Interaction of relativistic charged particles with strong laser fields in vacuum or in a plasma environment.

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[Varró S, Interaction of relativistic charged particles with strong laser fields in a plasma environment. [Talk presented at PP2023. Margaret Island Symposium 2023 on Particles & Plasmas. Jun 7 – 9, 2023, Budapest] File:"Varro_S_Interaction_PP223_07_June_v1"

Outline.

- **Introduction. Brief overview of the possible theoretical descriptions of photon–electron (or other charged particle) interaction.**

- **Relativistic dynamics of a charged particle in a strong laser field in vacuum. Analogies between the „figure–8–motion" and the Kepler–Coulomb problem.**

- **Relativistic motion of a charged particle interacting with a strong laser field in a plasma. Comparison of the high-harmonic spectrum in vacuum and in an underdense plasma.**

- **Dirac particle in external electromagnetic plane waves in vacuum or in a plasma. Optically induced band structure and exceptional solutions.**

- **Dirac particle interacting with quantized radiation fields in vacuum. Exact solutions, and considerations on squeezing and 'aberration' in quantum phase space of the photon.**

- **Summary and outlook.**

Possible description of (higher-order) processes taking place during the interaction of electrons (or other charged particles) and photons.The main purpose of the talk is to illustrate the interrelations and differences of some exact descriptions at various levels.

PHOTON / ELECTRON	Trajectory, Ray (Geometric Optics)	Field (Maxwell Theory)	Quantized Field (True Photon)
Trajectory, current [Point, charged dust, Mechanics, Hidrodynamics]	1.	**2. Classical Electrodynamics** Classical EM fields, Radiation reaction	3. Classical current, Classical (Poisson) photon
Field, Transition Currents [Wave Mechanics]	4.	**5. Semiclassical Theory.** [Schrödinger, KG, Dirac, Maxwell]	**6. Quantum Optics.** Quantum transitions + General Photon
Quantized Field [Electron-Positron (Hole) Field,Solid State Physics]	7.	**8. QED in External EM Fields** [e.g. e-e+ pair creation]	9. Full QED, pair creation and back- reaction of charges

$$du_\mu / d\tau = (e/m_0 c) F_{\mu\nu} u^\nu$$

$$[\gamma_\mu (i\partial - \varepsilon A_{rad})^\mu - \kappa]\Psi = 0$$

Exact trajectories of a relativistic charged particle interacting with a strong laser field in vacuum. Formal relations between the „figure–8–motion" in a plane wave electromagnetic radiation and the Kepler–Coulomb central force problem.

$$\frac{d}{dt}\left[\frac{m_0 d\boldsymbol{r}(t)/dt}{\sqrt{1-[d\boldsymbol{r}(t)/dt]^2/c^2}}\right] = e\boldsymbol{E}(\boldsymbol{r}(t),t) + \frac{e}{c}\frac{d\boldsymbol{r}(t)}{dt}\times\boldsymbol{B}(\boldsymbol{r}(t),t)$$

Classical considerations for a relativistic free electron. In vacuum, the argument of the e.m. plane wave at the electron's position is proportional to the proper time of the electron. The „figure–8 motion".

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{e}_x F_x(\theta) + \boldsymbol{e}_y F_y(\theta)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{n} \times \boldsymbol{E}(\theta)$$

$$\theta \equiv t - \boldsymbol{n} \cdot \boldsymbol{r} / c$$

$$m_0 du_\mu / d\tau = (e/c) F_{\mu\nu} u^\nu$$

$$\frac{d}{d\tau}\left[t - \frac{1}{c} \boldsymbol{n} \cdot \boldsymbol{r}(t) \right] = const. = \alpha$$

Along the polarization x-direction one receives formally a Newton equation in dipole approximation. By solving for the x- and y-components, the z-component (the longitudinal component, driven by the v × B force term) also satifies a Newton-like equation in dipole approcimation (mere τ-dependence).

$$\alpha = 1$$

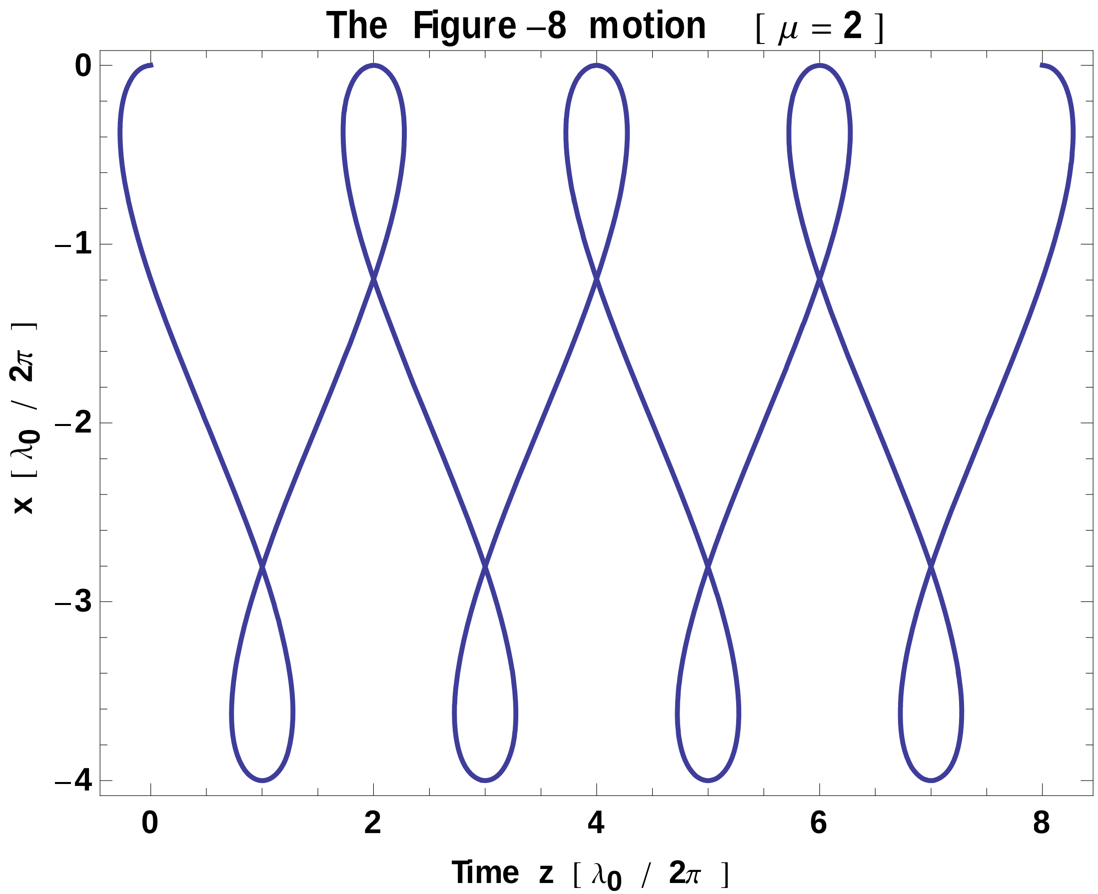
$$F_x(\tau) = F_{0x} \cos \omega_0 \tau$$

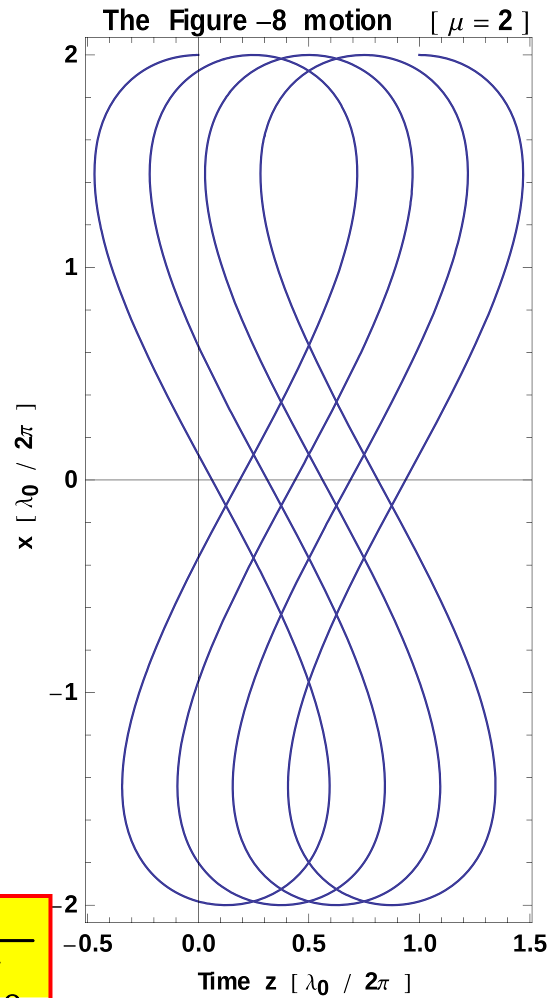
$$(\omega_0 / c) x(\tau) = \mu_{0x} (\cos \omega_0 \tau - 1)$$

$$\mu_{0x} = |e| F_{0x} / m_0 c \omega_0$$

$$(\omega_0 / c) z(\tau) = (\mu_{0x}^2 / 8)(2\omega_0 \tau - \sin 2\omega_0 \tau)$$

Illustration of the „figure–8 motion" for intensity parameter $\mu_{0x} = 2$ [$\lambda_0 = 1.24\ \mu$; $I_0 = 16 \times 10^{18}$ W / cm^2].





Here we have transformed out a part of the z – translation.

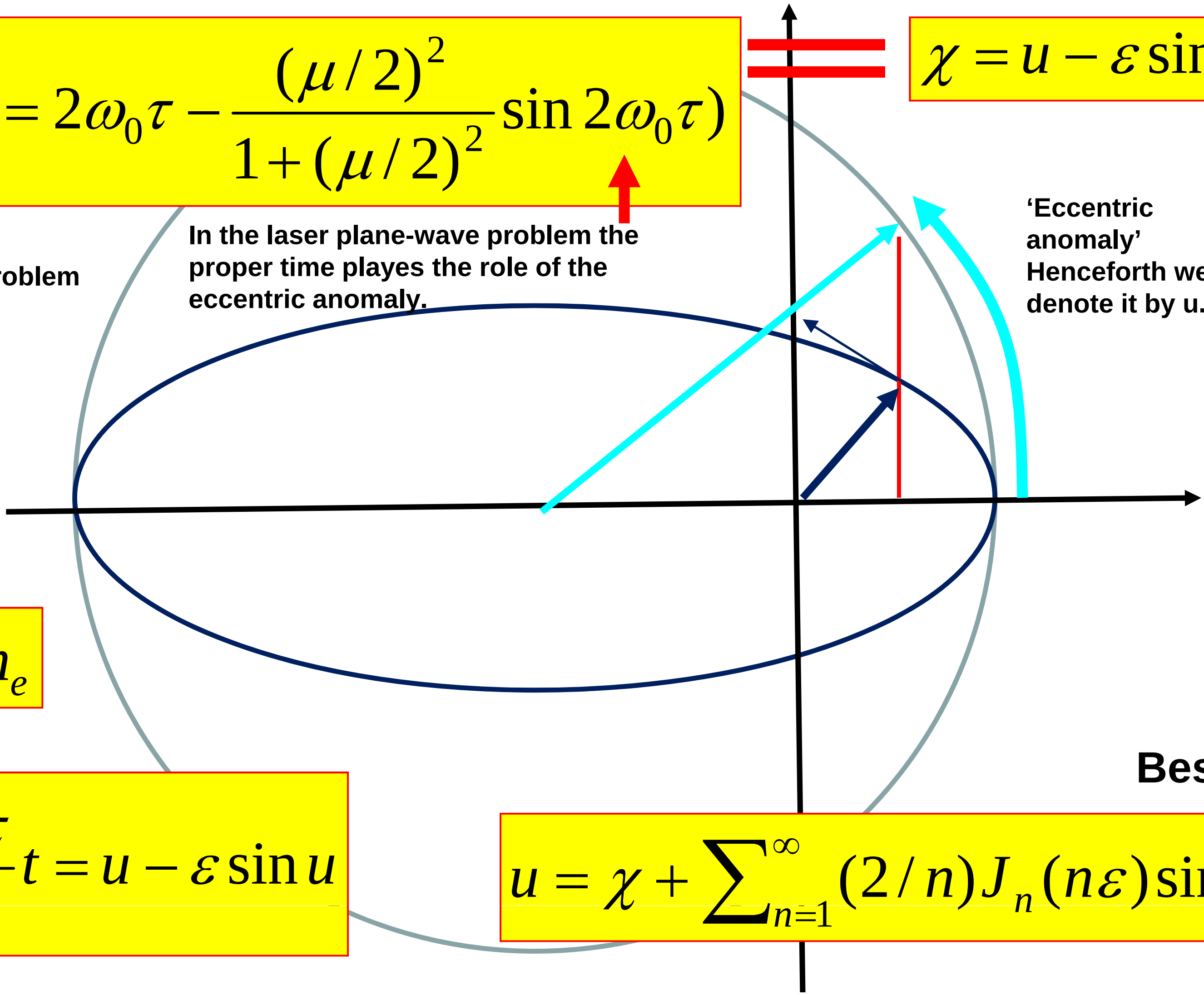
Dimensionless intensity parameter:

$$\mu_0 \equiv \frac{eF_0}{m_0 c \omega_0} = 8.5 \times 10^{-10}\, \lambda_0 \sqrt{I_0}$$

Mathematical connection between the „figure–8–motion” in a plane wave electromagnetic radiation and the Kepler–Coulomb central force problem. 1.

$$\frac{2\omega_0 t}{1+(\mu/2)^2} = 2\omega_0\tau - \frac{(\mu/2)^2}{1+(\mu/2)^2}\sin 2\omega_0\tau)$$

$$\chi = u - \varepsilon \sin u$$



Kepler – Coulomb problem

$$\ddot{\boldsymbol{r}} = -\frac{g}{r^3}\boldsymbol{r}$$

$$g = GM$$

$$g = Ze^2/m_e$$

(Kepler – III)

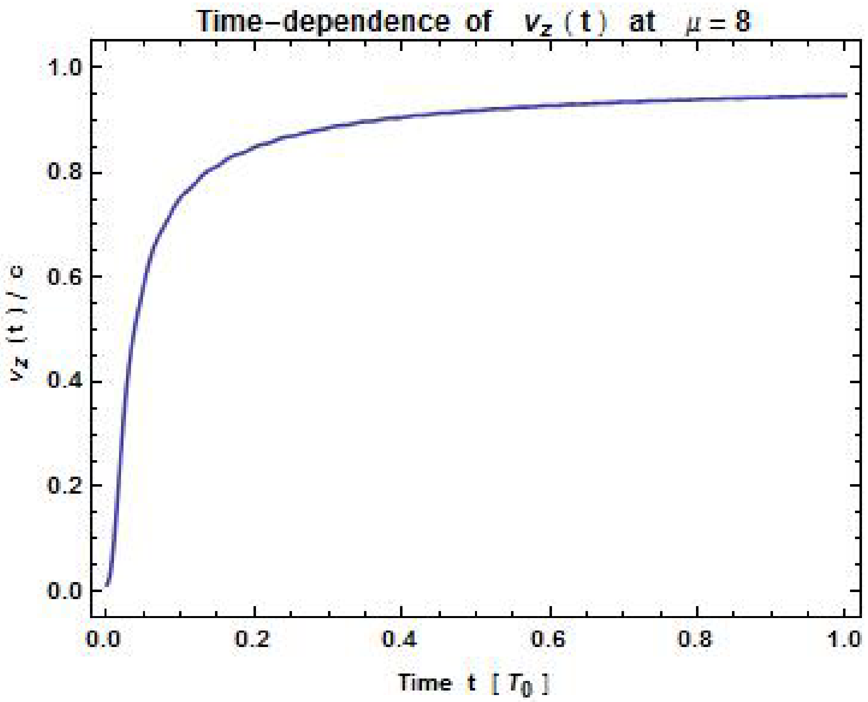
$$\sqrt{\frac{g}{a^3}}t = \frac{2\pi}{T}t = u - \varepsilon \sin u$$

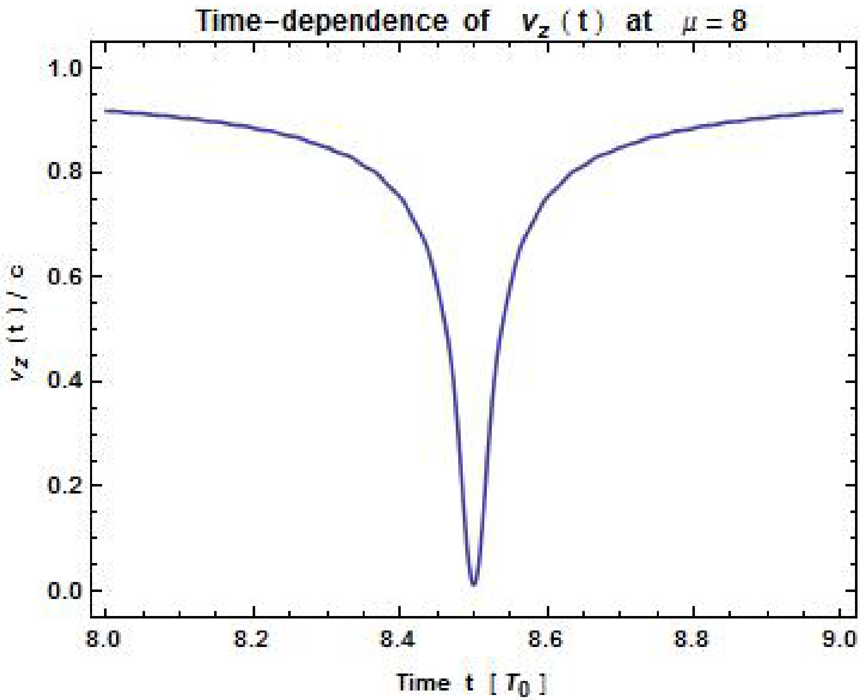
Bessel:

$$u = \chi + \sum_{n=1}^{\infty}(2/n)J_n(n\varepsilon)\sin n\chi$$

Intensity-dependent frequency down-shift,
Recurrence of the velocity.

$$\omega = \frac{\omega_0}{1+(\mu_0/2)^2}$$

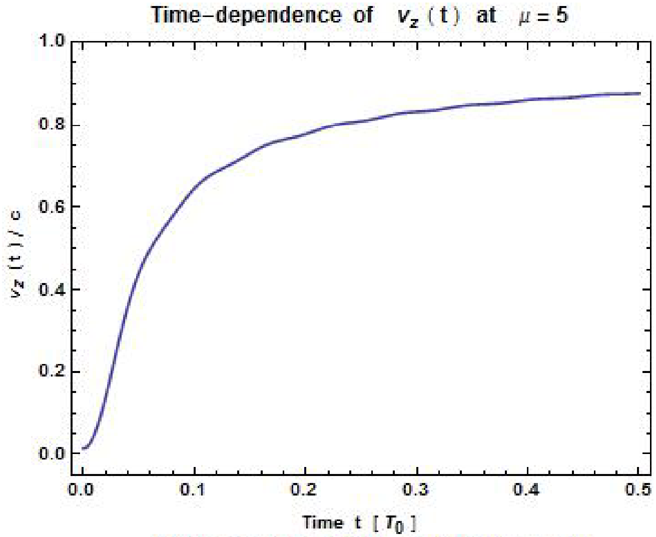


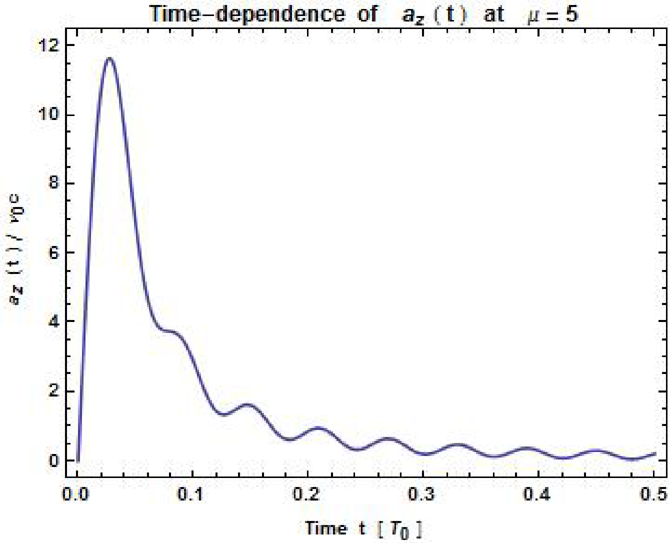


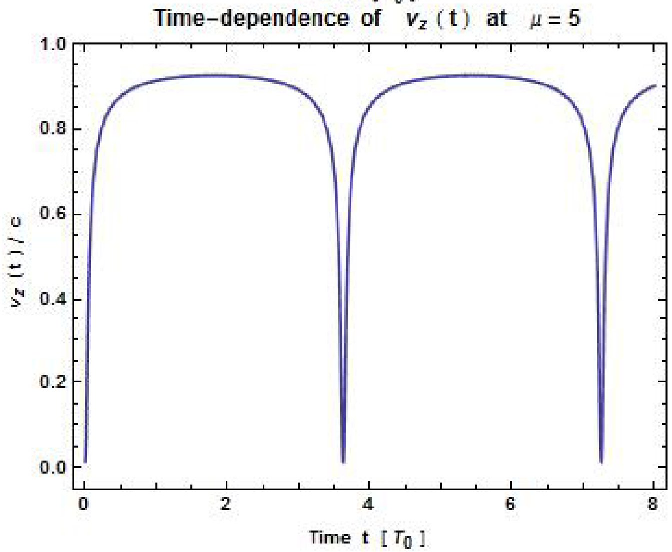
$$z(t) = \beta ct - (c/\omega_0)\sum_{n=1}^{\infty}\frac{1}{n}J_n(\beta n)\sin 2\omega t$$

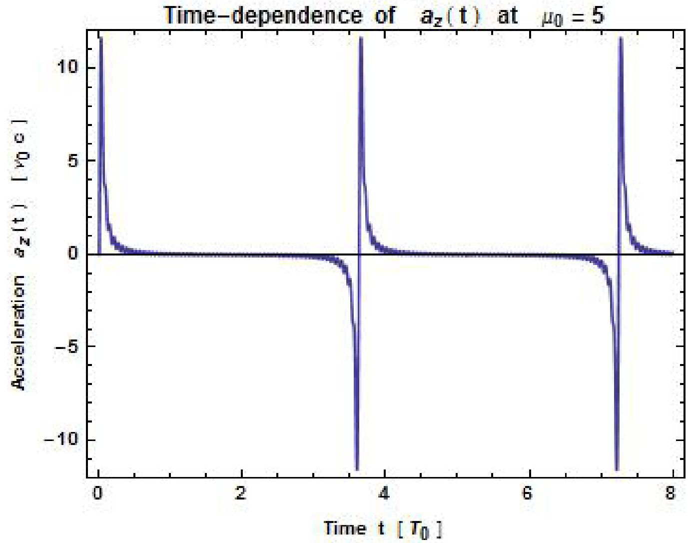
$$\beta = \frac{(\mu_0/2)^2}{1+(\mu_0/2)^2}$$

Recurrence of the velocity and acceleration. [‘collaps’ and ‘revival’.]









Relativistic motion of a charged particle interacting with a strong laser field in a plasma. Comparison of the high-harmonic spectrum in vacuum and in an underdense plasma.

For a relativistic free electron in a plasma, the argument of the e.m. plane wave at the electron's position is not any more proportional to the proper time of the electron, but a complicated function.

In vacuum

$$\theta \equiv t - \boldsymbol{n}\cdot\boldsymbol{r}/c$$

$$\theta = \alpha\tau$$

In plasma.

$$\theta \equiv t - n_m \boldsymbol{n}\cdot\boldsymbol{r}/c$$

$$\theta = \arcsin\left[\frac{sn(\sqrt{1+p^2}\,\omega_0'\tau)}{\sqrt{1+p^2 cn^2(\sqrt{1+p^2}\,\omega_0'\tau)}}\right]$$

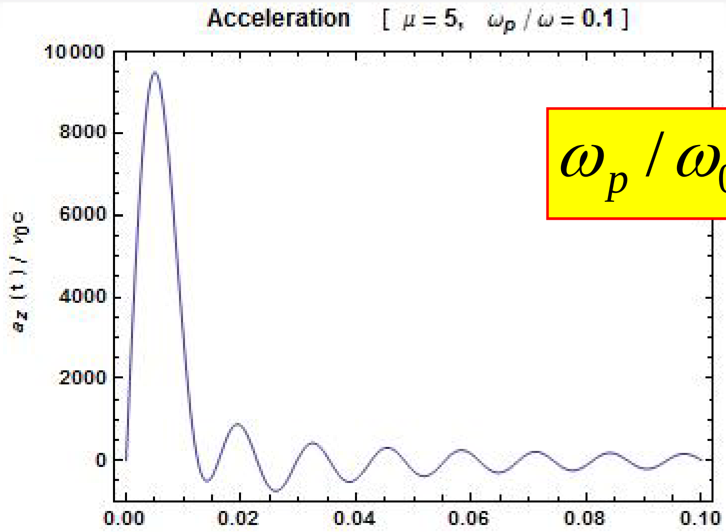
$$\omega = \frac{\omega_0}{1+(\mu_0/2)^2}$$

In laboratory time:'generalized Bessel function' can be approximately obtained, by using an analogous mathematical procedure with the eccentric anomaly.

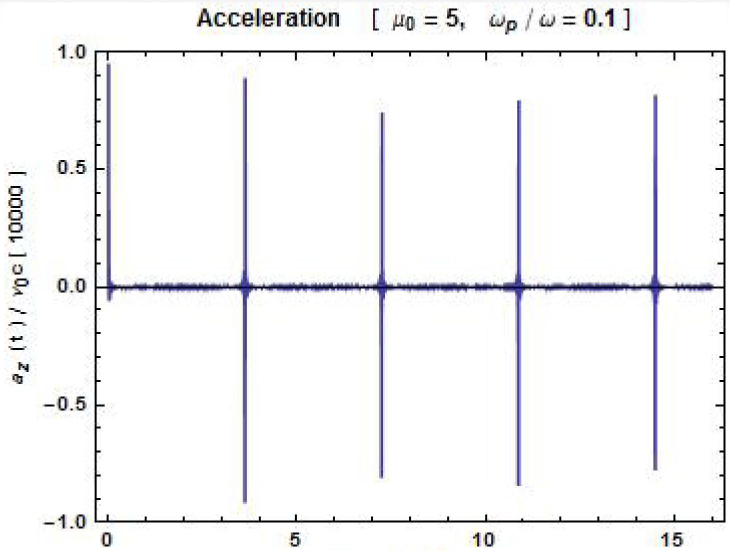
$$z(t) = \beta ct - (c/\omega_0)\sum_{n=1}^{\infty}\frac{1}{n}B_n(n\beta, np^2\beta)\sin 2\omega t$$

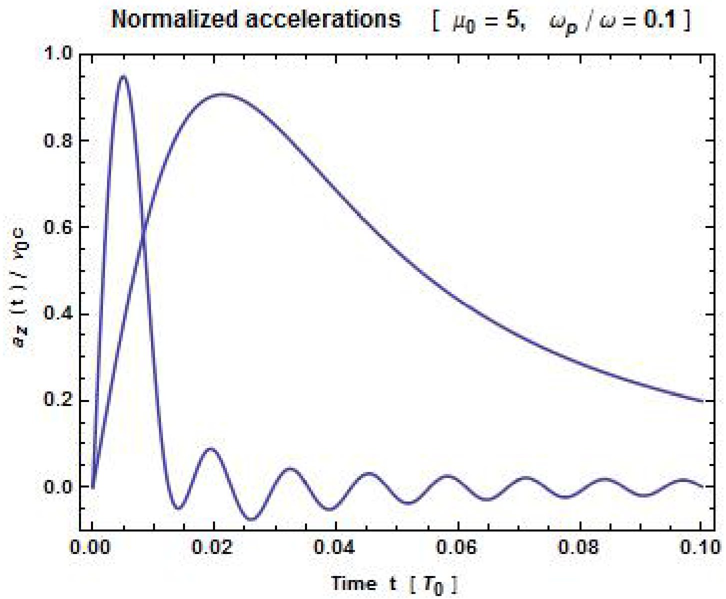
$$\beta = \frac{(\mu_0/2)^2}{1+(\mu_0/2)^2}$$

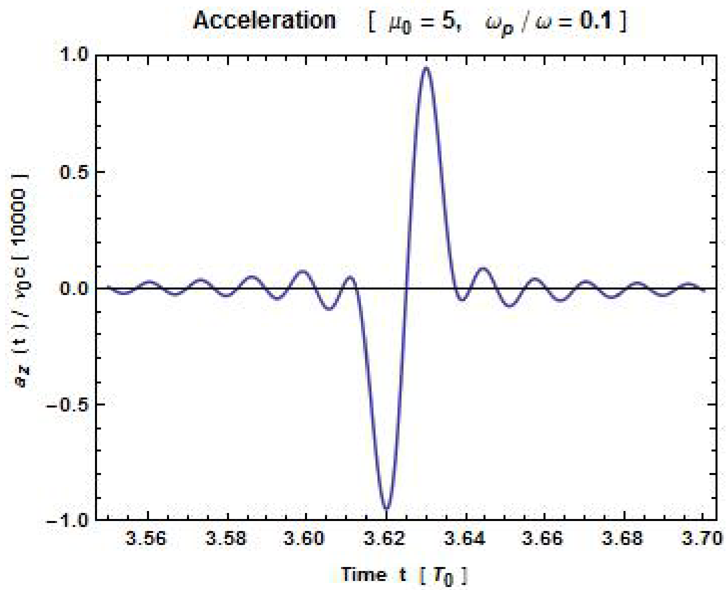
Temporal evolution of the electron's acceleration in plasma. [Comparison with the vacuum case.]



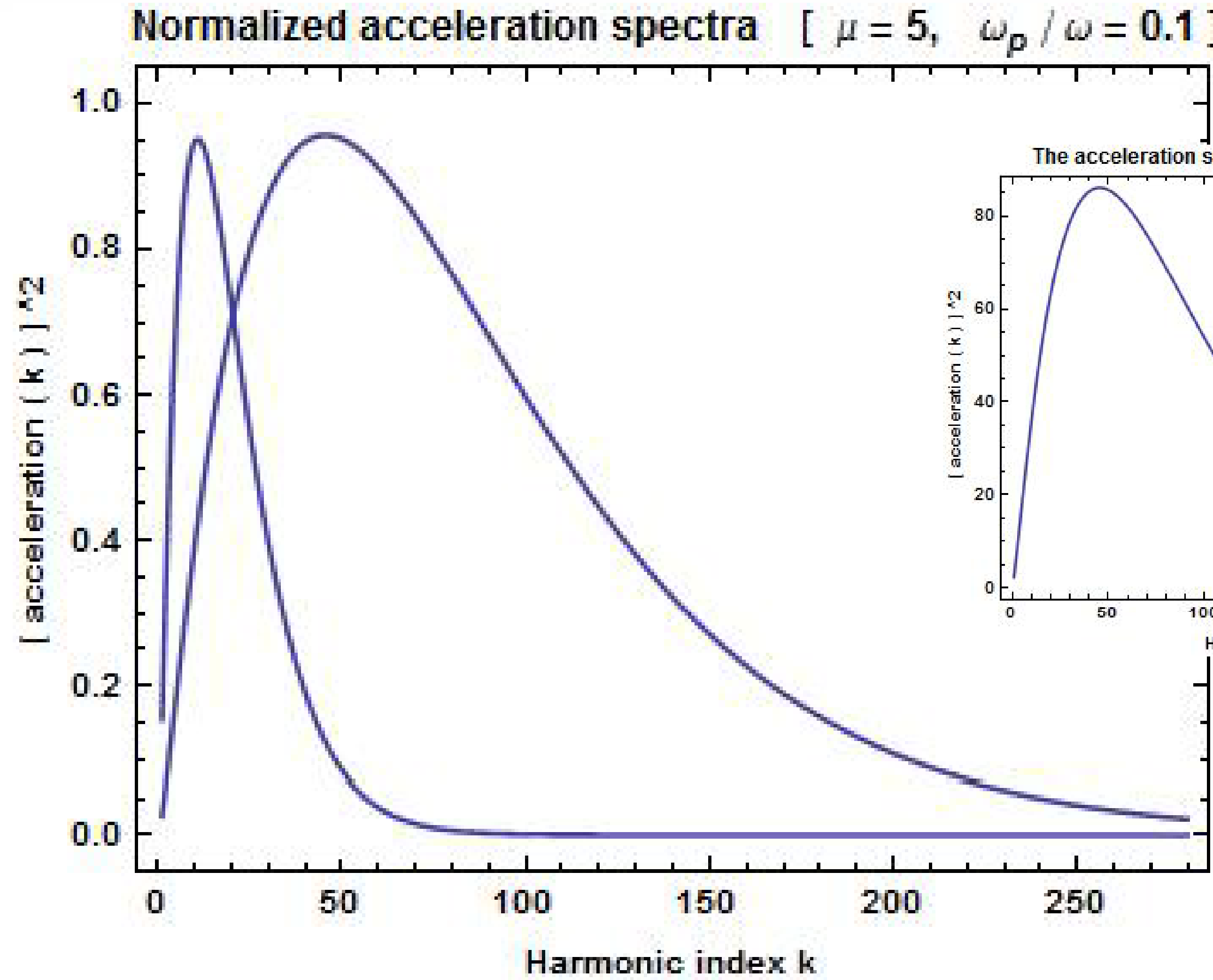
$$\omega_p / \omega_0 = 0.1$$

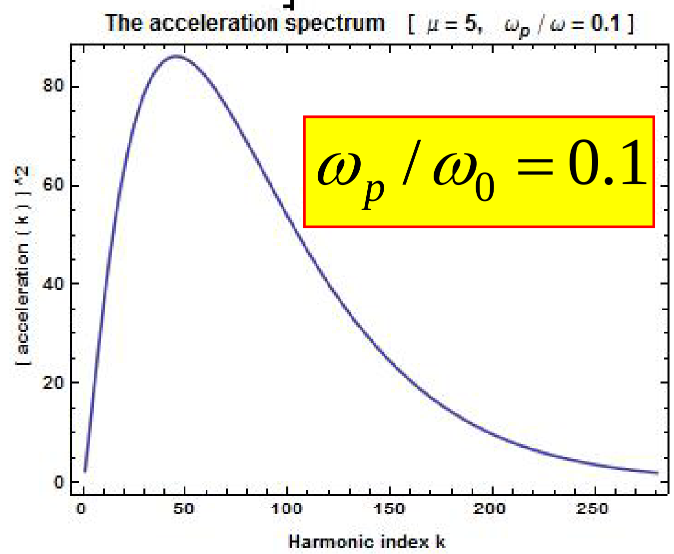






Comparison of the acceleration ('HHG') spectra in vacuum and in plasma.





Varró S, Klein-Gordon radio and laser acceleration of particles. Part I-II. ELI-ALPS Seminars ; 03, 17 May 2019, Szeged

Dirac particle interacting with external electromagnetic plane waves in vacuum or in a plasma. The Volkov states and beyond. Optically induced band structure and exceptional solutions.

Volkov states [1935]. One of the main tool for long...

$$[\gamma_\mu (i\partial - \varepsilon A_{rad})^\mu - \kappa]\left|\Psi\right\rangle = 0 \quad (\varepsilon\gamma_0 V\left|\Psi\right\rangle)$$

$$A_{rad}(\xi) = e_x A_0 f(\xi)$$

$$\xi = k_\mu x^\mu = \omega(t - z/c)$$

$$\Psi^{(\pm)}_{ps}(x) = \left[1 \pm \frac{\varepsilon(\gamma \cdot k)[\gamma \cdot A(\xi)]}{2k \cdot p}\right] u^{(\pm)}_{ps}$$
$$\times \exp\left[\mp i[p \cdot x + \int I^{(\pm)}_p(\xi) d\xi]\right]$$

$$I^{(\pm)}_p(\xi) = (1/2k \cdot p)[\pm 2\varepsilon p \cdot A(\xi) - \varepsilon^2 A^2(\xi)$$

Wolkow D M, Über eine Klasse von Lösungen der Diracschen Gleichung. *Zeitschrift für Physik* 94, 250-260 (1935). [Application to strong-field and multiphoton processes: from ~1960..]

The Gordon-Volkov states are modulated de Broglie plane waves.

$$\Phi = \Phi_p(\xi)e^{-ip\cdot x}$$

$$\xi = k_\mu x^\mu = \omega(t - z/c)$$

$$-k^2\frac{d^2\Phi_p}{d\xi^2} + 2ik\cdot p\frac{d\Phi_p}{d\xi} + (p^2 - \kappa^2 - 2\varepsilon p\cdot A + \varepsilon^2 A^2)\Phi_p = 0$$

In vacuum:

$$k^2 = 0$$

First-order ordinary differential equation for Φp.

Immediately integrable, yielding the Gordon-Volkov solutions.

In a medium:

$$k^2 = (\omega/c)^2(1 - n_m^2) \neq 0$$

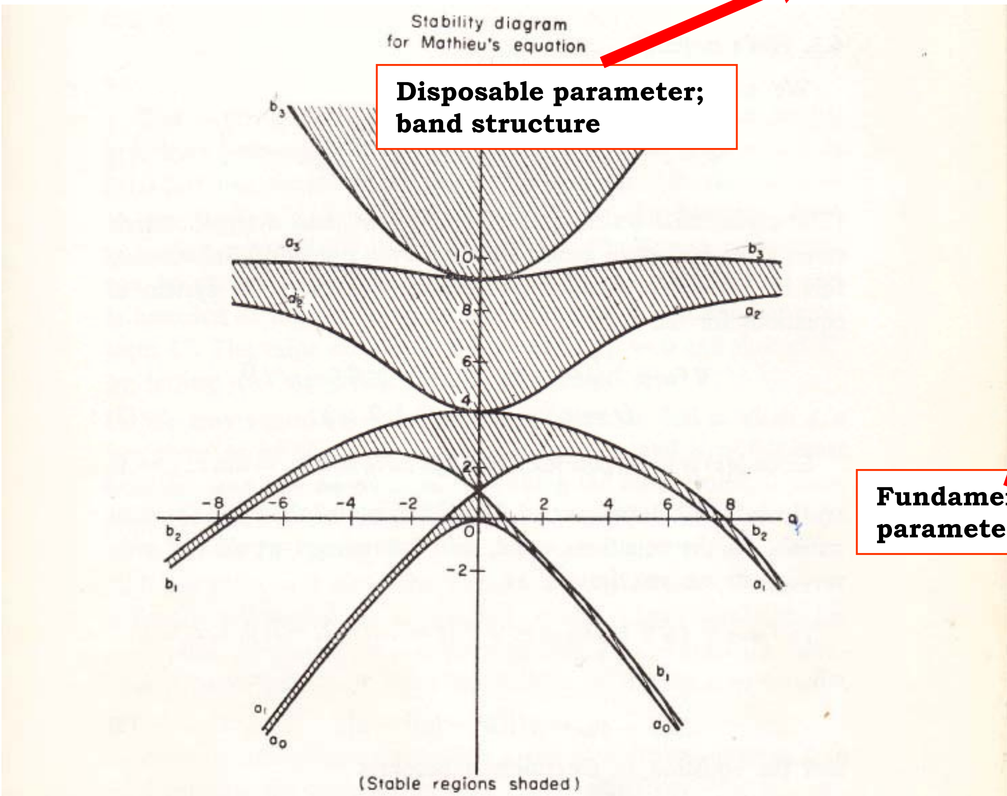
Second-order ordinary differential equation for Φp

E.g. Mathieu–type solutions. FEL. $\xi = k_\mu x^\mu = \omega(t - n_m y/c)$

$w'' + (\lambda - 2h^2 \cos 2z)w = 0$

$\lambda \propto k \cdot p, \quad h^2 \propto eA_0 p_\perp$

Disposable parameter; band structure



Fundamental parameter.

[Figure taken from Arscott F M, *Periodic differential equations* (Pergamon Press, Oxford, 1964) p.123.] . Nikishov & Ritus (1967), Nikishov (1970), Narozhny & Nikishov (1974), Becker (1977), Fedorov, McIver … FEL theories.

Exceptional solutions in plasma. $\xi = k_\mu x^\mu = \omega(t - n_m y / c)$

$$[(i\partial - \varepsilon A)^2 - \kappa^2 - \tfrac{1}{2}\varepsilon\sigma \cdot F]\Psi = 0$$

$$p_x = (n + \tfrac{1}{2})k_p$$

$$\Psi^{(e)}_{p1,2} = e^{i\overline{p}\cdot\overline{x}} e^{-(a/4)\cos\xi} g_n^k(\xi, a\,|+)$$

$$2(k \cdot p)^2 / k_p^4 = \eta_n^{(k)}$$

$$g'' + a\sin 2z(g' + ig) + (\eta - qa\cos 2z)g = 0$$

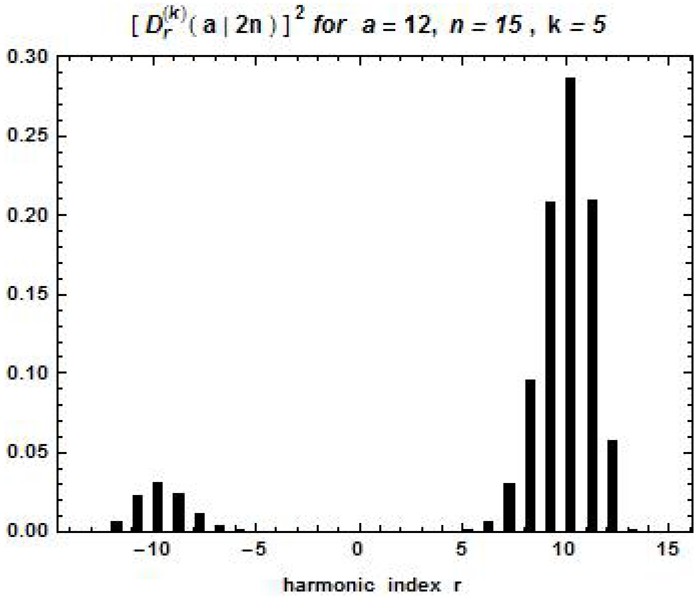
$$2p_x = (q+1)k_p$$

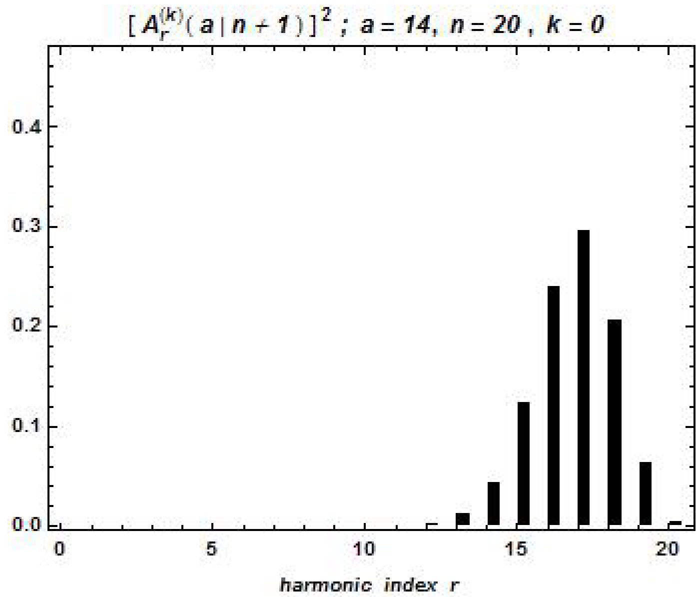
$$\begin{bmatrix} 4(-n+1)^2 & (+1)a & 0 & 0 & 0 \\ (2n-1)a & 4(-n+2)^2 & \cdots & 0 & 0 \\ 0 & (2n-2)a & \cdots & (2n-2)a & 0 \\ 0 & 0 & \cdots & 4(n-1)^2 & (2n-1)a \\ 0 & 0 & 0 & (+1)a & 4n^2 \end{bmatrix} \cdot \begin{bmatrix} D_{-n+1} \\ D_{-n+2} \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix} = \eta_n^{(k)} \cdot \begin{bmatrix} D_{-n+1} \\ D_{-n+2} \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix}$$

$$g = g_n^k(\xi, a\,|+) = \sum_{r=-n+1}^{n} D_r^{(k)}(a\,|\,2n)\exp(-ir\xi)$$

$$a = 4\frac{eF_0\lambda_p}{\hbar\omega_0}$$

Exceptional solutions in a plasma medium.





Dirac

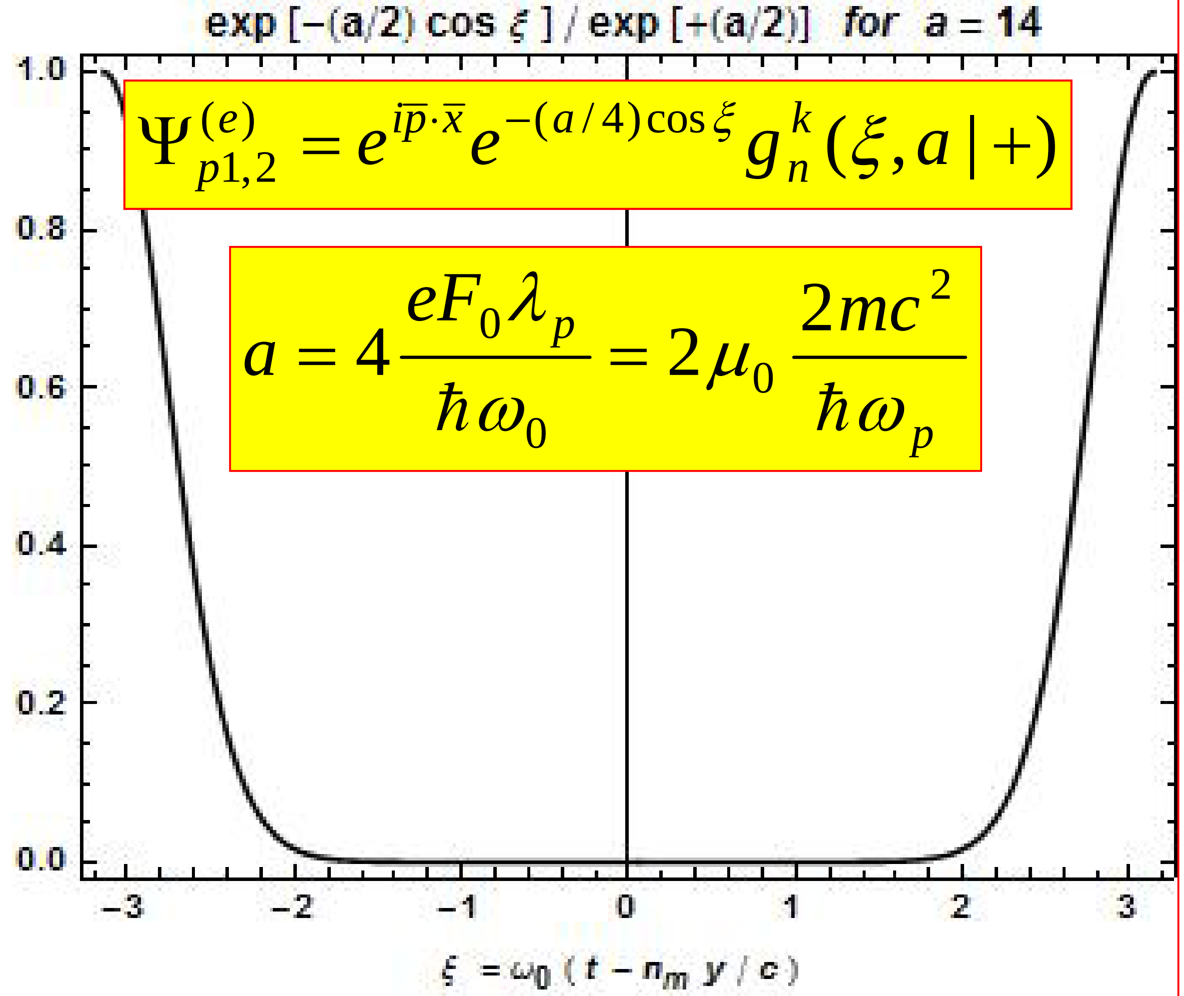
Klein-Gordon

Double peak structure. Single peak structure. Oscillatory spectrum.

[1] S. V. , New exact solutions of the Dirac equation of a charged particle interacting with an electromagnetic plane wave in a medium. *Laser Physics Letters* 10 (2013) 095301, E-print: arXiv:1305.4370 [quant-ph].

[2] S. V., A new class of exact solutions of the Klein-Gordon equation of a charged particle interacting with an electromagnetic plane wave in a medium. *Laser Physics Letters* 11 (2014) 016001, E-print: arXiv:1306.0097 [quant-ph].

Very large contrast charge density modulation.

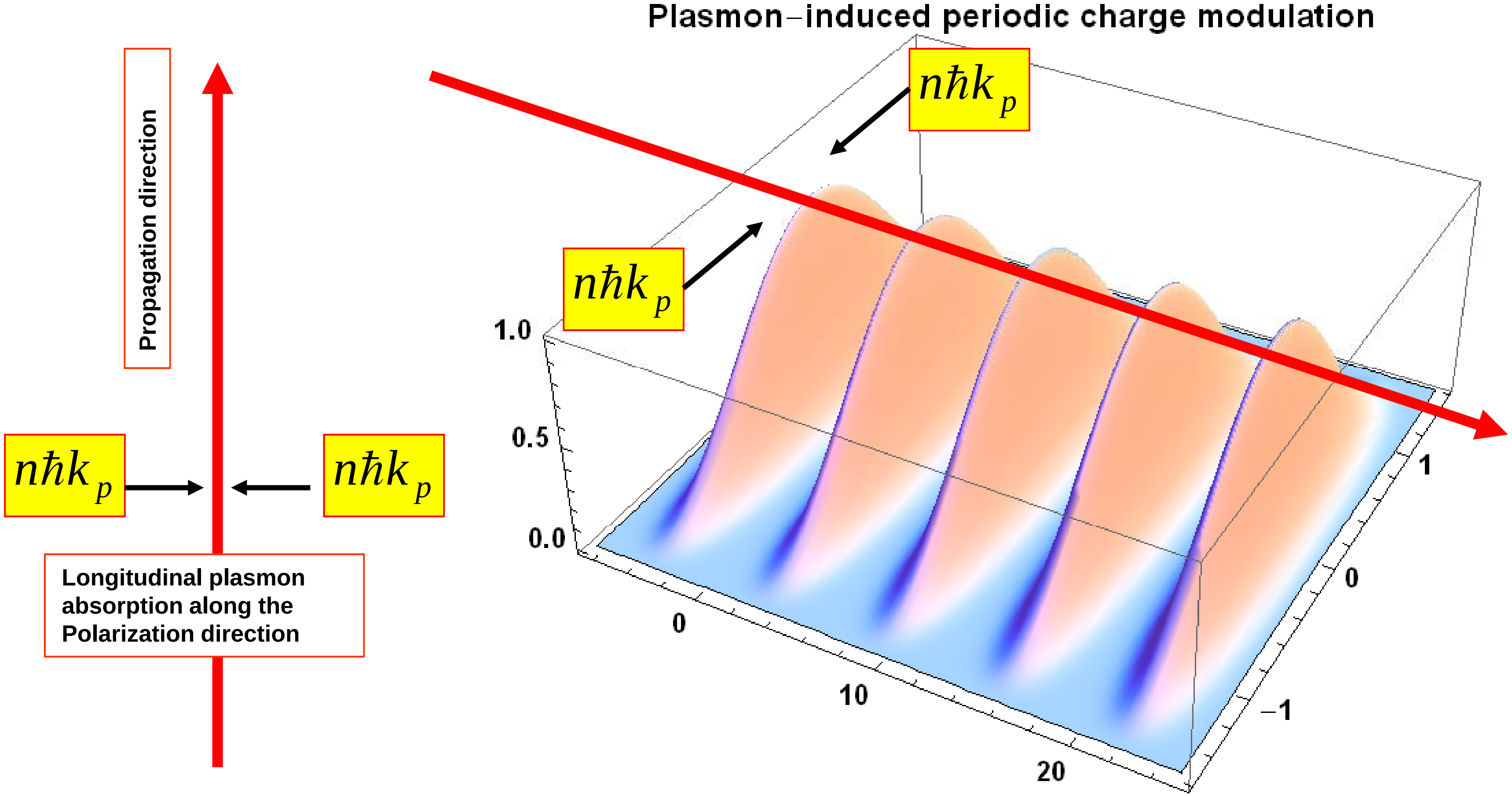


$$\Psi^{(e)}_{p1,2} = e^{i\bar{p}\cdot\bar{x}} e^{-(a/4)\cos\xi} g^{k}_{n}(\xi, a\,|+)$$

$$a = 4\frac{eF_0\lambda_p}{\hbar\omega_0} = 2\mu_0\frac{2mc^2}{\hbar\omega_p}$$

'Void regions' in the centre of the cycle. [This seems to be a 'Quantum bubble'? ...]

(Longitudinal) plasmon absorption along the (transverse) polarization (electric field) direction induces a high – contrast charge modulation along the propagation direction. [Ince polynomials with an exponential envelope]



S. V., *Laser Physics Letters* 10 095301 (2013). S. V. *ibid.* 11 (2014). S. V,. *Nucl. Instr. Meth. in Physics Res.* A 740 (2014) 280-283.

Dirac particle interacting with the quantized radiation field in vacuum. Exact solutions, and considerations on squeezing and ‘aberration’ in quantum phase space of the photon.

Quantized description of nonlinear Compton scattering (HHG) <u>beyond the semiclassical description</u> (1981). The generalization of the Klein–Nishina formula. The effect of <u>depletion</u> of the laser field; e.g. altered kinematics (spectrum) !

The calculation of the nonlinear Compton process was based on the Exact solutions for the 'Dirac electron + quantized e.m. radiation mode' system [1-3]:

$$\psi_{E,\boldsymbol{P}} = \left[1 + g\,\frac{\not{k}\not{\varepsilon}}{2Qk}\left(\hat{a}\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \hat{a}^{+}\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right) \right] u_Q\, e^{i[\boldsymbol{Q}+\boldsymbol{k}g(n)]\cdot\boldsymbol{r}}\, \hat{S}_\theta \hat{D}_\tau \left| n \right\rangle$$

$$\omega'_n = \frac{n\omega_0 + \omega_C \mu_0^2 \Delta}{1 + \left(2\dfrac{n\omega_0}{\omega_C} + \dfrac{\mu_0^2}{2} \right) \sin^2 \dfrac{\theta}{2}}$$

$$\Delta = \frac{n_0 - n}{n_0}$$

‚depletion factor' [1-3]

The generalization of the Klein–Nishina formula (complete depletion of the photon mode):

$$|\, t_{fi}^{(n)} \,|_{av}^2 = \frac{1}{4}\left[\frac{n\omega_0}{\omega'} + \frac{\omega'}{n\omega_0} - 2 + 4(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}')^2 \right] \frac{(nb)^n}{n!} e^{-nb}$$

$$b = \tfrac{1}{2}\mu_0^2 \,|\, \hat{\boldsymbol{k}}' \cdot \boldsymbol{\varepsilon} \,|^2$$

The matrix elements of the squeezing operator between photon number eigenstates. Expression in terms of classical Gegenbauer polynomials.

$$S = \exp(\tfrac{1}{2}\xi a^{+2} - \tfrac{1}{2}\xi^* a^2) \qquad H_{\mathrm{int}} = g(a^{+2}\mathrm{e}^{i\varphi} + a^2\mathrm{e}^{-i\varphi})$$

$$\langle m|S(\xi)|n\rangle = \mathrm{e}^{-\eta/4}(-2\mathrm{e}^{-i\varphi}\tanh|\xi|)^{\lambda}\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})}\sqrt{\frac{m!}{n!}}C_m^{(\lambda+\frac{1}{2})}\left(\frac{1}{\cosh|\xi|}\right) \qquad m \le n, \quad \lambda = \tfrac{1}{2}(n-m)$$

$$\langle m|S(\xi)|n\rangle = \mathrm{e}^{-\eta/4}(2\mathrm{e}^{i\varphi}\tanh|\xi|)^{\alpha}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})}\sqrt{\frac{n!}{m!}}C_n^{(\alpha+\frac{1}{2})}\left(\frac{1}{\cosh|\xi|}\right) \qquad m \ge n, \quad \alpha = \tfrac{1}{2}(m-n)$$

Remark: SU(1,1); SL(2,R) group. Lorentz group in 2 + 1 dimensions (osc. repr.).

$$K_+ = \tfrac{1}{2}(a^+)^2, \quad K_- = \tfrac{1}{2}(a)^2, \quad K_0 = \tfrac{1}{4}(aa^+ + a^+a), \quad [K_0, K_\pm] = \pm K_\pm \quad [K_-, K_+] = 2K_0$$

General transformation:

$$U(g) = \exp(-itK_0)S(\xi), \qquad U(g_1)U(g_2) = U(g), \qquad g = g_1 \circ g_2$$

Varró S, Coherent and incoherent superposition of transition matrix elements of the squeezing operator. ***New Journal of Physics*** **24, 053035 (2021).** ***E-print*****: arXiv: 2112.08430 [quant-ph] . 29th Int. Las. Phys. (LPHYS'21). J. Phys. Conf. Ser. 2249 012013 (2022).**

Transformation properties of the ‘disentengling operators’

Exact solutions for the ‘Dirac electron + quantized e.m. radiation mode’ system in case of two co-propagating circularly polarized quantized modes.

$$\psi_{E,\boldsymbol{P}} = \left[1 + \sigma \frac{\not{k}}{2Qk}(\not{A}_+ + \not{A}_-)\right] u_Q e^{i[\boldsymbol{Q}+\boldsymbol{k}g(n)]\cdot\boldsymbol{r}} U_g \hat{D}_1 \hat{D}_2 \left|n\right\rangle_1 \left|n+n_0\right\rangle_2$$

$$K_+ = a^+ b^+ \qquad K_- = ab \qquad K_0 = \tfrac{1}{2}(a^+ a + b^+ b + 1) \qquad \hat{C} = \tfrac{1}{4}(a^+ a - b^+ b)^2 - \tfrac{1}{4}$$

$$U(g) = \exp(-itK_0)S(\xi), \qquad UK_i U^+ = M_{ij} K_j, \qquad n_1 - n_2 = const.$$

Remark: SU(1,1); SL(2,R) group. Lorentz group in 2 + 1 dimensions (osc. repr.).

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 \\ -\sinh\beta & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}.$$

The expansion coefficients of the Dirac stationary states, i.e. The photon statistics is, at the same time, are the transition probabilities in the parametric down-conversion; generation of entangled photon pairs.

Photon statistics of the Dirac solution is essentially the same as the Transition probabilities in the parametric down-conversion; generation of entangled photon pairs. Connection with the Zernike polynomials.

$$\langle m|U(g)|n\rangle = (1-\rho^2)^{(\nu+1)/2} e^{i(m-n)\varphi} \rho^{|m-n|} \sqrt{\frac{n!(m+\nu)!}{m!(n+\nu)!}} P_n^{(|m-n|,\nu)}(1-2\rho^2) \qquad m \geq n$$

$$\langle m|U(g)|n\rangle = (1-\rho^2)^{(\nu+1)/2} e^{i(m-n)\varphi} (-\rho)^{|m-n|} \sqrt{\frac{m!(n+\nu)!}{n!(m+\nu)!}} P_m^{(|m-n|,\nu)}(1-2\rho^2) \qquad m \leq n$$

$$|n\rangle \leftrightarrow |n\rangle_1 |n+\nu\rangle_2, \qquad \rho = \tanh|\xi|$$

Here $P_n^{(\alpha,\beta)}(x)$ **are Jacobi polinomials. The case** $\nu = |n_2 - n_1| = 0$ **gives the**

Zernike polynomials: $R_{n+m}^{|m-n|}(\rho) = (-1)^n \rho^{|m-n|} P_n^{(|m-n,0)}(1-2\rho^2)$

[1] Varró S : Regular phase operator and SU(1,1) coherent states of the harmonic oscillator. *Physica Scripta* 90 (7) (2015) 074053. [2] Varró S, Coherent and incoherent superposition of transition matrix elements of the squeezing operator. *New Journal of Physics* 24, 053035 (2022).

Transition probabilities in the parametric down-conversion; generation of entangled photon pairs. Connection with the Zernike polynomials.

[We used the same normal ordering as in Ref. [2], but now with the SU(1,1) generators related to the quantum phase operator problem found in Ref. [1], and received similar hypergeometric functions as in Ref. [2]. These hypergeometric functions are in fact Jacobi polynomials.]

$$\langle m|U(g)|n\rangle = (1-\rho^2)^{(\nu+1)/2} e^{i(m-n)\varphi} \rho^{|m-n|} \sqrt{\frac{n!(m+\nu)!}{m!(n+\nu)!}} P_n^{(|m-n|,\nu)}(1-2\rho^2) \qquad m \geq n$$

$$\langle m|U(g)|n\rangle = (1-\rho^2)^{(\nu+1)/2} e^{i(m-n)\varphi} (-\rho)^{|m-n|} \sqrt{\frac{m!(n+\nu)!}{n!(m+\nu)!}} P_m^{(|m-n|,\nu)}(1-2\rho^2) \qquad m \leq n$$

$$|n\rangle \leftrightarrow |n\rangle_1 |n+\nu\rangle_2, \qquad \rho = \tanh|\xi|$$

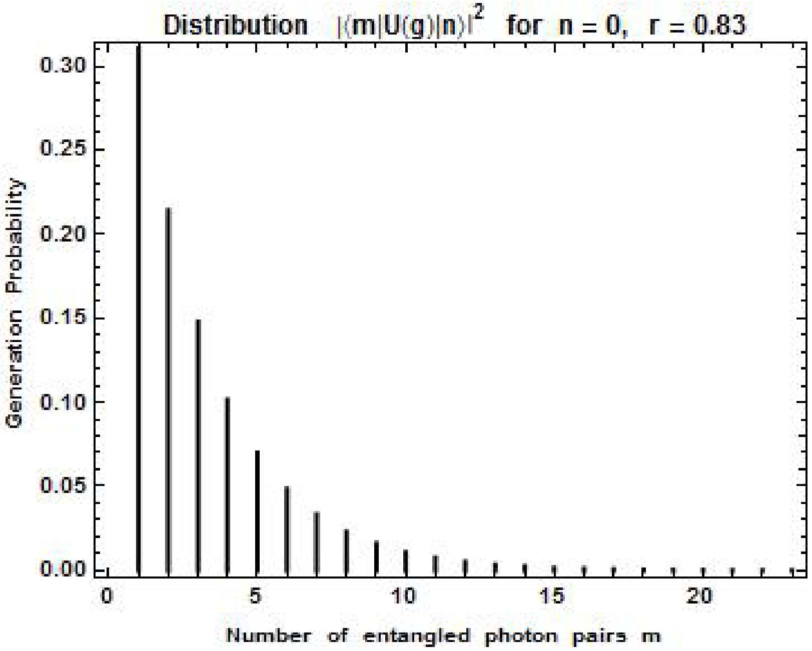
Here $P_n^{(\alpha,\beta)}(x)$ **are Jacobi polinomials. The case** $\nu = |n_2 - n_1| = 0$ **gives the**

Zernike polynomials: $R_{n+m}^{|m-n|}(\rho) = (-1)^n \rho^{|m-n|} P_n^{(|m-n,0)}(1-2\rho^2)$

[1] Varró S : Regular phase operator and SU(1,1) coherent states of the harmonic oscillator. *Physica Scripta* 90 (7) (2015) 074053. [2] Varró S, Coherent and incoherent superposition of transition matrix elements of the squeezing operator. *New Journal of Physics* 24, 053035 (2022).

Illustrations of the photon number distributions in parametric down-conversion.

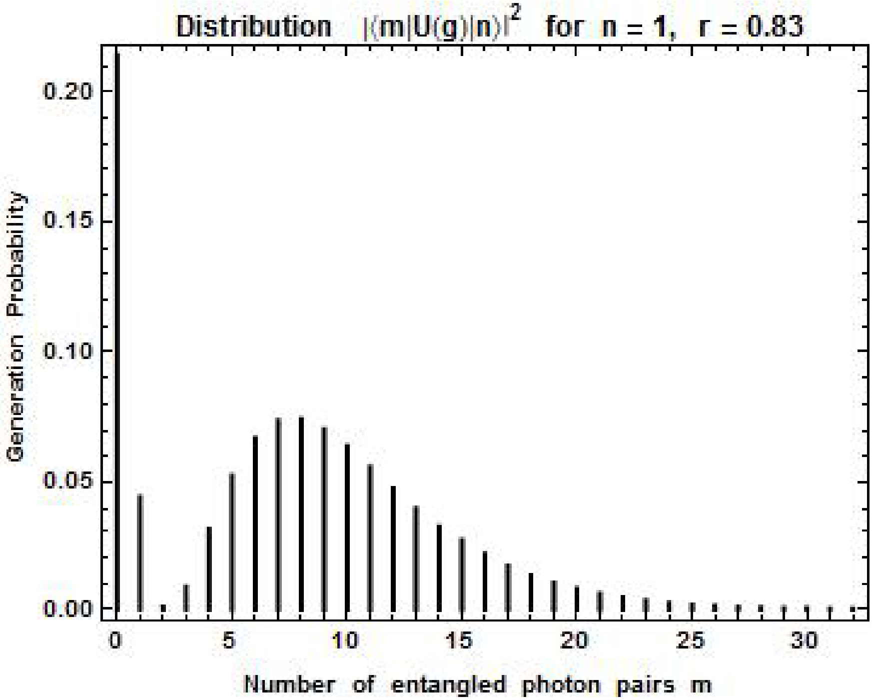
Special case ν=0, i.e. $n_2 = n_1 = n$ (~'Zernike polynomial distributions'). n = 0, 1.



$$\langle m|U(g)|0\rangle = \sqrt{1-\rho^2}\,(\rho e^{i\phi})^m$$

Yield for n=0 Bose distribution:

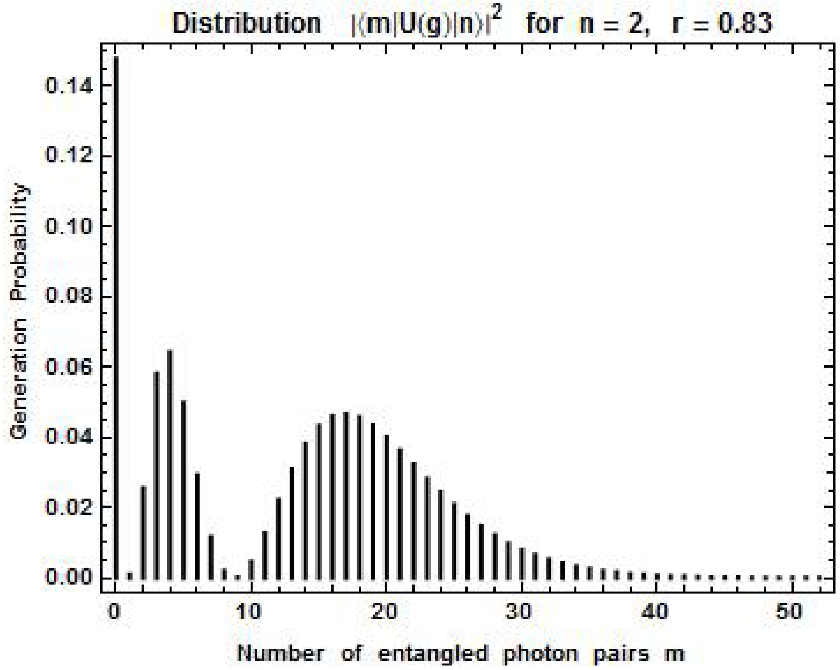
$$(1-\rho^2)\rho^{2m}.$$

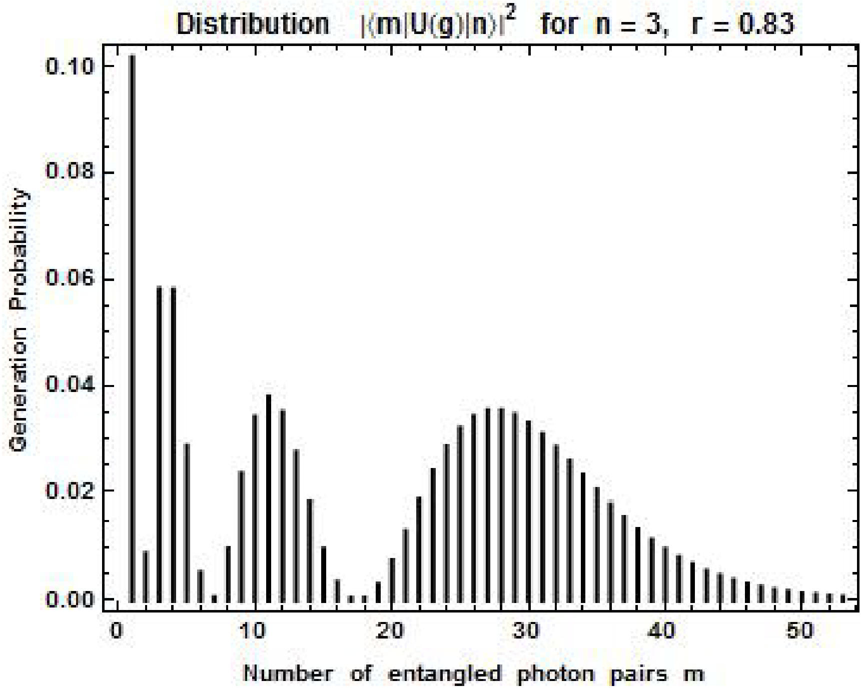


The left figure refers to the spontaneous process. The right figure refers to the initial n = 1.

Illustrations of the photon number distributions in parametric down-conversion.

Special case ν=0, i.e. $n_2 = n_1 = n$ (~'Zernike polynomial distributions'). n = 0, 1.





The left figure refers to the initial n = 2. The right figure refers to the initial n = 3.

Born M and Wolf E, Principles of Optics (Cambridge University Press, Cambridge, 2002).

Let R denote the radius $CP_1^\star$, of the Gaussian reference sphere and let s be the distance between Q and an arbitrary point P in the region of the image. The disturbance at Q is represented by $Ae^{ik(\Phi-R)}/R$, where A/R is the amplitude at Q. According to the Huygens–Fresnel principle the disturbance at P is given by

$$\sim e^{-(\boldsymbol{r}-\boldsymbol{r}_P)^2/\sigma^2}$$

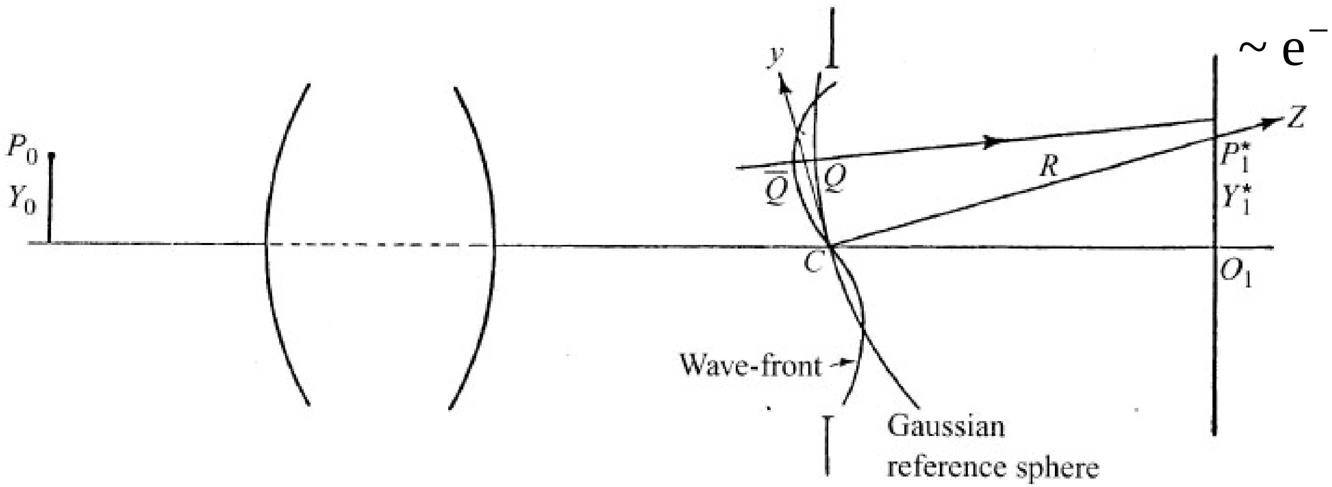


Fig. 9.1 Choice of reference system and notation.

* N. G. van Kampen, *Physica*, **14** (1949), 575; *ibid.*, **16** (1950), 817; *ibid.* **25** (1958), 437.

Aberration phase

$$U(P) = -\frac{i}{\lambda}\frac{Ae^{-ikR}}{R}\iint \frac{e^{ik[\Phi+s]}}{s}\,dS, \qquad (1)$$

Born M and Wolf E, Principles of Optics (Cambridge University Press, Cambridge, 2002).

Table 9.2. *Representation of the primary aberrations.*

Type of aberration	l	n	m	Representation in form (4)	Representation in form (6)
Spherical aberration	0	4	0	$A'_{040}\rho^4$	$\frac{1}{\sqrt{2}} A_{040} R_4^0(\rho) = \frac{1}{\sqrt{2}} A_{040}(6\rho^4 - 6\rho^2 + 1)$
Coma	0	3	1	$A'_{031}\rho^3 \cos\theta$	$\mathrm{A}_{031} R_3^1 \cos\theta = A_{031}(3\rho^3 - 2\rho)\cos\theta$
Astigmatism	0	2	2	$A'_{022}\rho^2 \cos^2\theta$	$A_{022} R_2^2 \cos 2\theta = A_{022}\rho^2(2\cos^2\theta - 1)$
Curvature of field	1	2	0	$A'_{120}\rho^2$	$\frac{1}{\sqrt{2}} A_{120} R_2^0(\rho) = \frac{1}{\sqrt{2}} A_{120}(2\rho^2 - 1)$
Distortion	1	1	1	$A'_{111}\rho\cos\theta$	$A_{111} R_1^1(\rho)\cos\theta = A_{111}\rho\cos\theta$

Born M and Wolf E, Principles of Optics (Cambridge University Press, Cambridge, 2002).

Some graphical illustrations of the Zernike polynomials. [Niu K and Tian C (2022)]

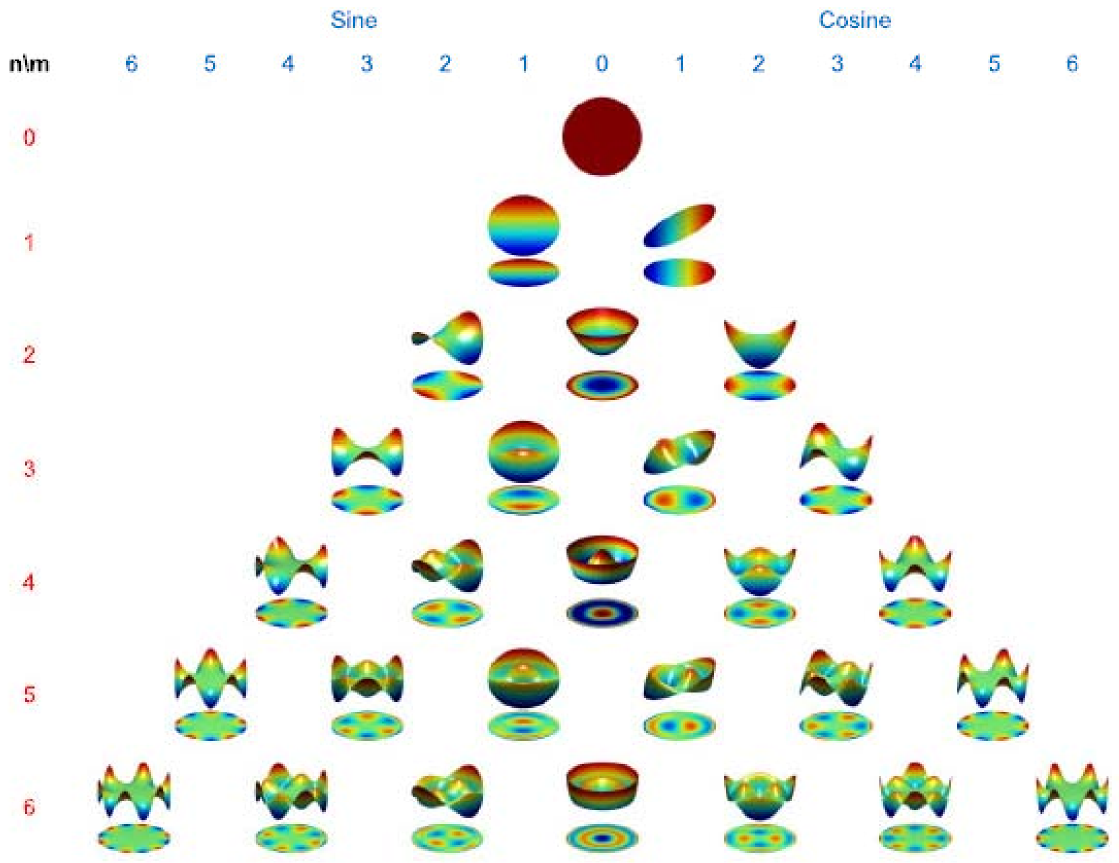
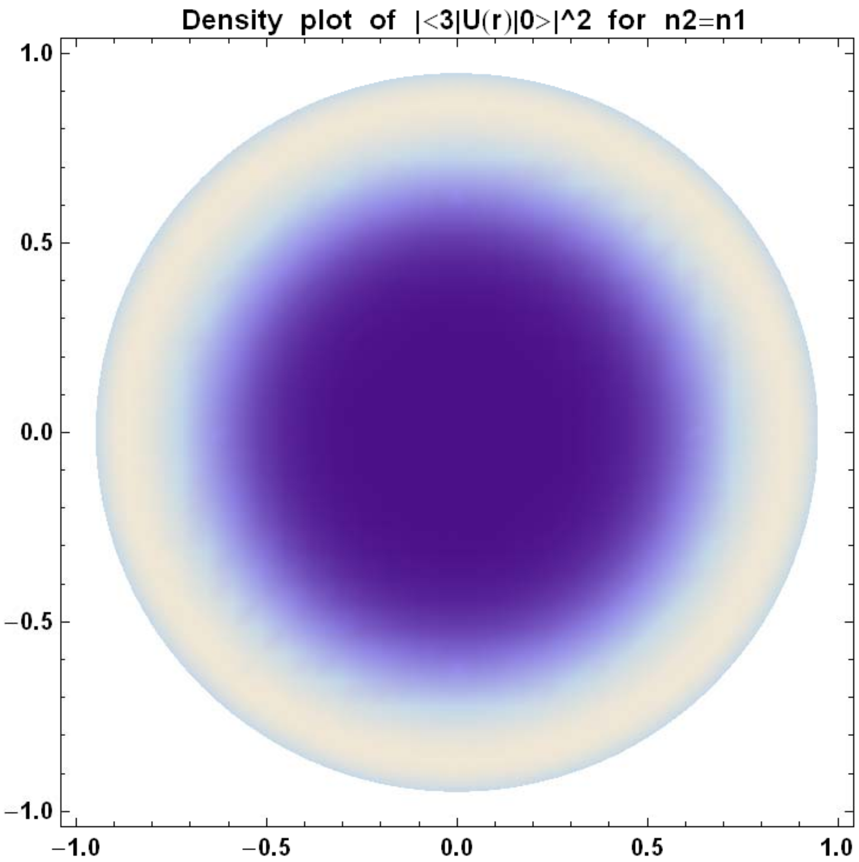


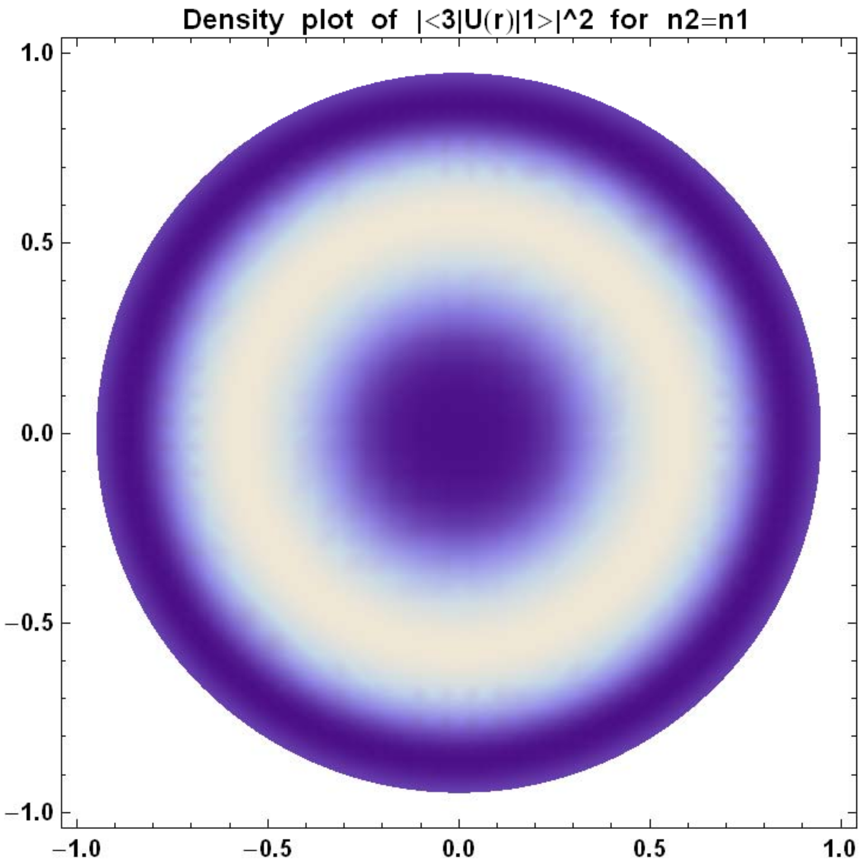
Figure 7. Pyramid of the non-normalized Zernike circle polynomials up to the sixth degree under the Noll indexing scheme.

Figure copied from [3] Niu K and Tian C 2022 Zernike polynomials and their applications. Journal of Optics 24, 123001 (2022).

Aberration of down-conversion probability in the generation of 3 photon pairs.

Special case ν=0, i.e. $n_2 = n_1 = n$ (~'Zernike polynomial distributions'). n = 0, 1.

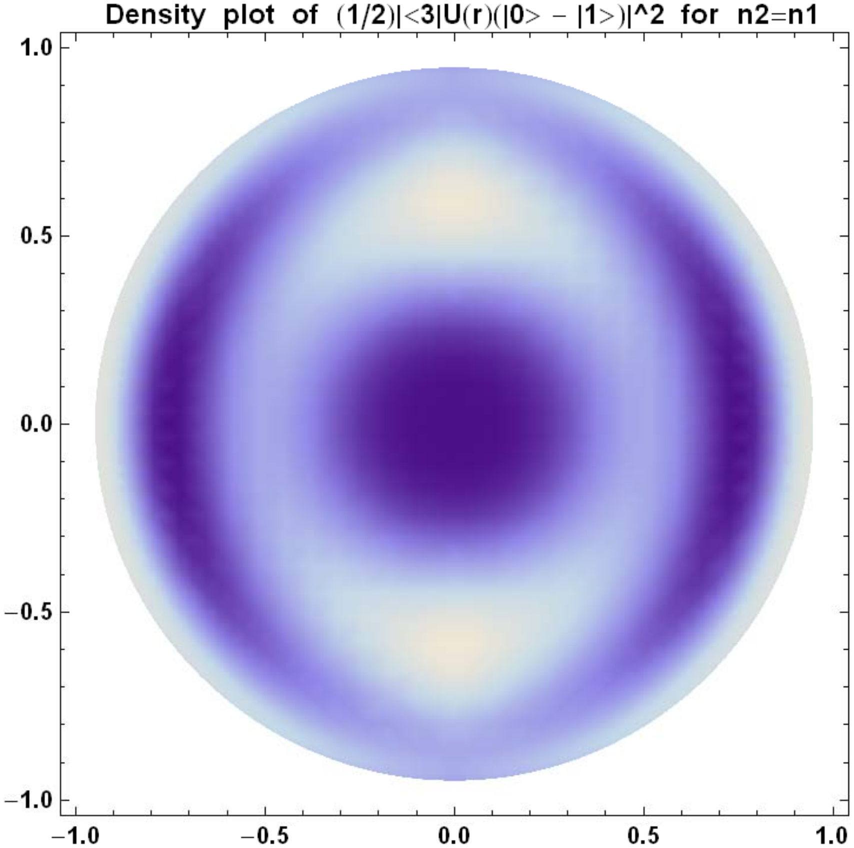


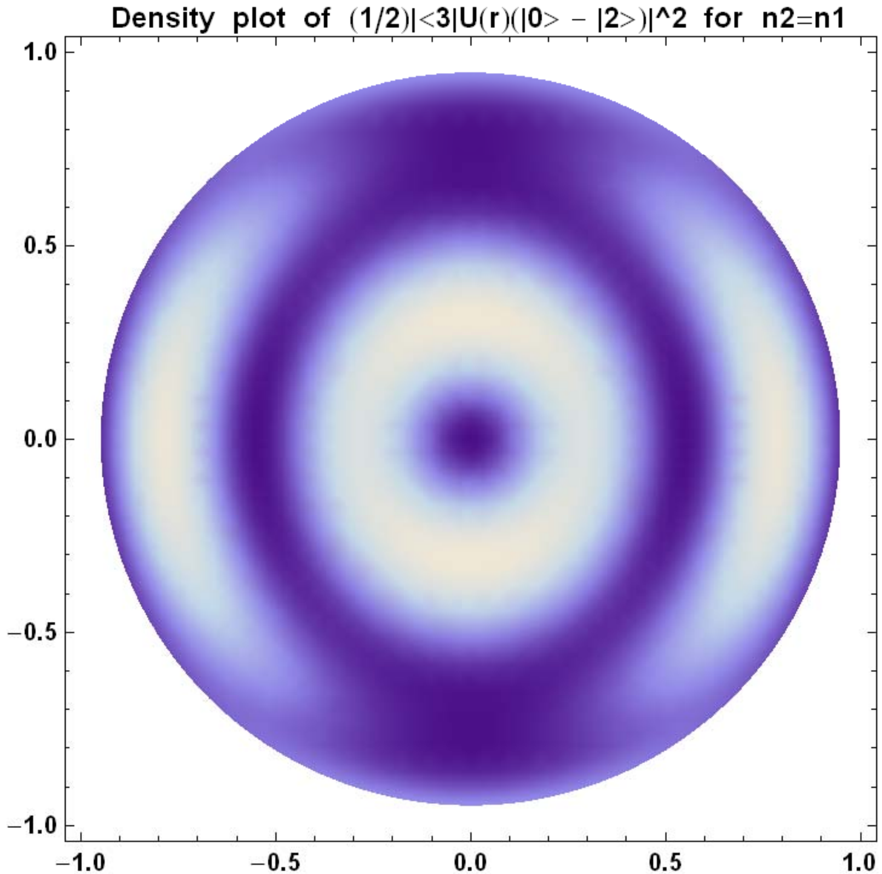


The left figure refers to the spontaneous process. The right figure refers to the initial n = 1.

Aberration of down-conversion probability in the generation of 3 photon pairs.

Special case ν=0, i.e. $n_2 = n_1 = n$ (~'Zernike distributions'). Superposition:n = 0, 1,

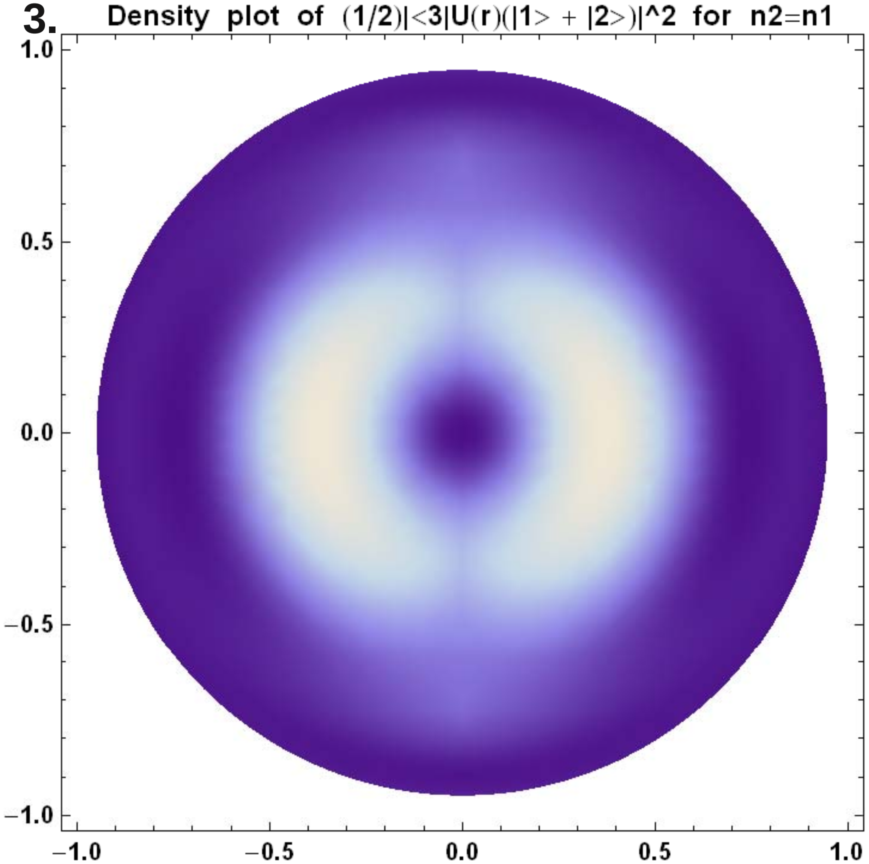


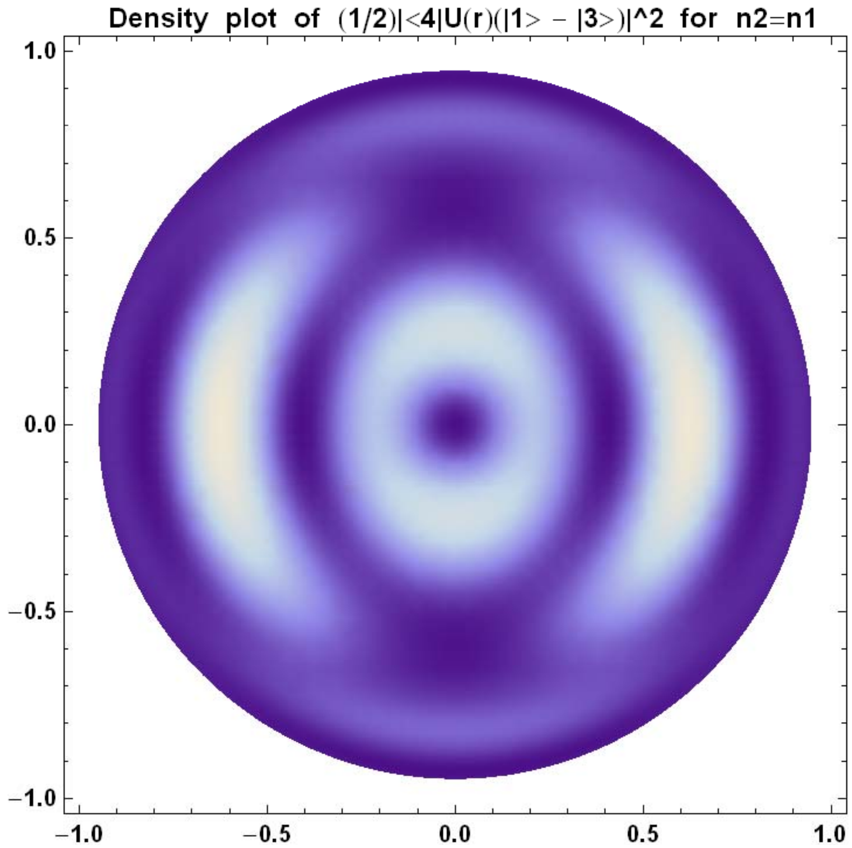


The left figure refers to the spontaneous process. The right figure refers to the initial n = 1.

Aberration of down-conversion probability in the generation of 3 photon pairs.

Special case ν=0, i.e. $n_2 = n_1 = n$ (~'Zernike distributions'). Superposition:n = 1, 2, 3.





The left figure refers to the spontaneous process. The right figure refers to the initial n = 1.

Summary

• Relativistic dynamics of a charged particle in a strong laser field in vacuum. Analogies between the „figure–8–motion" and the Kepler–Coulomb problem.

• Relativistic motion of a charged particle interacting with a strong laser field in a plasma. Comparison of the high-harmonic spectrum in vacuum and in an underdense plasma.

• Dirac particle in external electromagnetic plane waves in vacuum or in a plasma. Optically induced band structure and exceptional solutions.

• Dirac particle interacting with quantized radiation fields in vacuum. Exact solutions, and considerations on squeezing and 'aberration' in quantum phase space of the photon.

• Summary and outlook.

Appendix.

Interaction of relativistic charged particles with strong laser fields in vacuum or in a plasma environment.

Sándor Varró
Wigner FK, ELKH, Budapest; ELI-ALPS, Szeged

We present a comparative study of the exact solutions of both the classical and the quantum mechanical equations of motion of charged particles interacting with a laser field of arbitrary intensity. The well-known exact solutions in vacuum are represented by the figure-8 motion and the Volkov states. Here we point out the surprising mathematical connection between the figure-8 motion and the classical Coulomb-Kepler trajectories [1]. In an underdense plasma the particle motion depends on the stability charts of Mathieu or Hill equations. There are also exceptional solutions labelled by two integer numbers, which correspond to discrete particle momentum and energy spectra [2], even in the considered external field approximation for the laser field. These solutions correspond to very high contrast, propagating charge density modulations, which may perhaps be relevant for laser acceleration of particles. By going beyond the external field approximation, the Dirac (Klein-Gordon or Schrödinger) equation of the joint system of a charged particle interacting with quantized radiation modes in vacuum can also be solved exactly in various cases. The photon part of the single-mode stationary states are squeezed (coherent) number states [3], whose photon statistics has been determined quite recently in terms of Gegenbauer polynomials [4]. On the basis of the recently found exact solutions of the equation of motion, finally we discuss the interaction of a charged Dirac particle interacting with two co-propagating circularly polarized quantized modes. In this analysis entangled photon pairs naturally appear, and we will show that in a special case the derived probability amplitudes of the distribution of these pairs reduce to the Zernike functions, which are well-kown in the classical theory of aberration in optical imaging [5]. Thus, it is justified to introduce the concepts of aberration and Zernike moments on quantum phase space, which may give a new aspect in the non-perturbative theoretical study of high-order and parametric processes in laser-matter interactions.

References.

[1] Varró S; Intensity effects and absolute phase effects in nonlinear laser-matter interactions. In Laser Pulse Phenomena and Applications. (Duarte F J (Ed.); Rijeka, InTech, 2010). Ch. 12, pp 243-266.
[2] Varró S, New exact solutions of the Klein-Gordon and Dirac equations of a charged particle propagating in a strong laser field in an underdense plasma. Nuclear Instruments and Methods in Physics Research A 740 (2014) 280-283.
[3] Varró S, Quantum optical aspects of high-harmonic generation. Photonics 8, 269 (2021).
[4] Varró S, Coherent and incoherent superposition of transition matrix elements of the squeezing operator. New Journal of Physics 24, 053035 (2022).
[5] Born M and Wolf E, Principles of Optics (Cambridge University Press, Cambridge, 2002).

Classical considerations for a relativistic free electron. In vacuum, the argument of the e.m. plane wave at the electron's position is proportional to the proper time of the electron. The „figure–8 motion".

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{e}_x F_x(\theta)+\boldsymbol{e}_y F_y(\theta)$$

$$\boldsymbol{B}(\boldsymbol{r},t)=\boldsymbol{n}\times\boldsymbol{E}(\theta)$$

$$\theta\equiv t-\boldsymbol{n}\cdot\boldsymbol{r}/c$$

$$m_0 du_\mu/d\tau=(e/c)F_{\mu\nu}u^\nu$$

$$\frac{d}{d\tau}\left[t-\frac{1}{c}\boldsymbol{n}\cdot\boldsymbol{r}(t)\right]=const.=\alpha$$

$$\alpha=1$$

Along the polarization x-direction one receives formally a Newton equation in dipole approximation. By solving for the x- and y-components, the z-component (the longitudinal component, driven by the v × B force term) also satifies a Newton-like equation in dipole approcimation (mere τ-dependence).

$$m_0\frac{d^2x}{d\tau^2}=eF_x(\tau),$$

$$m_0\frac{d^2z}{d\tau^2}=\frac{e}{c}\left[\frac{dx(\tau)}{d\tau}F_x(\tau)+\frac{dy(\tau)}{d\tau}F_y(\tau)\right]$$

$$F_x(\tau)=F_{0x}\cos\omega_0\tau$$

$$(\omega_0/c)x(\tau)=\mu_{0x}(\cos\omega_0\tau-1)$$

$$\mu_{0x}=|e|F_{0x}/m_0c\omega_0$$

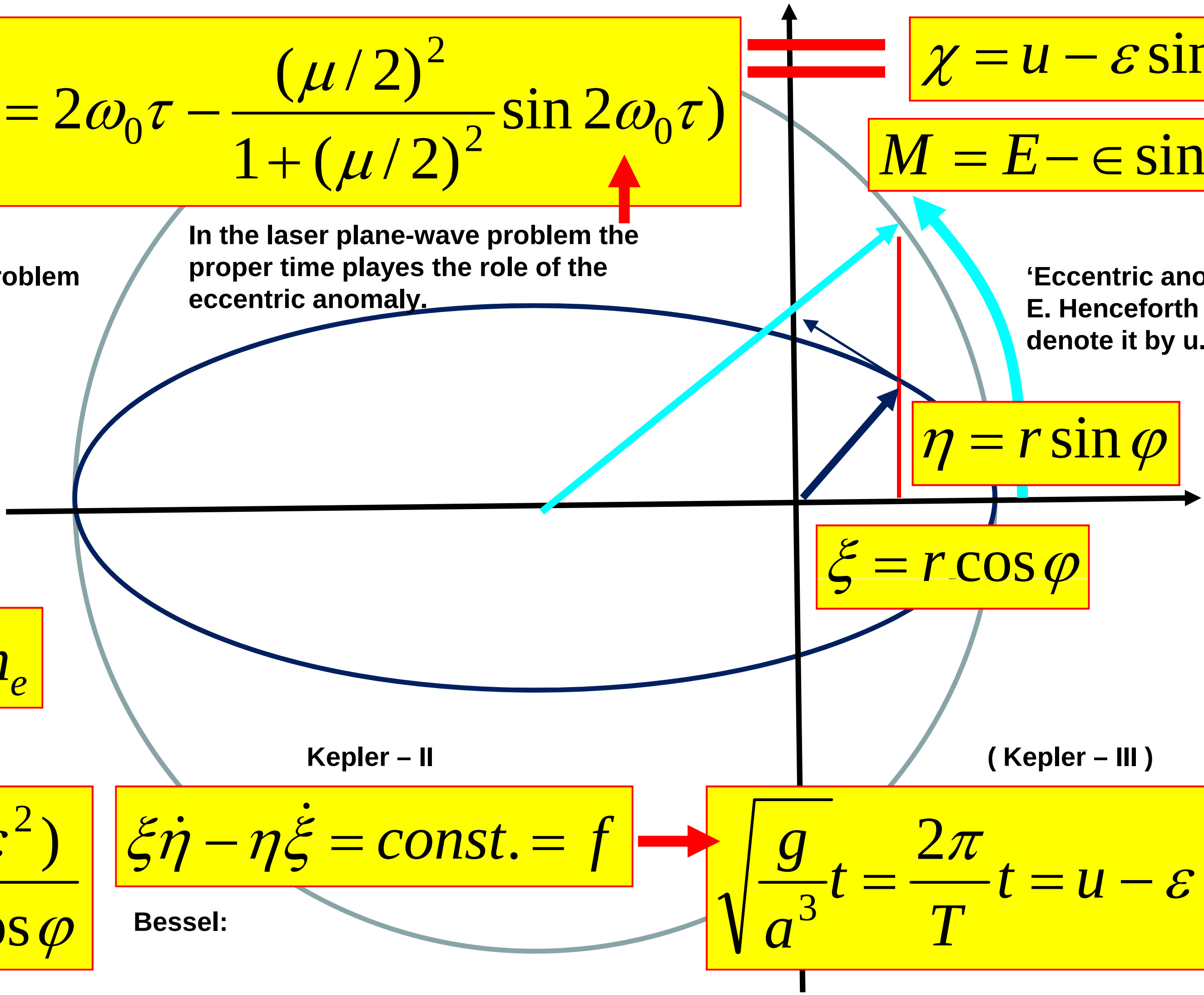
$$(\omega_0/c)z(\tau)=(\mu_{0x}^2/8)(2\omega_0\tau-\sin 2\omega_0\tau)$$

Mathematical connection between the „figure–8–motion” in a plane wave electromagnetic radiation and the Kepler–Coulomb central force problem. 1.

$$\frac{2\omega_0 t}{1+(\mu/2)^2} = 2\omega_0\tau - \frac{(\mu/2)^2}{1+(\mu/2)^2}\sin 2\omega_0\tau)$$

$$\chi = u - \varepsilon \sin u$$

$$M = E - \in \sin E$$



In the laser plane-wave problem the proper time playes the role of the eccentric anomaly.

‘Eccentric anomaly’ E. Henceforth we denote it by u.

$$\eta = r\sin\varphi$$

$$\xi = r\cos\varphi$$

Kepler – Coulomb problem

$$\ddot{\boldsymbol{r}} = -\frac{g}{r^3}\boldsymbol{r}$$

$$g = GM$$

$$g = Ze^2/m_e$$

Kepler – I

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\varphi}$$

Kepler – II

$$\xi\dot{\eta} - \eta\dot{\xi} = const. = f$$

Bessel:

(Kepler – III)

$$\sqrt{\frac{g}{a^3}}t = \frac{2\pi}{T}t = u - \varepsilon \sin u$$

Laser field in a homogeneous underdense plasma: Lánczos-Proca vector boson. „Massive photon"

$$k^2 = (\omega_0 / c)^2 (1 - n_m^2) \neq 0$$

$$n_m^2 = \varepsilon_m = 1 - \omega_p^2 / \omega^2$$

$$\omega_p^2 = \frac{4\pi e^2 n_e}{m_e} \rightarrow \mu = \hbar \omega_p / c^2$$

Effective mass [„Higgs-mechanism"]

$$L(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\mu^2 A_\nu A^\nu$$

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$$

Proca equation:

$$\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = 0$$

$$(\partial^2 + \mu^2) A_\nu = 0$$

(3 polarization!)

Lánczos C, Die tensoranalytischen Beziehungen der Diracschen Gleichung. *Zeitschrift für Physik* 57, 447 (1929).
Proca A, Sur la théorie ondulatoire desélectrons positifs et négatifs. J. Phys. Radium 7, 347 (1936).

Gordon's solutions [1927]

Der Comptoneffekt nach der Schrödingerschen Theorie.

Von **W. Gordon** in Berlin.

(Eingegangen am 29. September 1926.)

Die beim Comptoneffekt ausgestrahlten Frequenzen und Intensitäten werden nach der Schrödingerschen Theorie berechnet. Die quantentheoretischen Größen ergeben sich als geometrische Mittel aus den klassischen Größen des Anfangs- und Endzustandes des Prozesses.

1. Aufstellung der Differentialgleichung für ψ. Heisenberg und Schrödinger haben Methoden angegeben zur Bestimmung der Quantenfrequenzen und Intensitäten. Der Comptoneffekt ist bereits von Dirac[1]) nach der Heisenbergschen Methode gerechnet worden. Hier soll dasselbe Problem nach Schrödinger behandelt werden. Das Verfahren von Schrödinger hat den Vorzug, sich der gebräuchlichen mathematischen Hilfsmittel zu bedienen. Es beruht auf der Ermittlung einer Größe ψ, die für ein einzelnes Elektron eine Funktion der kartesischen Raumkoordinaten x_1, x_2, x_3 und der Zeit t ist. Schrödinger hat zwei Regeln aufgestellt zur Gewinnung der linearen partiellen Differentialgleichung zweiter Ordnung, der ψ zu genügen hat. Beide

Gordon W, Der Comptoneffekt nach der Schrödingerschen Theorie. *Zeitschrift für Physik* 40, 117-133 (1927). [Application to strong-field: From ~1960..]

Volkov's solutions [1935]

Über eine Klasse von Lösungen der Diracschen Gleichung.

Von **D. M. Wolkow** in Leningrad.

(Eingegangen am 12. Februar 1935.)

1. Der Fall eines sinusoidalen Feldes. — 2. Lösung für den Fall, daß das äußere Feld aus polarisierten, in einer bestimmten Richtung fortschreitenden Wellen besteht, die ein *abzählbares* Spektrum nach Frequenz und Anfangsphasen haben.

1. Der Fall des sinusoidalen Feldes.

Es sei das skalare Potential des auf das relativistische Quantenelektron wirkenden äußeren Feldes gleich Null, das Vektorpotential sei

$$A = a \cos 2\pi\nu\left[t - \frac{n x}{c} + \alpha\right] = a \cos\varphi \ \text{ mit } \ \varphi = 2\pi\nu\left[t - \frac{n x}{c} + \alpha\right],$$

ν ist hier eine konstante Zahl (die Frequenz), t die Zeit, c die Lichtgeschwindigkeit, n ein Einheitsvektor (der die Richtung der Ausbreitung einer dem gegebenen A zugeordneten elektromagnetischen Welle anzeigt); x bedeutet den Vektor, der vom Anfangspunkt des fest gewählten rechtwinkligen Cartesischen Koordinatensystems nach dem veränderlichen Punkt geht,

Wolkow D M, Über eine Klasse von Lösungen der Diracschen Gleichung. *Zeitschrift für Physik* 94, 250-260 (1935). [Application to strong-field: From ~1960..]

Gordon-Volkov states (1927, 1935): Exact solutions of the Klein–Gordon and Dirac equations of an electron in an arbitrary intense 'laser field' propagating in vacuum. After ~ 80 years; the only new exact, closed form solutions for the 'monochromatic problem' in a medium [S. V. (2013, 2014)].

Der Comptoneffekt nach der Schrödingerschen Theorie.

Von W. Gordon in Berlin.

(Eingegangen am 29. September 1926.)

Über eine Klasse von Lösungen der Diracschen Gleichung.

Von D. M. Wolkow in Leningrad.

(Eingegangen am 12. Februar 1935.)

$$e^{-iz\sin\omega_0 t} = \sum_n J_n(z) e^{-in\omega_0 t}$$

$$e^{\frac{i}{\hbar}(E_f - E_i)t - in\omega_0 t} = e^{\frac{i}{\hbar}(E_f - E_i - n\hbar\omega_0)t}$$

IOP PUBLISHING

Laser Phys. Lett. **10** (2013) 095301 (13pp)

LETTER

New exact solutions of the Dirac equation of a charged particle interacting with an electromagnetic plane wave in a medium

Sándor Varró

Nuclear Instruments and Methods in Physics Research A 740 (2014) 280–283

Contents lists available at ScienceDirect

Nuclear Instruments and Methods in Physics Research A

journal homepage: www.elsevier.com/locate/nima

ELSEVIER

New exact solutions of the Dirac and Klein–Gordon equation of a charged particle propagating in a strong laser field in an underdense plasma

Sándor Varró

IOP Publishing | Astro Ltd Laser Physics Letters

Laser Phys. Lett. **11** (2014) 016001 (14pp) doi:10.1088/1612-2011/11/1/016001

Letter

A new class of exact solutions of the Klein–Gordon equation of a charged particle interacting with an electromagnetic plane wave in a medium

Sándor Varró

Diagonalization: elimination of the p.A and A² terms. The appearance of the „quantized space-translated potential”.

$$\left|\Psi(t)\right\rangle = \hat{S}\hat{D}\left|\Phi_{SD}(t)\right\rangle$$

$$\hat{S} = \Pi_k \hat{S}_k(\theta_k)$$

$$\hat{D} = \Pi_k \hat{D}_k(\sigma_k)$$

$$\hat{S}_k(\theta_k) = \exp[\tfrac{1}{2}\theta_k(\hat{a}_k^2 - \hat{a}_k^{+2})]$$

$$\theta_k = \tfrac{1}{4}\log(1+2\beta_k)$$

(**squeezing transformation**)

$$\hat{D}_k(\sigma_k) = \exp[\sigma_k(\hat{a}_k^+ - \hat{a}_k)]$$

$$\sigma_k = -\sqrt{\beta_k / m\hbar\omega_k}\, e^{-3\theta_k}(\hat{\boldsymbol{p}}\cdot\boldsymbol{\varepsilon}_k)$$

(**displacement** op.)

„quantized space-translated potential”

$$\left[\frac{\hat{\boldsymbol{p}}^2}{2m} + V(\boldsymbol{r}+\hat{\boldsymbol{\alpha}}) + \tilde{H}_{rad} + \hat{D}^+\hat{S}^+\hat{M}\hat{S}\hat{D}\right]\left|\Phi_{SD}(t)\right\rangle = i\hbar\partial_t\left|\Phi_{SD}(t)\right\rangle$$

New equation:

where

$$\hat{\boldsymbol{\alpha}} = \sum_k \hat{\boldsymbol{\alpha}}_k$$

$$\hat{\boldsymbol{\alpha}}_k = -\sqrt{\tilde{\beta}_k / m\hbar\tilde{\omega}_k}\,\boldsymbol{\varepsilon}_k(\hat{a}_k^+ - \hat{a}_k) = \frac{e}{mc^2}\hat{\boldsymbol{Z}}_k$$

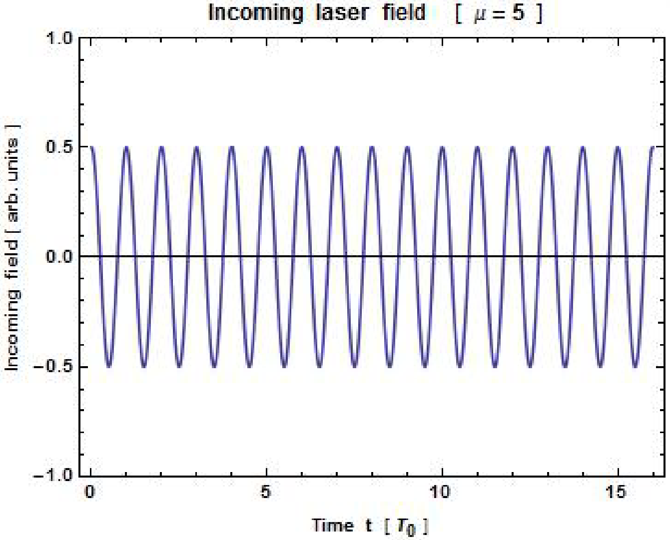
Hertz vector

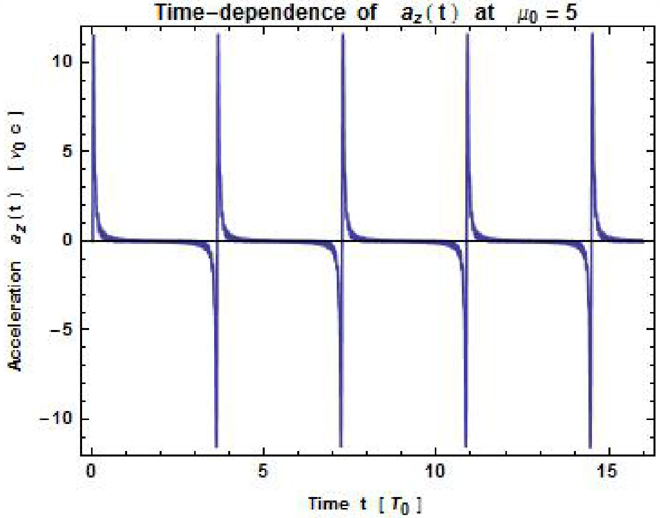
$$\tilde{\omega}_k = \sqrt{c^2|\boldsymbol{k}|^2 + \omega_p^2}$$

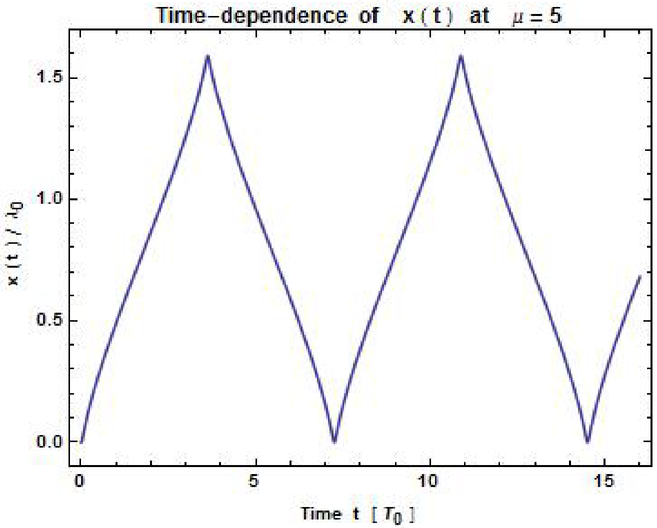
plasmon dispersion, „blue-shift”, „attochirp”...

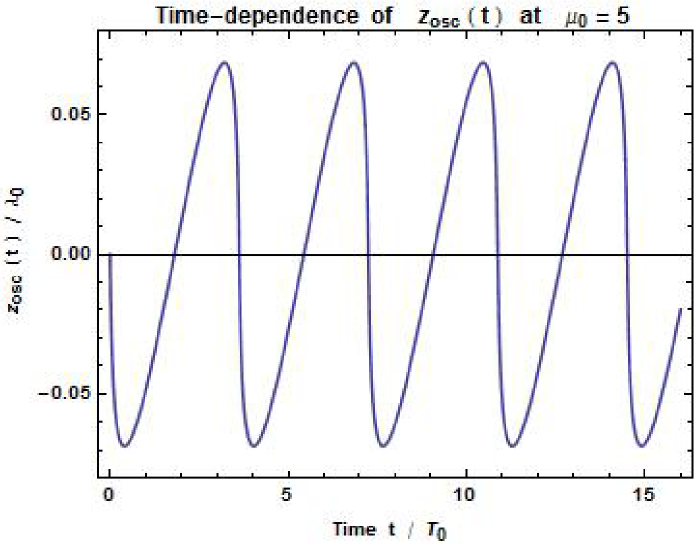
$$\tilde{H}_{rad} = \sum_{\boldsymbol{k},s}\hbar\tilde{\omega}_k(\hat{a}_{\boldsymbol{k},s}^+\hat{a}_{\boldsymbol{k},s} + \tfrac{1}{2})$$

Varró S, Quantum optical aspects of high-harmonic generation. *Photonics* 2021, *8* (7), 269 (2021). [https://doi.org/10.3390/photonics8070269]. Special Issue “Quantum Optics in Strong Laser Fields”. The elimination technique is the same as in Bergou J and Varró S, J. Phys. A 14, 1469 (1981), ibid. 14, 2281 (1981) used for free Schrödinger and Dirac electrons, resp.









Varró S, Klein-Gordon radio and attosecond light pulses. SZFI-Wigner Seminar 07 May 2019.

Irreducible representation of the Lorentz group.

UNITARY REPRESENTATIONS OF THE LORENTZ GROUP

By I. GELFAND and M. NEUMARK

Steklov Mathematical Institute, Academy of Sciences of the USSR

(Received October 6, 1945)

The usual (finite-dimensional) representations of the Lorentz group do not leave invariant any positive definite form, *i. e.*, are not unitary. In this note there are indicated all unitary (hence infinite-dimensional) irreducible representations of the Lorentz group.

ANNALS OF MATHEMATICS
Vol. 48, No. 3, July, 1947

IRREDUCIBLE UNITARY REPRESENTATIONS OF THE LORENTZ GROUP

BY V. BARGMANN

Added in proof. In the meantime the interesting note by L. Gelfand and M. Neumark (*Journal of Physics* (USSR), Vol. X, pp. 93-94, 1946) on the irreducible unitary representations of the Lorentz group ($\mathfrak{S}_4$ in our notation) has arrived in this country. The results on the classification of the representations which the authors announce are stronger than ours (cf. the introduction to the present paper) since no assumptions about the *infinitesimal* representations are introduced. There is no discussion, however, of the matrix elements as functions on the group manifold.

The representations obtained by the authors are the same as those mentioned in our introduction, and even the realization of the representing linear operators by functional operators is the same—if the space of light rays (or the unit sphere) is used, as described at the end of their note.

PUKÁNSZKY, L.
Math. Annalen 156, 96—143 (1964)

The Plancherel Formula for the Universal Covering Group of $SL(R, 2)$*

By

L. PUKÁNSZKY in Los Angeles (USA)

Herrn Professor GÁBOR SZEGŐ zu seinem 70. Geburtstage mit tiefster Verehrung zugeeignet

Similar SU(1,1) structure of the degenerate and non-degenerate interaction.

SU(1,1) generators in the degenerate case.

$$K_+ = \tfrac{1}{2}(a^+)^2 \qquad K_- = \tfrac{1}{2}(a)^2 \qquad K_0 = \tfrac{1}{4}(aa^+ + a^+a)$$

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_0$$

SU(1,1) generators in the non-degenerate case: a=a$_+$ and b=a$_-$

$$K_+ = a^+b^+ \qquad K_- = ab \qquad K_0 = \tfrac{1}{2}(a^+a + b^+b + 1)$$

Parametric interaction: $H_I = \chi(a^+b^+\mathrm{e}^{i\varphi} + ab\mathrm{e}^{-i\varphi})$ $\quad \hat{C} = \tfrac{1}{4}(a^+a - b^+b)^2 - \tfrac{1}{4}$ $\quad n_1 - n_2 = const.$

SU(1,1) phase operator formalism: $ab|n\rangle_1|n+\nu\rangle_2 = \sqrt{n}\sqrt{n+\nu}|n-1\rangle_1|n+\nu-1\rangle_2 \leftrightarrow A\sqrt{N+\nu}|n\rangle$ $\quad \nu = |n_1 - n_2|$

$$K_- = A\sqrt{N+\nu} = F(N+\nu) \qquad K_+ = \sqrt{N+\nu}A^+ = (N+\nu)F^+ \qquad K_0 = N + \tfrac{1}{2}(\nu+1) = N + \kappa$$

Evolution operator: $U(g) = \mathrm{e}^{-iK_0\theta}\exp(\xi K_+ - \xi^* K_-),$ $\quad \xi = \chi\tau/\hbar$

$$\chi := 2\pi\bar{\sigma}(\boldsymbol{e}_p\chi^{(2)} : \boldsymbol{e}_s\boldsymbol{e}_i)(\omega_s\omega_i)^{1/2}/n_s n_i \qquad \bar{\sigma} = (1/\tau)\int dtdVf_s(\boldsymbol{r})f_i(\boldsymbol{r})E_p(\boldsymbol{r},t)$$

[1] Varró S : Regular phase operator and SU(1,1) coherent states of the harmonic oscillator. *Physica Scripta* 90 (7) (2015) 074053 [2] Varró S, Coherent and incoherent superposition of transition matrix elements of the squeezing operator. *New Journal of Physics* 24, 053035 (2022).

Zernike polynomials from Gram-Schmidt orthogonalization. [[1] Lakshminarayanan and Fleck (2011)]

Table 4. Functions generated from Gram–Schmidt orthogonalization of a power series [2].

Functions	Series	Interval	Weight	Norm
Legendre	$\{1, r, r^2, r^3, \ldots\}$	$-1 \le r \le 1$	1	$2/(2n+1)$
Shifted Legendre	"	$0 \le r \le 1$	1	$1/(2n+1)$
Chebyshev I	"	$-1 \le r \le 1$	$(1-x^2)^{-1/2}$	$\pi/(2-\delta_0^n)$
Shifted Chebyshev I	"	$0 \le r \le 1$	$[x(1-x^2)]^{-1/2}$	$\pi/(2-\delta_0^n)$
Chebyshev II	"	$-1 \le r \le 1$	$(1-x^2)^{-1/2}$	$\pi/2$
Associated Laguerre	"	$0 \le r < \infty$	$r^k e^{-r}$	$(n+k)!/n!$
Hermite	"	$-\infty < r < \infty$	e^{-r^2}	$2^n \pi^{1/2} n!$
Zernike radial	$\{r^m, r^{m+2}, r^{m+4}, \ldots\}$	$0 \le r \le 1$	r	$1/(2n+2)$

$$R_{n+m}^{|m-n|}(\rho) = \sum_{k=0}^{n} \frac{(-1)^k (n+m-k)!}{k!(m-k)!(n-k)!} \rho^{n+m-2k},$$

We have seen that the Zernike polynomials came out from the down-conversion matrix elements through the special cases of the Jacobi polynomials:

$$R_{n+m}^{|m-n|}(\rho) = (-1)^n \rho^{|m-n|} P_n^{(|m-n,0)}(1-2\rho^2)$$

The table has been copied from [1] Lakshminarayanan V and Fleck A 2011 Zernike polynomials: a guide. Journal of Modern Optics, 58 (7), 545-561 (2011).

Correlations of photons stemming from parametric down-conversion.

SIMULTANEOUS NEAR-FIELD AND FAR-FIELD . . . PHYSICAL REVIEW A **69**, 023802 (2004)

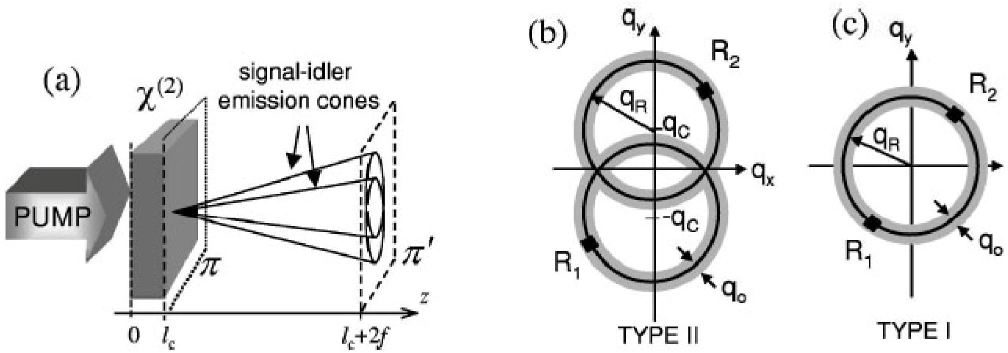


FIG. 1. Scheme for the observation of down-conversion in the far-field zone (a). The lens (not shown in the figure) is located at $z=l_c+f$. (b) and (c) display the phase-matching curves (25) in the spatial frequency plane for a type-II (b) and a type-I (c) crystal, respectively. The symmetrical black squares R_1 and R_2 indicate the locations of the detectors from which maximal signal-idler correlation can be measured.

Figure copied from: E. Brambilla, A. Gatti, M. Bache, and L. A. Lugiato(2004) **Simultaneous near-field and far-field spatial quantum correlations in the high-gain regime of parametric down-conversion**. PHYSICAL REVIEW A **69, 023802 ~2004!**

Some graphical illustrations of the Zernike polynomials. [Niu K and Tian C (2022)]

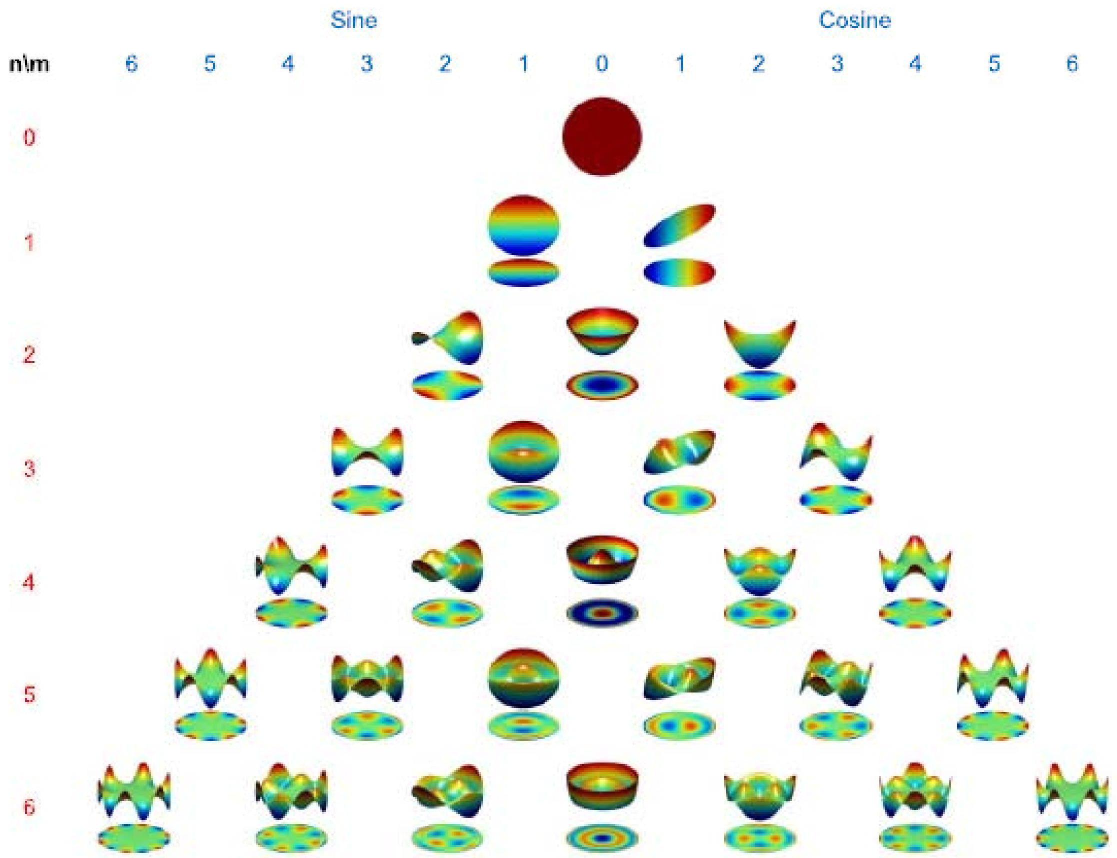
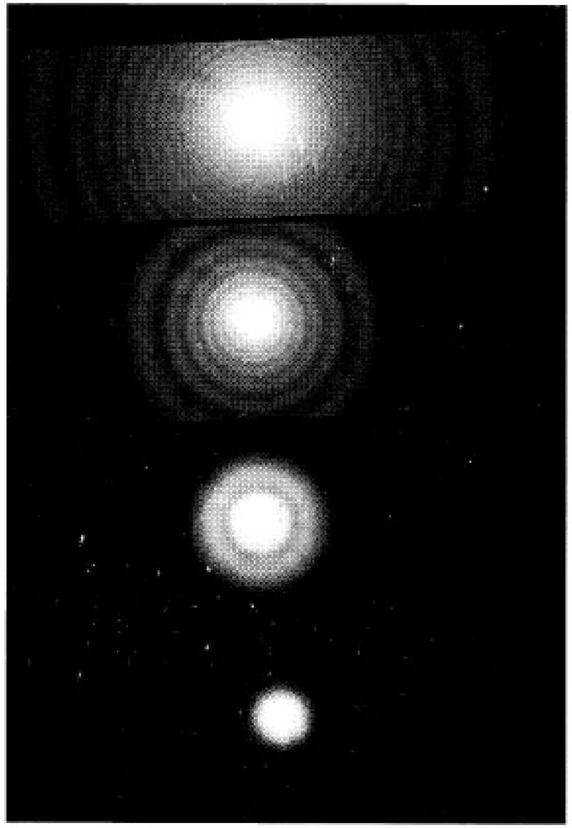


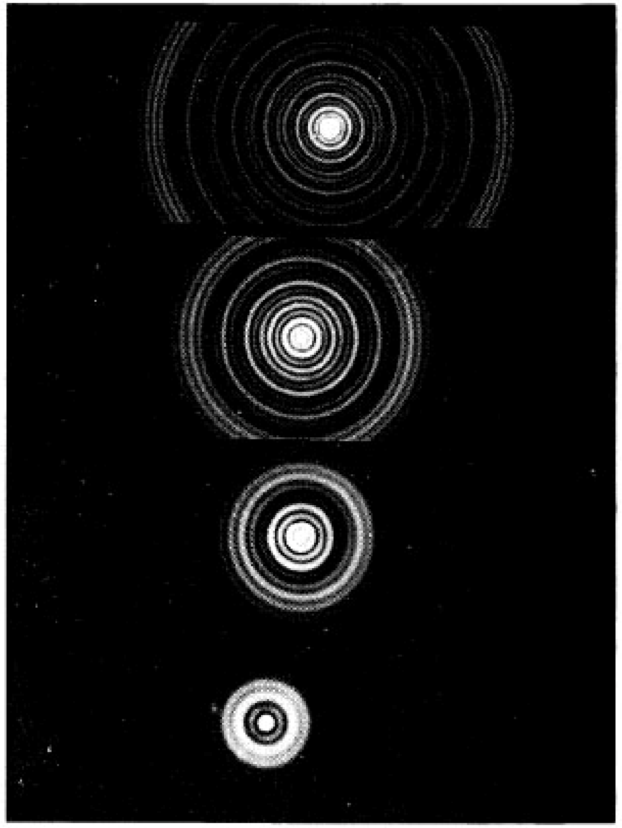
Figure 7. Pyramid of the non-normalized Zernike circle polynomials up to the sixth degree under the Noll indexing scheme.

Figure copied from [3] Niu K and Tian C 2022 Zernike polynomials and their applications. Journal of Optics 24, 123001 (2022).

Born M and Wolf E, Principles of Optics (Cambridge University Press, Cambridge, 2002).



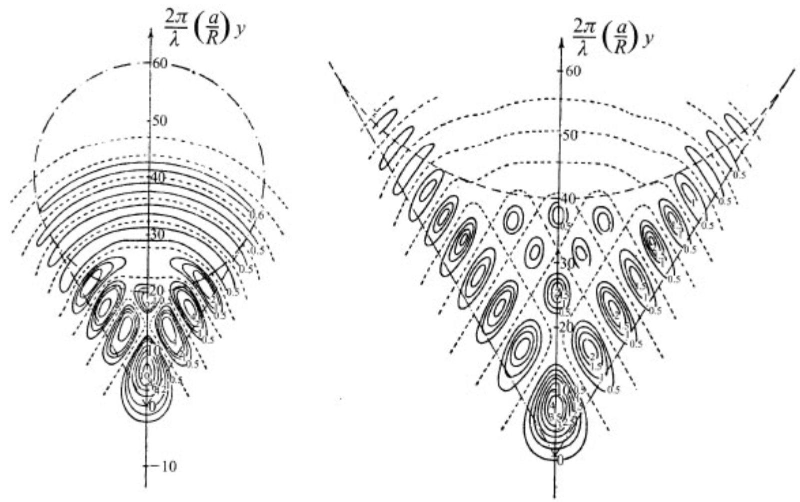
(a)



(b)

Fig. 9.5 Images in the Gaussian focal plane (a), and in the plane of the geometrical circle of least confusion (b), in the presence of primary spherical aberration $\Phi = 17.5\lambda\rho^4$, $8.4\lambda\rho^4$, $3.7\lambda\rho^4$, and $1.4\lambda\rho^4$. (Scale of (b) is three times that of (a).) (After K. Nienhuis, Thesis (University of Groningen, 1948), p. 56.)

Born & Wolf; . Intensity distribution and images in the presence of coma.



(a) $\Phi = 3.2\lambda\rho^3\cos\theta$ (b) $\Phi = 6.4\lambda\rho^3\cos\theta$

Fig. 9.7 Isophotes in the plane $z = 0$ in the presence of primary coma. The intensity is normalized to 100 at the centre of the aberration-free image. (After R. Kingslake, *Proc. Phys. Soc.*, **61** (1948), 147.)

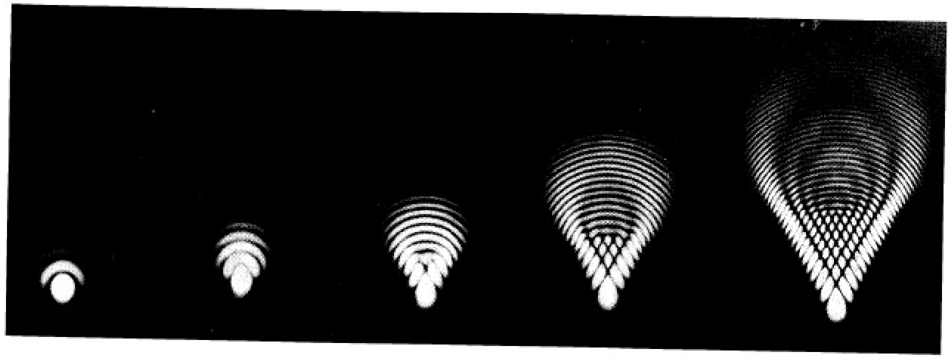
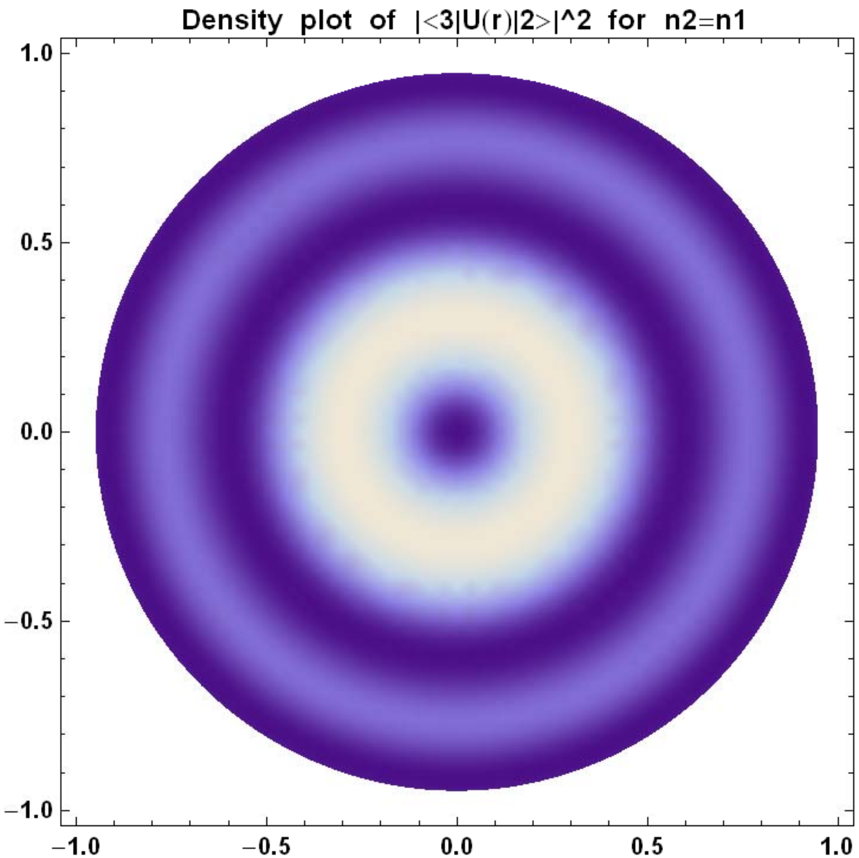
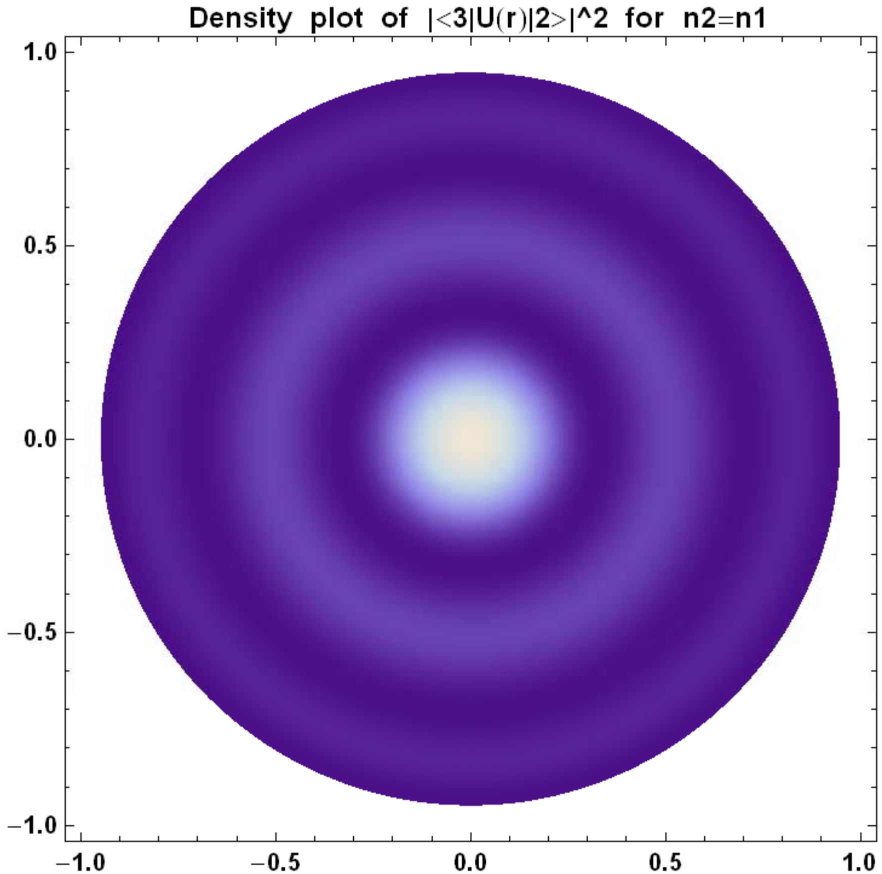


Fig. 9.8 Images in the Gaussian focal plane in the presence of coma $\Phi = 0.3\lambda\rho^3\cos\theta$, $\lambda\rho^3\cos\theta$, $2.4\lambda\rho^3\cos\theta$, $5\lambda\rho^3\cos\theta$, $10\lambda\rho^3\cos\theta$. (After K. Nienhuis, Thesis (University of Groningen, 1948), p. 40.)

Aberration of down-conversion probability in the generation of 3 photon pairs.

Special case ν=0, i.e. $n_2 = n_1 = n$ (~'Zernike polynomial distributions'). n = 2, 3.

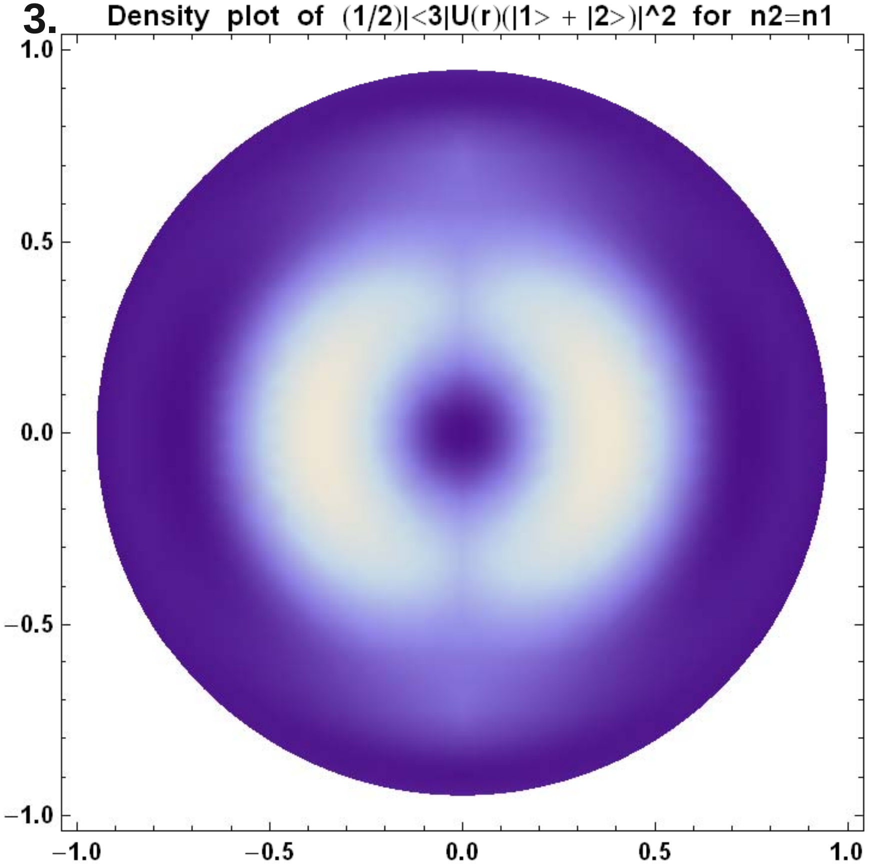


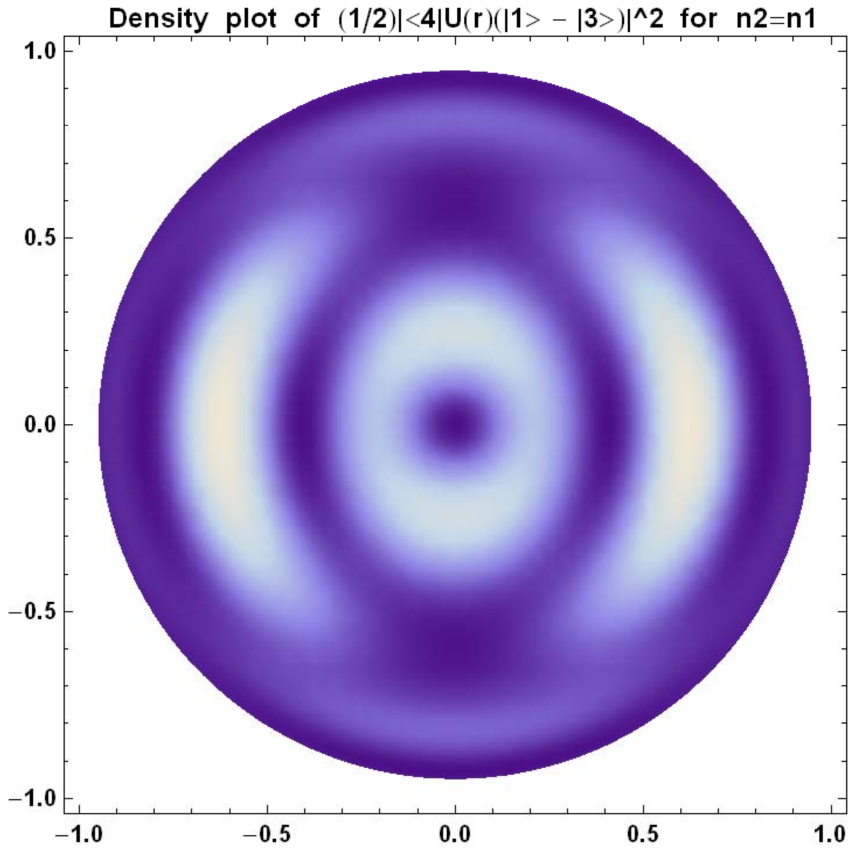


The left figure refers to the initial n = 2. The righ figure refers to the initial n = 3.

Aberration of down-conversion probability in the generation of 3 photon pairs.

Special case ν=0, i.e. $n_2 = n_1 = n$ (~'Zernike distributions'). Superposition:n = 1, 2, 3.





The left figure refers to the spontaneous process. The right figure refers to the initial n = 1.