 0.2$	$0.03941(28)$	$-0.109(13)$	$-0.0142(17)$	$-0.0205408(80)$	$-0.0828399(80)$	$-0.062299(11)$	$-0.0686(17)$
	$0.7, 10^{-6}$	$1.440(15)10^{-4}$	$-7.33(17)10^{-4}$	$-2.834(46)10^{-5}$	$-1.23(33)10^{-4}$	$-1.482311(33)$	$-1.48219(35)$	$-1.482283(34)$
	$0.7, 0.1$	$0.004783(50)$	$-0.0082(23)$	$-0.00278(69)$	$-0.002791(31)$	$-1.526766(31)$	$-1.52398(35)$	$-1.52399(72)$
	$0.7, 0.2$	$0.006013(38)$	$-0.00873(96)$	$-0.00414(45)$	$-0.002488(29)$	$-1.568899(29)$	$-1.56641(20)$	$-1.56476(48)$
n	$0.2, 10^{-6}$	$4.8(1.6)10^{-5}$	$-0.00021(124)$	$1.3(3.7)10^{-5}$	$1.36(31)10^{-5}$	$1.36(31)10^{-5}$	$-1.2(1.1)10^{-8}$	$0.89(7.65)10^{-6}$
	$0.2, 0.1$	$-0.01147(15)$	$0.0286(32)$	$0.00460(24)$	$0.0058177(41)$	$0.0058182(41)$	$4.9(2.1)10^{-7}$	$0.00122(69)$