Inverse Reynolds-Dominance approach to transient fluid dynamics

David Wagner[1,3], Andrea Palermo[1,2], Victor E. Ambruș[1,3]

[1]Institut für Theoretische Physik, Goethe Universität Frankfurt am Main, Germany
[2]Dipartimento di Fisica e Astronomia, Universitá di Firenze and INFN Sezione di Firenze, Italy
[3]Department of Physics, West University of Timișoara, Romania

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Outline

Motivation and goal

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Hydrodynamics: Conservation equations

$$\partial_\mu T^{\mu\nu} = 0 \,, \qquad \partial_\mu N^\mu = 0 \tag{1}$$

- Hydrodynamics: based on $(4+1=5)$ conservation equations
  - Ideal case: Sufficient (if equation of state is supplied)
    - $\rightarrow$ Variables: ϵ, n, u^μ
  - Dissipative case: Underdetermined
    - $\rightarrow$ Variables: ϵ, n, u^μ, Π, n^μ, $\pi^{\mu\nu}$
- **Fundamental question of dissipative hydrodynamics**: How to obtain information about the dissipative components of N^μ and $T^{\mu\nu}$?

Decomposition of conserved currents (Landau frame)

$$N^\mu = n u^\mu + n^\mu \tag{2}$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \tag{3}$$

Projectors: $\Delta^{\mu\nu} := g^{\mu\nu} - u^\mu u^\nu$, $\Delta^{\mu\nu}_{\alpha\beta} := (\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha)/2 - \Delta^{\mu\nu}\Delta_{\alpha\beta}/3$

First- and second-order hydrodynamics

- First-order hydro: Relate dissipative quantities to fluid-dynamical gradients

$$\Pi = -\zeta\theta\,, \quad n^\mu = \kappa I^\mu\,, \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} \tag{4}$$

- (In Eckart or Landau frame): **Acausal**!
- Second-order hydro: Treat dissipative quantitites as dynamical, provide relaxation equations

Relaxation equations

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \text{h.o.t.} \tag{5a}$$

$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa I^\mu + \text{h.o.t.} \tag{5b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \text{h.o.t.} \tag{5c}$$

- Needs input from **microscopic theory**
- This talk: Take **kinetic theory** as the foundation

$\theta := \partial^\mu u_\mu$, $\sigma^{\mu\nu} := \nabla^{\langle\mu} u^{\nu\rangle}$, $\nabla^\mu := \Delta^{\mu\nu}\partial_\nu$, $I^\mu := \nabla^\mu(\mu/T)$, $A^{\langle\mu}B^{\nu\rangle} := \Delta^{\mu\nu}_{\alpha\beta}A^\alpha B^\beta$

Kinetic Theory: Basics

- Describe system in (x,k)-phase space through one-particle distribution function $f(x,k)$
- Connection to hydrodynamics through conserved currents

Conserved quantities

$$N^\mu = \int \mathrm{d}K k^\mu f(x,k)\ , \quad T^{\mu\nu} = \int \mathrm{d}K k^\mu k^\nu f(x,k) \tag{6}$$

- Dynamics of $f(x,k)$ determine evolution of hydrodynamic quantities
  - Governed by Boltzmann equation $k^\mu \partial_\mu f(x,k) = C[f]$
- Separate into equilibrium part $f_0(x,k)$ and deviation $\delta f(x,k)$
  - $f_0(x,k)$ determined by $C[f_0] = 0$
- Binary elastic collisions: $f_0(x,k) = \left[e^{-\alpha_0(x)+\beta_0(x)u^\mu(x)k_\mu} + a\right]^{-1}$
  - $a \in \{-1, 0, 1\}$ determined by statistics of particles
  - α_0, β_0, u^μ: Lagrange multipliers

$\mathrm{d}K := \mathrm{d}^3k/[(2\pi)^3 k^0]$, $E_{\mathbf{k}} := u^\mu k_\mu$

Moment expansion

- Question: Which parts of $\delta f(x,k)$ in momentum space are important for hydrodynamics?
- Expand in terms of complete and orthogonal basis of irreducible tensors $1,\, k^{\langle\mu\rangle},\, k^{\langle\mu}k^{\nu\rangle},\, \cdots$
  - Equivalent to spherical harmonics (angular part) and a radial part

Expansion of δf

$$\delta f(x,k) = f_0\tilde{f}_0 \sum_{\ell=0}^{\infty}\sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\rho_{n,\mu_1\cdots\mu_\ell}(x) \tag{7}$$

- Irreducible moments $\rho_n^{\mu_1\cdots\mu_\ell}$ carry all information

Irreducible moments

$$\rho_r^{\mu_1\cdots\mu_\ell}(x) \coloneqq \int \mathrm{d}K\, E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f(x,k) \tag{8}$$

$\tilde{f}_0 \coloneqq 1 - af_0$

Equations of motion

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Boltzmann equation

$$u^\mu \partial_\mu \delta f = E_{\mathbf{k}}^{-1} C - u^\mu \partial_\mu f_0 - E_{\mathbf{k}}^{-1} k^\mu \nabla_\mu (f_0 + \delta f) \tag{9}$$

- Boltzmann equation determines evolution of all moments
  - Infinite set of ordinary differential equations
  - Coupled (linearly) through generalized collision term $\mathcal{A}_{rn}^{(\ell)}$

Moment equations

$$(\ell = 0) \quad \dot\rho_r + \sum_{n=0,\neq 1,2}^{N_0} \mathcal{A}_{rn}^{(0)} \rho_n = \alpha_r^{(0)} \theta + \text{h.o.t.} \tag{10a}$$

$$(\ell = 1) \quad \dot\rho_r^{\langle\mu\rangle} + \sum_{n=0,\neq 1}^{N_1} \mathcal{A}_{rn}^{(1)} \rho_n^\mu = \alpha_r^{(1)} I^\mu + \text{h.o.t.} \tag{10b}$$

$$(\ell = 2) \quad \dot\rho_r^{\langle\mu\nu\rangle} + \sum_{n=0}^{N_2} \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu} = 2\alpha_r^{(2)} \sigma^{\mu\nu} + \text{h.o.t.} \tag{10c}$$

$$(\ell > 2) \quad \dot\rho_r^{\langle\mu_1\cdots\mu_\ell\rangle} + \sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell} = \text{h.o.t.} \tag{10d}$$

- **How to close this system?**

Matching conditions: $\rho_1 = \rho_2 = \rho_1^\mu = 0$

Truncation and power counting

- Basic idea: Power-counting scheme to **second order** in two small quantities:
  1. Knudsen number $\mathrm{Kn} := \lambda_{\mathrm{mfp}}/\lambda_{\mathrm{hydro}}$, and
  2. inverse Reynolds numbers $\mathrm{Re}^{-1} := \delta f/f_0$
- Interested in the evolution of $T^{\mu\nu}$ and N^{μ}
  - → Benchmark: Evolution equations for $\Pi = -(m^2/3)\rho_0$, $n^\mu = \rho_0^\mu$, $\pi^{\mu\nu} = \rho_0^{\mu\nu}$
  - → Only interested in moments with $\ell \leq 2$
- $\rho_r^{\mu_1\cdots\mu_{\ell>2}}$ give rise to corrections of order $\mathcal{O}(\mathrm{Kn}^2\mathrm{Re}^{-1}, \mathrm{Kn}^3)$

Moment equations

$$\sum_{n=0,\neq 1,2}^{N_0} \tau_{rn}^{(0)} \dot{\rho}_n + \rho_r = \zeta_r \theta + \mathrm{h.o.t.} \tag{11a}$$

$$\sum_{n=0,\neq 1}^{N_1} \tau_{rn}^{(1)} \dot{\rho}_n^{\langle\mu\rangle} + \rho_r^\mu = \kappa_r I^\mu + \mathrm{h.o.t.} \tag{11b}$$

$$\sum_{n=0}^{N_2} \tau_{rn}^{(2)} \dot{\rho}_n^{\langle\mu\nu\rangle} + \rho_r^{\mu\nu} = 2\eta_r \sigma^{\mu\nu} + \mathrm{h.o.t.} \tag{11c}$$

- Still coupled system of $N_0 + 3N_1 + 5N_2$ equations
- **How to decouple the remaining equations?**

$\tau^{(\ell)} := (\mathcal{A}^{(\ell)})^{-1}$

IReD: Idea

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DW, A. Palermo, V. E. Ambruș, Phys. Rev. D **106**, 016013 (2022)

- General idea: Relate moments through their Navier-Stokes solutions

IReD: Asymptotic matching

$$\rho_r = \zeta_r \theta + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) \quad \Rightarrow \quad \rho_r = \frac{\zeta_r}{\zeta_n}\rho_n + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) \tag{12}$$

$$\rho_r^\mu = \kappa_r I^\mu + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) \quad \Rightarrow \quad \rho_r^\mu = \frac{\kappa_r}{\kappa_n}\rho_n^\mu + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) \tag{13}$$

$$\rho_r^{\mu\nu} = 2\eta_r \sigma^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) \quad \Rightarrow \quad \rho_r^{\mu\nu} = \frac{\eta_r}{\eta_n}\rho_n^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) \tag{14}$$

- **Crucial:** No terms $\sim \mathcal{O}(\mathrm{Kn})$ appear in asymptotic matching ($\to \mathrm{Re}^{-1}$ dominance)
- Equations of motion can be closed in terms of any set of moments ρ_n, ρ_n^μ, $\rho_n^{\mu\nu}$

Also known as "order-of-magnitude approximation" J. A. Fotakis, E. Molnár, H. Niemi, C. Greiner, D. H. Rischke, Phys. Rev. D 106, 036009 (2022)

IReD: Closure

DW, A. Palermo, V. E. Ambruș, Phys. Rev. D **106**, 016013 (2022)

- Choose $n = 0$ to obtain closure in terms of hydrodynamic quantities

Hydrodynamic asymptotic matching

$$\rho_r = -\frac{\zeta_r}{\zeta_0}\frac{3}{m^2}\Pi + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) \tag{15}$$

$$\rho_r^{\mu} = \frac{\kappa_r}{\kappa_0} n^{\mu} + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) \tag{16}$$

$$\rho_r^{\mu\nu} = \frac{\eta_r}{\eta_0}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) \tag{17}$$

- Use asymptotic matching to absorb effects of higher-order moments into (resummed) transport coefficients while staying accurate to second order

Replacement (example)

$$\theta \sum_{n=0}^{N_2} \tau_{0n}^{(2)} \rho_n^{\mu\nu} = \theta \pi^{\mu\nu} \sum_{n=0}^{N_2} \tau_{0n}^{(2)} \frac{\eta_r}{\eta_0} + \mathcal{O}(\mathrm{Kn}^2\mathrm{Re}^{-1}) \, . \tag{18}$$

Obtaining hydrodynamics

- Use asymptotic matching conditions to express all irreducible moments through dissipative quantities
- Discard terms of order $\mathcal{O}(\mathrm{Kn}^2\mathrm{Re}^{-1})$ or higher

Hydrodynamic relaxation equations

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta_0 \theta + \mathcal{J} \tag{19a}$$

$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa_0 n^\mu + \mathcal{J}^\mu \tag{19b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta_0 \sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} \tag{19c}$$

- First-order contributions $\sim \mathcal{O}(\mathrm{Re}^{-1})$ and $\sim \mathcal{O}(\mathrm{Kn})$
- Second-order contributions $\sim \mathcal{O}(\mathrm{Re}^{-1})$, $\sim \mathcal{O}(\mathrm{Kn})$, $\sim \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1})$
  - No terms $\sim \mathcal{O}(\mathrm{Kn}^2)$ which could lead to parabolic equations
    - $\rightarrow$ Advantage over famous DNMR approach

      G. S. Denicol, H. Niemi, E. Molnar, D. H. Rischke, Phys. Rev. D **85**, 114047 (2012)

Transport coefficients

Second-order terms

$$\mathcal{J} = -\ell_{\Pi n}\nabla_\mu n^\mu - \tau_{\Pi n} n_\mu \dot{u}^\mu - \delta_{\Pi\Pi}\Pi\theta - \lambda_{\Pi n} n_\mu \nabla^\mu \alpha + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} \,, \tag{20}$$

$$\begin{aligned}\mathcal{J}^\mu = &-\tau_n n_\nu \omega^{\nu\mu} - \delta_n n^\mu \theta - \ell_{n\Pi}\nabla^\mu \Pi + \ell_{n\pi}\Delta^{\mu\nu}\nabla_\lambda \pi^\lambda{}_\nu + \tau_{n\Pi}\Pi \dot{u}^\mu \\ &- \tau_{n\pi}\pi^{\mu\nu}\dot{u}_\nu - \lambda_{nn} n_\nu \sigma^{\mu\nu} + \lambda_{n\Pi}\Pi\nabla^\mu\alpha - \lambda_{n\pi}\pi^{\mu\nu}\nabla_\nu \alpha \,,\end{aligned} \tag{21}$$

$$\begin{aligned}\mathcal{J}^{\mu\nu} = &\, 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} \\ &- \tau_{\pi n} n^{\langle\mu}\dot{u}^{\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu} n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha \,.\end{aligned} \tag{22}$$

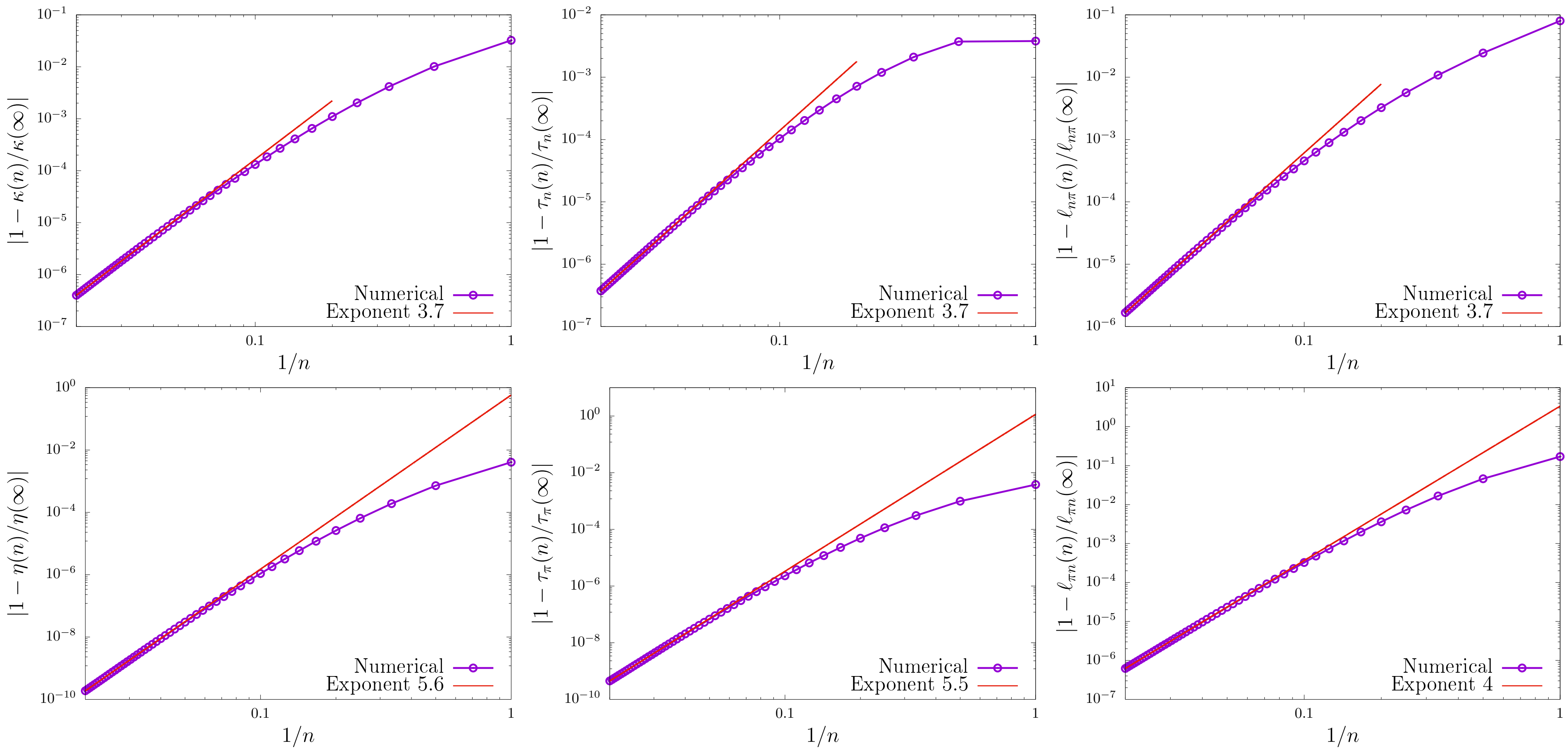
- All transport coefficients calculable to any finite order $N_0 \geq 2,\, N_1 \geq 1,\, N_2 \geq 0$
- Red ones determine the behaviour of the linearized theory

Interlude: Convergence of the expansion

- Simple model with constant cross-section: Generalized collision terms can be calculated analytically

  DW, V. E. Ambruș, E. Molnár, in preparation

- All coefficients converge, but at different rates



Entropy analysis

Entropy current

$$S^\mu = S^\mu_{(0)} + S^\mu_{(1)} + S^\mu_{(2)} + \cdots \ , \quad (23)$$

$$S^\mu_{(0)} = su^\mu \ , \quad (24)$$

$$S^\mu_{(1)} = -\alpha n^\mu \ , \quad (25)$$

$$S^\mu_{(2)} = -\frac{1}{2} u^\mu (\delta_0 \Pi^2 + \delta_1 n^\alpha n_\alpha + \delta_2 \pi^{\alpha\beta}\pi_{\alpha\beta}) - \gamma_0 \Pi n^\mu - \gamma_1 \pi^{\mu\nu} n_\nu \ . \quad (26)$$

- Idea: Construct entropy current up to second order in dissipative quantities
- Take divergence and assert $\partial_\mu S^\mu \geq 0$
- Guaranteed by bringing the divergence into quadratic form,

$$\partial_\mu S^\mu \sim \Pi^2 \ , -n^\mu n_\mu \ , \pi^{\mu\nu}\pi_{\mu\nu} \quad (27)$$

- Forces dissipative quantities to obey relaxation equations
  - Coefficients are related!
- **Which conditions do we get?**

2^{nd} law: Conditions

- Simple setup:
  - Constant cross section
  - Massless limit

URHS conditions

- Automatically fulfilled

$$\delta_{nn} = \tau_n \;, \quad \delta_{\pi\pi} = 4\tau_\pi/3 \;, \quad \frac{\tau_{n\pi}}{\ell_{n\pi}} + \frac{\tau_{\pi n}}{\ell_{\pi n}} = \frac{5}{\epsilon + P} \tag{28}$$

Nontrivial conditions

- Fulfilled in IReD **in the limit** $N_1, N_2 \to \infty$
- **Not** fulfilled in DNMR

$$\frac{\ell_{n\pi}}{\kappa} = -\frac{\ell_{\pi n}}{2\eta T} \;, \qquad \frac{\ell_{n\Pi}}{\kappa} = -\frac{\ell_{\Pi n}}{\zeta T} \tag{29}$$

- Coefficients obtained using IReD are compatible with second law

  $\rightarrow$ Thermodynamic completion of DNMR

  L. Gavassino, arXiv: 2210.05067

Conclusion and Outlook

- The IReD approach to relativistic dissipative hydrodynamics relates irreducible moments $(\rho_r,\ \rho_r^{\mu},\ \rho_r^{\mu\nu})$ directly to dissipative quantities $(\Pi,\ n^{\mu},\ \pi^{\mu\nu})$
  - $\rightarrow$ No terms $\sim \mathcal{O}(\mathrm{Kn}^2)$ appear in equations of motion
  - $\rightarrow$ Equations stay **hyperbolic**
- When an interaction is specified, all first- and second-order transport coefficients can be calculated
- Values of the transport coefficients are consistent with the second law of thermodynamics

Appendix

Hard spheres collision matrix

- The collision matrix is linked with the expansion of $\delta f_{\mathbf{k}}$ with respect to a complete basis,

$$\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \mathcal{H}_{\mathbf{k}n}^{(\ell)},$$

where $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$ is defined such that $\rho_n^{\mu_1 \cdots \mu_\ell} \equiv \int dK E_{\mathbf{k}}^n k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \delta f_{\mathbf{k}}$.

- The linearized collision integrals are given by

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu(2\ell+1)} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^{r-1} k^{\langle \nu_1} \cdots k^{\nu_\ell \rangle}$$
$$\times \left(\mathcal{H}_{\mathbf{k}n}^{(\ell)} k_{\langle \nu_1} \cdots k_{\nu_\ell \rangle} + \mathcal{H}_{\mathbf{k}'n}^{(\ell)} k'_{\langle \nu_1} \cdots k'_{\nu_\ell \rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)} p_{\langle \nu_1} \cdots p_{\nu_\ell \rangle} - \mathcal{H}_{\mathbf{p}'n}^{(\ell)} p'_{\langle \nu_1} \cdots p'_{\nu_\ell \rangle} \right) ,$$

- In the case of the UR ideal HS gas, $W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} = s(2\pi)^6 \delta^{(4)}(k + k' - p - p') \frac{\sigma_T \nu}{4\pi}$ and

$$\mathcal{A}_{r=0,n}^{(1)} = \frac{16(-\beta)^n g^2}{\lambda_{\text{mfp}}(n+3)!} \left[S_n^{(1)}(N_1) - \frac{\delta_{n0}}{2} \right], \qquad \mathcal{A}_{r=0,n}^{(2)} = \frac{432 g^2 (-\beta)^n}{\lambda_{\text{mfp}}(n+5)!} S_n^{(2)}(N_2),$$

$$\mathcal{A}_{r>0,n\le r}^{(1)} = \frac{g^2 \beta^{n-r} (r+2)! [n(r+4) - r]}{\lambda_{\text{mfp}} (n+3)! r} \times \left(\delta_{nr} + \delta_{n0} - \frac{2}{r+1} \right),$$

$$\mathcal{A}_{r>0,n\le r}^{(2)} = \frac{g^2 \beta^{n-r} (r+4)! (n+1)}{\lambda_{\text{mfp}} (n+5)! r (r+1)} \times (9n + nr - 4r) \left(\delta_{nr} - \frac{2}{r+2} \right),$$

while $\mathcal{A}_{r>0,n>r}^{(1)} = \mathcal{A}_{r>0,n>r}^{(2)} = 0$ and $S_n^{(\ell)}(N_\ell) = \sum_{m=n}^{N_\ell} \binom{m}{n} \frac{1}{(m+\ell)(m+\ell+1)}$.

DNMR: Idea

G. S. Denicol, H. Niemi, E. Molnar, D. H. Rischke, Phys. Rev. D **85**, 114047 (2012)

- Idea: Only the slowest microscopic timescales are of macroscopic importance (*Separation of scales*)
- Program to follow:
  1. Find the eigenmodes $X_r^{(\ell)}$ of the linearized collision kernel $\mathcal{A}^{(\ell)}$
  2. Retain dynamics only of slowest eigenmodes
  3. Express dynamics of hydrodynamic quantities through eigenmodes
- First step: **Diagonalize** (inverse) collision matrices $\tau^{(\ell)} \equiv (\Omega^{(\ell)})^{-1}\mathrm{diag}(\tau_1^{(\ell)}, \tau_2^{(\ell)}, \cdots)\Omega^{(\ell)}$
- **Sort** eigenvalues in **decreasing** order
  - Lowest-order eigenmodes relax slowest

Relaxation equation of eigenmodes

$$\tau_r^{(0)}\dot{X}_r + X_r = -\sum_{n=0}^{N_0}\Omega_{rn}^{(0)}\zeta_n\theta + \mathrm{h.o.t.} \quad (30a)$$

$$\tau_r^{(1)}\dot{X}_r^{\langle\mu\rangle} + X_r^{\mu} = \sum_{n=0}^{N_1}\Omega_{rn}^{(1)}\kappa_n I^{\mu} + \mathrm{h.o.t.} \quad (30b)$$

$$\tau_r^{(2)}\dot{X}_r^{\langle\mu\nu\rangle} + X_r^{\mu\nu} = 2\sum_{n=0}^{N_2}\Omega_{rn}^{(2)}\eta_n\sigma^{\mu\nu} + \mathrm{h.o.t.} \quad (30c)$$

DNMR: Separation of timescales

- Apply the *separation of scales* idea and retain dynamics of X_0, X_0^μ and $X_0^{\mu\nu}$
- **Crucial step:** Higher moments are approximated by their Navier-Stokes solutions

$$X_{r>2} = -\sum_{n=0}^{N_0} \Omega_{rn}^{(0)} \zeta_n \theta\,, \;\; X_{r>1}^{\mu} = \sum_{n=0}^{N_1} \Omega_{rn}^{(1)} \kappa_n I^\mu\,, \;\; X_{r>0}^{\mu\nu} = 2\sum_{n=0}^{N_2} \Omega_{rn}^{(2)} \eta_n \sigma^{\mu\nu}$$

- Relate irreducible moments back to dissipative quantities via $\rho_r^{\mu_1 \cdots \mu_\ell} = \sum_{n=0}^{N_\ell} \Omega_{rn}^{(\ell)} X_n^{\mu_1 \cdots \mu_\ell}$ and apply approximation

DNMR: Asymptotic matching

$$m^2/3\rho_r = -\Omega_{r0}^{(0)} \Pi - \left(\zeta_r - \Omega_{r0}^{(0)} \zeta_0\right) \theta + \mathcal{O}(\mathrm{KnRe}^{-1}) \tag{31a}$$

$$\rho_r^\mu = \Omega_{r0}^{(1)} n^\mu + \left(\kappa_r - \Omega_{r0}^{(1)} \kappa_0\right) I^\mu + \mathcal{O}(\mathrm{KnRe}^{-1}) \tag{31b}$$

$$\rho_r^{\mu\nu} = \Omega_{r0}^{(2)} \pi^{\mu\nu} + \left(\eta_r - \Omega_{r0}^{(2)} \eta_0\right) \sigma^{\mu\nu} + \mathcal{O}(\mathrm{KnRe}^{-1}) \tag{31c}$$

- This closes the system of equations

DNMR: Obtaining hydrodynamics

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- Use asymptotic matching to express all irreducible moments through dissipative quantities and fluid-dynamical gradients
- Discard terms of order $\mathcal{O}(\mathrm{Kn}^2\mathrm{Re}^{-1})$ or higher

Hydrodynamic relaxation equations (DNMR)

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta_0\theta + \mathcal{J} + \mathcal{K} \tag{32a}$$

$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa_0 n^\mu + \mathcal{J}^\mu + \mathcal{K}^\mu \tag{32b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta_0\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} \tag{32c}$$

- First-order contributions $\sim \mathcal{O}(\mathrm{Re}^{-1})$ and $\sim \mathcal{O}(\mathrm{Kn})$
- Second-order contributions $\sim \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1})$ and $\sim \mathcal{O}(\mathrm{Kn}^2)$
- Contributions of order $\mathcal{O}(\mathrm{Kn}^2)$ result directly from asymptotic matching
  - Example: $\theta\rho_r \to \theta\Pi\,,\ \theta^2$

The issue: Parabolic terms

- Consider the second-order terms of tensor-rank two:

$$\mathcal{J}^{\mu\nu} = 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n} n^{\langle\mu}F^{\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}I^{\nu\rangle}\,, \tag{33}$$

$$\mathcal{K}^{\mu\nu} = \tilde{\eta}_1 \omega^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda + \tilde{\eta}_2\theta\sigma^{\mu\nu} + \tilde{\eta}_3\sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \tilde{\eta}_4\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \tilde{\eta}_5 I^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_6 F^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_7 I^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_8\nabla^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_9\nabla^{\langle\mu}F^{\nu\rangle} \tag{34}$$

- Second derivatives of fluid-dynamical quantities appear
  - → Equations become **parabolic**!
  - → Theory becomes acausal and thus unstable
- Usual procedure: **Ignore** terms of order $\mathcal{O}(\mathrm{Kn}^2)$
  - → Equations are hyperbolic again
- IReD is a way to ensure $\mathcal{K} = \mathcal{K}^\mu = \mathcal{K}^{\mu\nu} = 0$ from the beginning!

$F^\mu := \nabla^\mu P_0$, $\omega^{\mu\nu} := (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$