Higher-rank sectors and marginal deformations in the hexagon formalism

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based on 2212.03211 and 2302.02961
with Burkhard Eden and Anne Spiering

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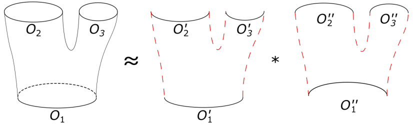
Hexagon program

- **Spectrum** is fairly well-understood
- Three-point functions by **hexagon operators**
  for AdS_5 [Basso, Komatsu, Vieira '15]
  for AdS_3 [Eden, DIP, Sonfdrini '21]



- In principle:
  - → **higher-point functions** [Eden, Sfondrini '17] [Fleury, Komatsu '17]
  - → **non-planar correlators** [Eden, Jiang, DIP, Sfondrini '17] [Bargheer, Caetano, Fleury, Komatsu, Vieira '17] [Bargheer, Coronado, Vieira '19] . . .
  - → **wrapping corrections** [Basso, Komatsu, Vieira '15][Eden, Sfondrini '15] [Fleury, Komatsu '17] . . .

Outline

The $\mathrm{su}(2)$ spin chain

Anomalous dimension $\leftrightarrow$ Spin chain energy [Minahan, Zarembo '02]

Spin chain with vacuum Z ($\downarrow$) and excitations X ($\uparrow$)

$\mathrm{su}(2)$ sector **BMN-operator** with two scalar excitations $\mathrm{Tr}(Z^{L-k-2}XZ^kX)$

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Planar one-loop **dilatation operator** on single-trace operators ↔ Spin chain Hamiltonian $H_0 = 1 - \mathbb{P}$

$$H_0 \left|n_1, n_2, \ldots\right\rangle_L = \sum_{j=1}^{M} \left(2\left|\ldots, n_j, \ldots\right\rangle - \left|\ldots, n_j - 1, \ldots\right\rangle - \left|\ldots, n_j + 1, \ldots\right\rangle\right) ,$$

yields the energy and S matrix

$$E(p) = 4\sin\left(\frac{p}{2}\right)^2 , \quad S(p_j, p_k) = -\frac{e^{i(p_j+p_k)} - 2e^{ip_k} + 1}{e^{i(p_j+p_k)} - 2e^{ip_j} + 1} .$$

The $\mathrm{su}(2)$ Bethe equations

Introducing the **rapidity** $u = \frac{1}{2}\cot\frac{p}{2}$, the S matrix can be written as

$$S(u_j, u_k) = \frac{u_j - u_k - i}{u_j - u_k + i}\,.$$

The **Energy** or anomalous dimension is

$$E = \sum_{j=1}^{M} \frac{1}{u_j^2 + \frac{1}{4}}\,.$$

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For M excitations, the **Bethe equations** are given by:

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L \prod_{j\neq k} \frac{u_j - u_k - i}{u_j - u_k + i} = 1\,, \quad \text{and} \quad \prod_{j=1}^{M}\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right) = 1\,.$$

Example with $L = 4$, $M = 2$: $u_1 = -u_2 = \frac{1}{\sqrt{12}}$

Hexagon-like formula from the spin chain

Bethe state:

$$|\Psi(p_1, p_2)\rangle = \sum_{1\leq n<m\leq L} \underbrace{\left(e^{ip_1 n + ip_2 m} + S(p_1, p_2) e^{ip_2 n + ip_2 m}\right)}_{\psi(n,m)} |n, m\rangle$$

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Normalized cyclic state given by [Gaudin '76][Korepin '82]

$$\mathcal{O}_L = \frac{|\Psi(p_1,p_2)\rangle}{\sqrt{\mathcal{G}\, L\, S_{12} \prod_j (u_j^2+\frac{1}{4})}}$$

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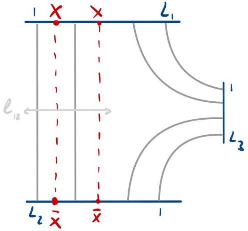
Overlap:

$$c_{123} \propto \sum_{1\le n<m\le \ell_{12}} \psi_1(n,m) \quad \psi_2(L_2-m+1, L_2-n+1)$$



Symmetries of the three-point function

Choosing Z as the vacuum

Take 1/2-BPS operator $\mathcal{O}(0)$ at $x = 0$

$\rightarrow$ want to construct *three* translated operators $\mathcal{O}(x)$

$\rightarrow$ should preserve as much (super)symmetry as possible

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Introduce the **supertranslation generator** [Basso, Komatsu, Vieira '15]

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Use $\mathcal{T}_\kappa$ to construct one parameter family of operators starting from $\mathcal{O}(0)$

$$\mathcal{O}_{t,\kappa} = e^{t\,\mathcal{T}_\kappa}\, \mathcal{O}(0)\, e^{-t\,\mathcal{T}_\kappa} \,.$$

Constraining the hexagon form factor by symmetry

Charges commuting with $\mathcal{T}_\kappa$ form diagonal subalgebra $\mathfrak{psu}(2|2)_D$

Write $\mathfrak{psu}(2|2)^2$ excitations as $\chi^{a\dot{a}} = \xi^a \otimes \dot{\xi}^{\dot{a}}$

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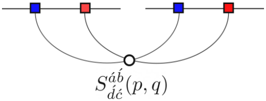
→ two-particle form factors Beisert S matrix elements [Beisert '06]

$$\langle \mathbf{h} | \chi^{a_1 \dot{a}_1} \chi^{a_2 \dot{a}_2} \rangle = (-1)^f \langle \xi^{a_2} \xi^{a_1} | \mathcal{S} | \dot{\xi}^{\dot{a}_1} \dot{\xi}^{\dot{a}_2} \rangle$$
$$= (-1)^f \dot{S}_{\dot{a}_1 \dot{a}_2}^{\dot{b}_1 \dot{b}_2} \, h_{\chi^{a_1 \dot{b}_1}} \, h_{\chi^{a_2 \dot{b}_2}} \, .$$



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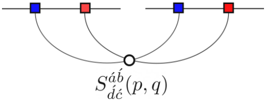
$$= (-1)^f \, \dot{S}^{\dot{b}_1\dot{b}_2}_{\dot{a}_1\dot{a}_2} \, h_{\chi^{a_1\dot{b}_1}} \, h_{\chi^{a_2\dot{b}_2}} \, .$$



→ Multi-particle form factor:

$$\langle \mathbf{h}|\chi^{a_1\dot{a}_1}\chi^{a_2\dot{a}_2}\dots\chi^{a_N\dot{a}_N}\rangle = (-1)^f \, \langle \xi^{a_N}\dots\xi^{a_2}\xi^{a_1}|\mathcal{S}|\dot{\xi}^{\dot{a}_1}\dot{\xi}^{\dot{a}_2}\dots\dot{\xi}^{\dot{a}_N}\rangle \, .$$

[Basso, Komatsu, Vieira '15]

Constraining the scalar h-factor

Scalar factor h in the hexagon $\longleftrightarrow$ dressing phase S_0 in the S matrix

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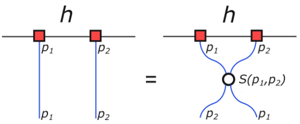
Scalar factor h in the hexagon $\longleftrightarrow$ dressing phase S_0 in the S matrix

- **Watson equation**
  Scattering with the full S matrix

$$\langle \mathbf{h}|\,\mathbf{S}\,|\chi^{A\dot{A}}(p_1)\chi^{B\dot{B}}(p_2)\rangle = \langle \mathbf{h}|\chi^{A\dot{A}}(p_1)\chi^{B\dot{B}}(p_2)\rangle$$



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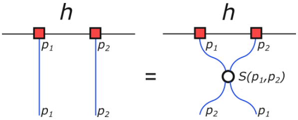
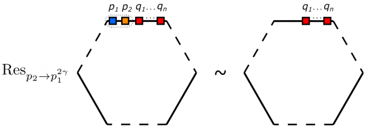


- **Decoupling condition** for a singlet



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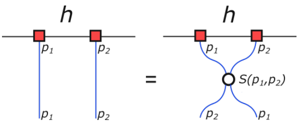

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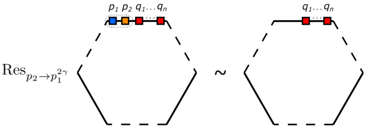
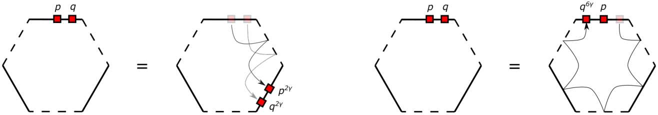


- **Cyclicity**



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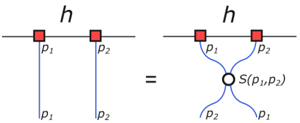

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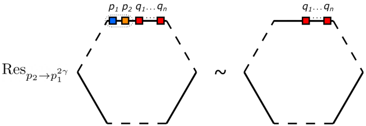

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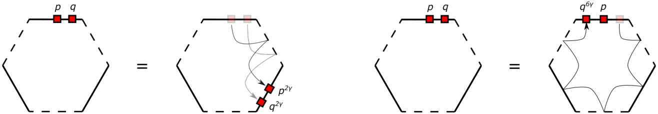


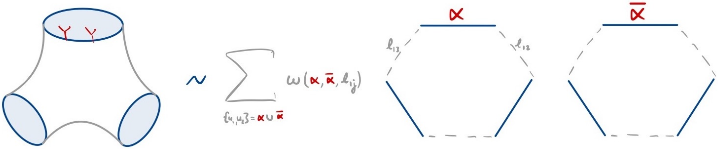
$\Rightarrow$ Fixes the h-factor!

$\Rightarrow$ Similar construction in AdS_3

[Basso, Komatsu, Vieira '15]

[Eden, DIP, Sonfdrini '21]

Simple example



$$\mathcal{A} \;=\; \sum_{\alpha\cup\bar{\alpha}} \;\omega(\alpha,\bar{\alpha},\ell)\;\;\langle \mathbf{h}|\alpha\rangle\;\;\langle \mathbf{h}|\bar{\alpha}\rangle\;.$$

The **splitting factor** $\omega(\alpha,\bar{\alpha},\ell)$ is given by

$$\omega(\alpha,\bar{\alpha},\ell) = (-1)^{|\bar{\alpha}|}\prod_{j\in\bar{\alpha}} e^{ip_j\ell}\prod_{\substack{k\in\alpha\\ j<k}} S(p_j,p_k)\,.$$

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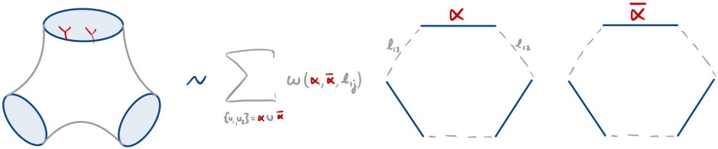

$$\mathcal{A} \quad = \quad \sum_{\alpha\cup\bar\alpha} \quad \omega(\alpha,\bar\alpha,\ell) \quad \langle \mathbf{h}|\alpha\rangle \quad \langle \mathbf{h}|\bar\alpha\rangle \; .$$

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How to generalize formalism to **higher-rank** sectors?
→ replace by nested wave function [Basso, Coronado, Komatsu, Lam, Vieira, Zhong '17]

Can we maintain the hexagon operator?

Plan

Higher-rank models

Consider $\mathrm{SU}(3)$ sector with excitations X and Y

Consider the wave function $\psi_{\{X_1, Y_2\}}$, with the scattering

$$|X_1\, Y_2\rangle \quad \to \quad T_{12}\, |Y_2\, X_1\rangle + R_{12}\, |X_2\, Y_1\rangle\ ,$$

with **transmission** and **reflection** amplitudes

$$T_{12} = \frac{A_{12} - B_{12}}{2} \qquad \text{and} \qquad R_{12} = \frac{A_{12} + B_{12}}{2}\ .$$

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$$T_{12} = \frac{A_{12} - B_{12}}{2} \qquad \text{and} \qquad R_{12} = \frac{A_{12} + B_{12}}{2} \ .$$

Introduce a second wave function $\psi_{\{Y_1,X_2\}}$ with scattering

$$|Y_1\, X_2\rangle \quad \to \quad T_{12}\, |X_2\, Y_1\rangle + R_{12}\, |Y_2\, X_1\rangle \ ,$$

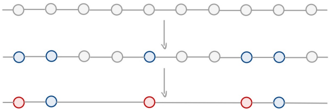
and consider the sum

$$\psi(L) = g_{XY}\, \psi_{\{X_1,Y_2\}} + g_{YX}\, \psi_{\{Y_1,X_2\}} \ ,$$

with yet to be determined **coefficients** g_{XY} and g_{YX}.

Extracting the coefficients from nesting

- **Level-0** vacuum of length L
- M **level-1** excitations move on level-0 vacuum with $S^{10} = e^{ip}$ and $S^{11}_{jk} = S(u_j, u_k)$
- k **level-2** excitations move on level-1 vacuum of length M with S^{21}, are scattered by S^{22} and have a creation amplitude f^{21}



$$|Y(v)\rangle^2 = f^{21}(v, u_1)\,|Y_1 X_2\rangle + f^{21}(v, u_2)S^{21}(v, u_1)\,|X_1 Y_2\rangle \ .$$

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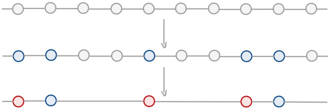

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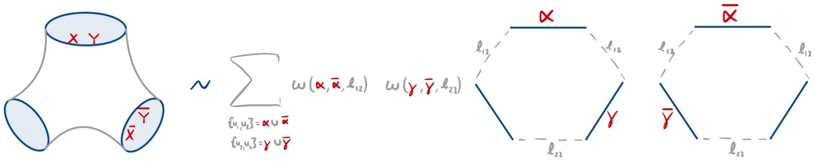
Scattering leads to

$$g_{XY} T_{12} + g_{YX} R_{12} = f^{21}(v, u_1) S^{21}(v, u_2) S^{11}(u_1, u_2)\,,$$
$$g_{XY} R_{12} + g_{YX} T_{12} = f^{21}(v, u_2) S^{11}(u_1, u_2)\,.$$

$\Rightarrow$ **Coefficients** g_{XY} and g_{YX} inherit dependence on the auxiliary Bethe roots v.

The nested hexagon



Cutting the $\mathrm{SU}(3)$ state

$$
\omega(\alpha,\bar{\alpha},\ell)\psi_{\{\alpha\}}\psi_{\{\bar{\alpha}\}} = \begin{cases} g_{XY}\,\psi_{\{X_{u_1},Y_{u_2}\}}\psi_{\{\}} + g_{YX}\,\psi_{\{Y_{u_1},X_{u_2}\}}\psi_{\{\}}\,, \\ e^{ip_2\ell}\left(g_{XY}\,\psi_{\{X_{u_1}\}}\psi_{\{Y_{u_2}\}} + g_{YX}\,\psi_{\{Y_{u_1}\}}\psi_{\{X_{u_2}\}}\right)\,, \\ e^{ip_1\ell}\left(T_{12}g_{YX} + R_{12}g_{XY}\right)\psi_{\{X_{u_2}\}}\psi_{\{Y_{u_1}\}} + \\ e^{ip_1\ell}\left(T_{12}g_{XY} + R_{12}g_{YX}\right)\psi_{\{Y_{u_2}\}}\psi_{\{X_{u_1}\}}\,, \\ e^{i(p_1+p_2)\ell}\left(g_{XY}\psi_{\{\}}\psi_{\{X_{u_1},Y_{u_2}\}} + g_{YX}\psi_{\{\}}\psi_{\{Y_{u_1},X_{u_2}\}}\right)\,. \end{cases}
$$

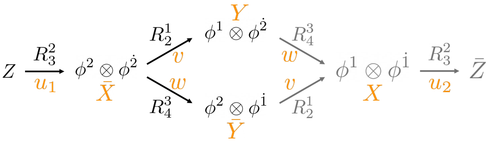
$\Rightarrow$ Agreement with free field theory

Double excitations

Consider $\mathrm{Tr}(X\bar{X} + Y\bar{Y} + Z\bar{Z})$

How can we describe $\bar{Z}$?

$\longrightarrow$ **double** excitations!



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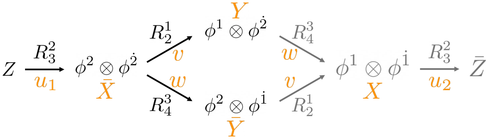

Can introduce **creation amplitude** in nested/matrix ansatz

Computations makes no further reference to the local structure of the state, i.e.

$$\begin{aligned}\psi^L_{\{X_1,\bar{X}_2\}} = &\psi_{\{X_1,\bar{X}_2\}}\,\psi_{\{\}} - e^{i\,p_2\,\ell}\,\psi_{\{X_1\}}\,\psi_{\{\bar{X}_2\}} - \\ &e^{i\,p_1\,\ell}\Big[\quad T^2_{12}\,\psi_{\{\bar{X}_2\}}\,\psi_{\{X_1\}} + \quad R^2_{12}\,\psi_{\{X_2\}}\,\psi_{\{\bar{X}_1\}}\Big] - \\ &e^{i\,p_1\,\ell}\Big[\,T_{12}R_{12}\,\psi_{\{\bar{Y}_2\}}\,\psi_{\{Y_1\}} + R_{12}T_{12}\,\psi_{\{Y_2\}}\,\psi_{\{\bar{Y}_1\}}\Big] + \\ &e^{i(p_1+p_2)\ell}\,\psi_{\{\}}\,\psi_{\{X_1,\bar{X}_2\}}\,.\end{aligned}$$

Konishi

Let us evaluate $\langle \mathcal{K}\, \mathcal{O}^{L_2}\, \mathcal{O}^{L_3} \rangle$ with $\mathcal{K} = \frac{1}{\sqrt{3}} \mathrm{Tr}(X\bar{X} + Y\bar{Y} + Z\bar{Z})$.

This yields (tree-level)

$$\mathcal{A}_{\mathsf{QFT}} = \frac{1}{\sqrt{3}} \sqrt{L_2 L_3}\,.$$

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Using $g_{X\bar{X}} = g_{\bar{X}X} = -g_{Y\bar{Y}} = -g_{\bar{Y}Y}$ and $u_2 = -u_1 = \frac{1}{\sqrt{12}}$, $v = 0$, $w = 0$

$$\mathcal{A}^{\ell_{12}=1}_{\mathrm{hexagon}}(-u, u) = \frac{8\, g_{X\bar{X}}\, u}{(u - \frac{i}{2})(u + \frac{i}{2})^2} = \frac{\sqrt{3}}{2}\,.$$

Konishi

Let us evaluate $\langle \mathcal{K}\, \mathcal{O}^{L_2}\, \mathcal{O}^{L_3} \rangle$ with $\mathcal{K} = \frac{1}{\sqrt{3}} \mathrm{Tr}(X\bar{X} + Y\bar{Y} + Z\bar{Z})$.

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We find agreement

$$\mathcal{A}_{\mathrm{QFT}} = \left(u^2 + \frac{1}{4}\right) L_1 \sqrt{L_2 L_3}\, \mathcal{A}_{\mathrm{hexagon}}\,.$$

$\rightarrow$ Analogous results for $L_1 = 3, 4, \ldots$ with $u = \frac{1}{2}, \frac{1}{2}\sqrt{1 \pm \frac{2}{\sqrt{5}}}, \ldots$

Deformed $\mathrm{su}(2)$ Bethe equations

Introduce twist factors into the Bethe equations [Beisert, Roiban '05]

For two excitations, the **Bethe equations** are given by:

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L \frac{u_j - u_k - i}{u_j - u_k + i} = e^{2iL\beta}, \quad \text{and} \quad \left(\frac{u_1 + \frac{i}{2}}{u_1 - \frac{i}{2}}\right)\left(\frac{u_2 + \frac{i}{2}}{u_2 - \frac{i}{2}}\right) = e^{4i\beta}.$$

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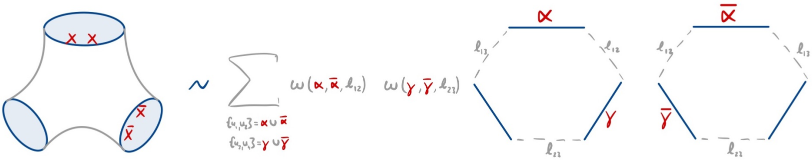
Solutions for $L=4$:

- BMN-like: $u_4^\pm = \pm\frac{1}{2\sqrt{3}} - \frac{2\beta}{3} \pm \frac{8\beta^2}{9\sqrt{3}} - \frac{16\beta^3}{27} \pm \frac{112\beta^4}{81\sqrt{3}} + \mathcal{O}(\beta^5)$
- Vacuum descendant-like:

$$u_4^\pm = \frac{3\pm i\sqrt{3}}{8}\frac{1}{\beta} + \frac{-3\pm i\sqrt{3}}{9}\beta - \frac{4}{405}(21\mp 8\sqrt{3}i)\beta^3 + \mathcal{O}(\beta^5)$$

Deformed SU(2) sector



- Correlators can involve **BMN**-like and **vacuum descendant**-like operators

$$\langle \mathcal{B}_1 \mathcal{B}_2 \mathcal{O}_3 \rangle \ , \quad \langle \mathcal{B}_1 \mathcal{O}_2'' \mathcal{O}_3 \rangle$$

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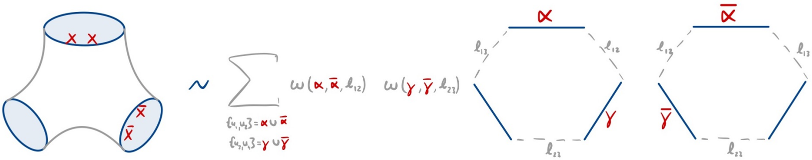

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- Splitting factor

$$\omega(\alpha, \bar{\alpha}, \ell) = \prod_{\tilde{u}_i \in \bar{\alpha}} e^{2i\beta(d_\alpha - d_{\bar{\alpha}})} \left(\frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}} \right)^\ell e^{-2i\beta\ell} \prod_{u_1 \in \bar{\alpha}, u_2 \in \alpha} \frac{u_1 - u_2 - i}{u_1 - u_2 + i}$$

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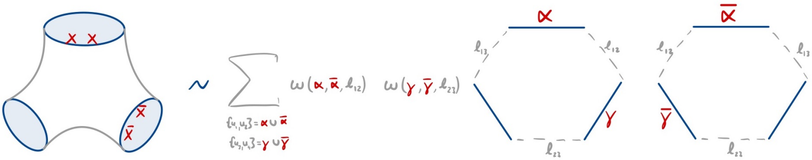

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- Hexagon normalization $\mathcal{N} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{G}_{12} S_{12} \mathcal{G}_{34} S_{34}}}$

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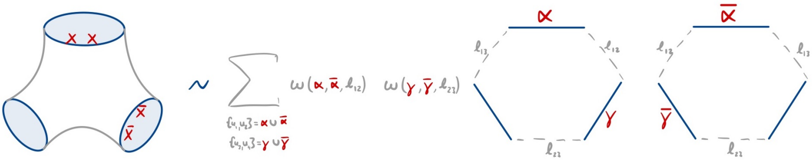

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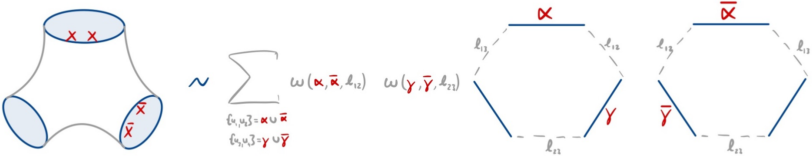
- Hexagon normalization $\mathcal{N} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{G}_{12} S_{12} \mathcal{G}_{34} S_{34}}}$
- Special role of **longitudinal excitations**

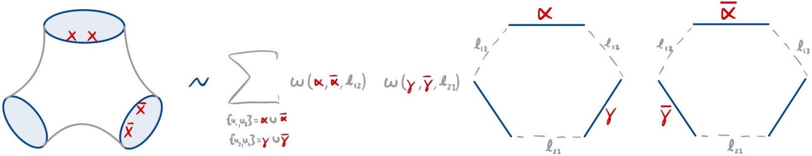
⇒ **Agreement** with field theory results

Asymptotic hexagon at one-loop



- Consider three-point functions with *all* $\ell_{ij} \geq 1$ and disregard wrapping corrections

Asymptotic hexagon at one-loop



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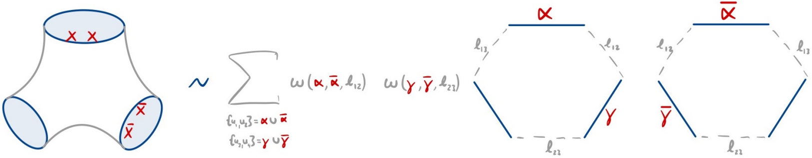

- Consider three-point functions with *all* $\ell_{ij} \geq 1$ and disregard wrapping corrections
- Rapidities from asymptotic Bethe equations
- Need to include the **measure factor**

$$\mu(u) = \frac{-i}{\mathrm{res}_{v=u} \langle \widetilde{\mathbf{h}} | \bar{X}(v^{2\gamma}) X(u) \rangle} = 1 - \frac{g^2}{(u^2 + \frac{1}{4})^2} + \mathcal{O}(g^4)$$

$\Rightarrow$ **Agreement** with field theory results

Plan

Lagrangian insertion method

Consider n-point function

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \int D\phi \, DA \, D\psi \; e^{i \int d^4x_0 \mathcal{L}(x_0)} \; \mathcal{O}_1 \dots \mathcal{O}_n \, .$$

It follows that

$$g^2 \frac{\partial}{\partial g^2} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = -i \int d^4x_0 \; \langle \mathcal{L}_0 \; \mathcal{O}_1 \dots \mathcal{O}_n \rangle \; .$$

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It follows that

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Introduce **Lagrange operator** as $L = 2$ vacuum descendant

Integrability picture: Yang-Mills Lagrangian $\mathrm{Tr}(F^2)$ build from the excitations

$$\Psi_1^{4\dot{2}} , \quad \Psi_2^{4\dot{1}} , \quad \Psi_3^{3\dot{2}} , \quad \Psi_4^{3\dot{1}} ,$$

with rapidities $u_1, \dots u_4$ and auxiliary rapidities v_1, v_2 and w_1, w_2

Lagrangian insertion: A first test

Two-point function of BPS-operators is protected: $\langle \mathcal{L}_0 \, \mathcal{O}_1^L \, \mathcal{O}_2^L \rangle = 0$

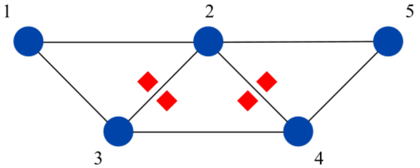
$$
\begin{aligned}
\langle \mathcal{L}_0 \, \mathcal{O}_1^L \, \mathcal{O}_2^L \rangle \;=\; & 2 \Big[\langle \mathbf{h} | \Psi_1^{4\dot{2}} \, \Psi_2^{4\dot{1}} \, \Psi_3^{3\dot{2}} \, \Psi_4^{3\dot{1}} \rangle + \langle \mathbf{h} | \Psi_1^{4\dot{2}} \, \Psi_4^{3\dot{1}} \rangle \, \langle \mathbf{h} | \Psi_2^{4\dot{1}} \, \Psi_3^{3\dot{2}} \rangle \Big] + \\
& \tilde{g} \Big[\langle \mathbf{h} | D_1^{4\dot{3}} \rangle \, \langle \mathbf{h} | \Psi_2^{4\dot{1}} \, \Psi_3^{3\dot{2}} \, D_4^{3\dot{4}} \rangle + \langle \mathbf{h} | D_2^{4\dot{3}} \rangle \, \langle \mathbf{h} | \Psi_1^{4\dot{2}} \, D_3^{3\dot{4}} \, \Psi_4^{3\dot{1}} \rangle + \\
& \quad \langle \mathbf{h} | D_3^{3\dot{4}} \rangle \, \langle \mathbf{h} | \Psi_1^{4\dot{2}} \, D_2^{4\dot{3}} \, \Psi_4^{3\dot{1}} \rangle + \langle \mathbf{h} | D_4^{3\dot{4}} \rangle \, \langle \mathbf{h} | D_1^{4\dot{3}} \, \Psi_2^{4\dot{1}} \, \Psi_3^{3\dot{2}} \rangle + \\
& \quad \langle \mathbf{h} | Y_1 \rangle \, \langle \mathbf{h} | \Psi_2^{4\dot{1}} \, \Psi_3^{3\dot{2}} \, \bar{Y}_4 \rangle + \langle \mathbf{h} | \bar{Y}_2 \rangle \, \langle \mathbf{h} | \Psi_1^{4\dot{2}} \, Y_3 \, \Psi_4^{3\dot{1}} \rangle + \\
& \quad \langle \mathbf{h} | Y_3 \rangle \, \langle \mathbf{h} | \Psi_1^{4\dot{2}} \, \bar{Y}_2 \, \Psi_4^{3\dot{1}} \rangle + \langle \mathbf{h} | \bar{Y}_4 \rangle \, \langle \mathbf{h} | Y_1 \, \Psi_2^{4\dot{1}} \, \Psi_3^{3\dot{2}} \rangle \Big] + \\
& \tilde{g}^2 \Big[\langle \mathbf{h} | D_1^{4\dot{3}} \, D_2^{4\dot{3}} \rangle \, \langle \mathbf{h} | D_3^{3\dot{4}} \, D_4^{3\dot{4}} \rangle + \langle \mathbf{h} | D_1^{4\dot{3}} \, \bar{Y}_2 \rangle \, \langle \mathbf{h} | Y_3 \, D_4^{3\dot{4}} \rangle + \\
& \quad \langle \mathbf{h} | Y_1 \, D_2^{4\dot{3}} \rangle \, \langle \mathbf{h} | D_3^{3\dot{4}} \, \bar{Y}_4 \rangle + \langle \mathbf{h} | Y_1 \, \bar{Y}_2 \rangle \, \langle \mathbf{h} | Y_3 \, \bar{Y}_4 \rangle \Big]
\end{aligned}
$$

$$
\langle \mathcal{L}_0 \, \mathcal{O}_1^L \, \mathcal{O}_2^L \rangle \to 4 \, \tilde{g}^2 \, (1 - 2 + 1) = 0
$$

Future Applications?

Example: five-point function involving five protected operators

Process at one-loop involving two **mirror excitations**



Hard to evaluate [Fleury, Komatsu '17], [de Leeuw, Eden, DIP, Meier, Sfondrini '19]

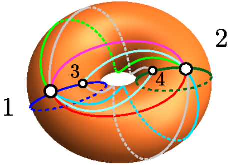
$$I = \sum_{a,b=1}^{\infty} \sum_{k,l=0}^{a-1,b-1} \int \frac{du_1\, du_2\, a\, b}{(u_1 + \frac{a}{4})^2 (u_2 + \frac{b}{4})^2} W_1 W_2 \Sigma^{ab} \mathcal{X}_k^{k,l}$$

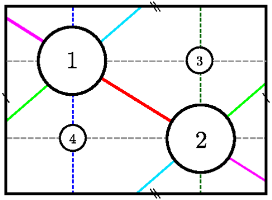
→ Can the Lagrangian insertion method simplify the evaluation?

Future Applications?

Colour-dressed hexagons allow to tessellate the **torus**

→ Non-planar two point functions [Eden, DIP, Sfondrini, Jiang '17]





→ Many length 0 edges

→ Can the Lagrangian insertion method help?

Plan

Conclusions and Outlook

- Powerful tool to calculate correlation functions in $\mathcal{N} = 4$ SYM
- Maintain the hexagon operator for **higher-rank** sectors
  - importing the g-coefficients from the nested Bethe ansatz
  - local details of the wave functions eclipsed
- **Marginal deformations** for certain classes of correlators involving $\mathfrak{psu}(1,1|2)$ operators
  $\rightarrow$ Is there a hexagon operator for deformed theories?
- Lagrangeoperator using **double excitations**
  - Four fermions on the hexagon
- Simplest test of **Lagrangian insertion** method with hexagons $\rightarrow$ Loop corrections for less simple two- and three-point functions?
  $\rightarrow$ Non-planar corrections?