GPU accelerated parallel computing of iterated function systems in mechatronic applications

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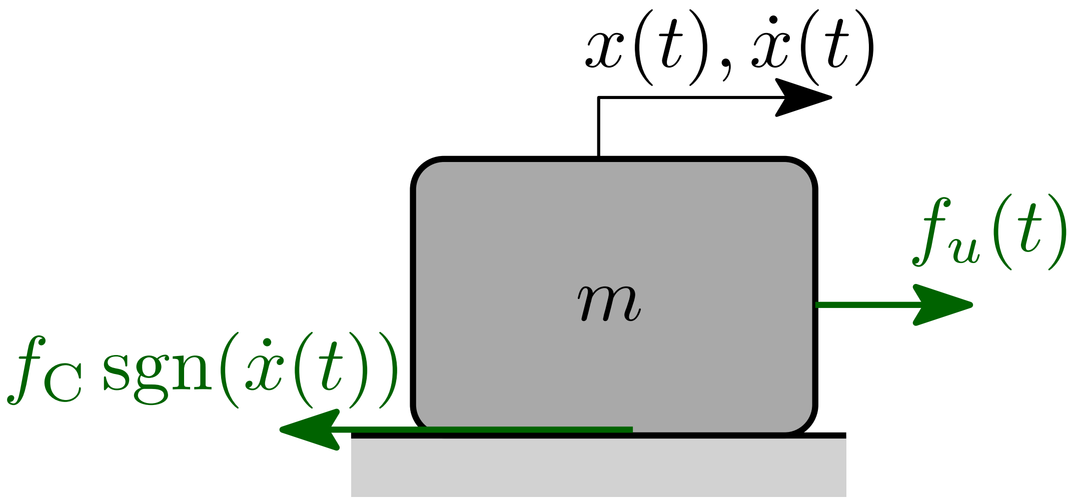
Introduction

- Main topics of the talk:
  1. Position control problem in mechatronics
  2. Analyzing the problem with Iterated Function Systems (IFS's)
  3. Efficient IFS evaluation on GPU
  4. Results and conclusions
- Somewhat unusual approach of a practical problem involving fractals and GPU programming

Position control task

Initial problem:

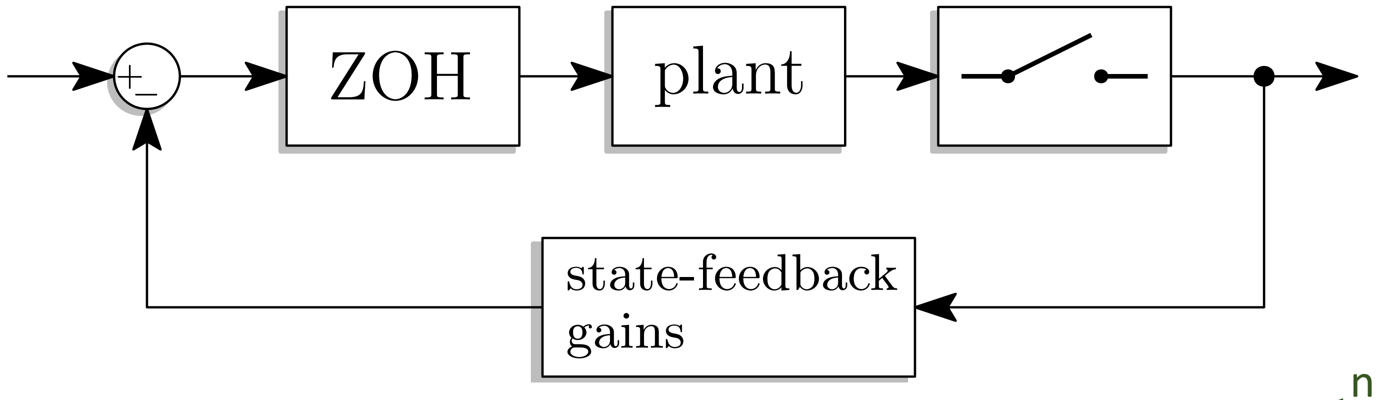
Find a control input $f_u(t)$ which drives the following one-degree-of-freedom (1DoF) system to the $x(t) = 0, \dot{x}(t) = 0$ state!



Solution: use closed-loop control

Position control with sampled-data state-feedback controller

- Closed-loop control structure



- Plant: equation of motion

$$m\,\ddot{x}(t) = f_u(t) - f_C\,\mathrm{sgn}\big(\dot{x}(t)\big)$$

non-smooth friction characteristics

- Controller: control law with sampling and zero order hold

$$f_u(t) = -k_p\,x(k\tau) - k_d\dot{x}(k\tau)$$

for $k\tau \le t < (k+1)\tau$ and $k = 0,1,2,\ldots$

piecewise-smooth control input

Dimension analysis

- Original system: 5 free parameters (m, f_C, k_p, k_d, τ)

$$m\,\ddot{x}(t) = -k_p\,x(k\tau) - k_d\dot{x}(k\tau) - f_C\,\mathrm{sgn}\big(\dot{x}(t)\big)$$

for $k\tau \le t < (k+1)\tau$ and $k = 0,1,2,0\ldots$

- Introducing dimensionless time $T = t/\tau$

$$x''(T) = -p\,x_k - d v_k - \sigma\,\mathrm{sgn}\big(x'(T)\big)$$

for $k \le T < k+1$ and $k = 0,1,2,0\ldots$

Where $\square' \coloneqq \frac{\mathrm{d}}{\mathrm{d}T}\square, \square'' \coloneqq \frac{\mathrm{d}^2}{\mathrm{d}T^2}\square, x_k = x(k), v_k = x'(k)$

- Only 3 free parameters left: $p = \frac{k_p\tau^2}{m}, d = \frac{k_d\tau}{m}, \sigma = \frac{f_C\tau}{m}$

Discretized state-space model

- Assuming no motion reversal between sampling instants
- Discrete-time model

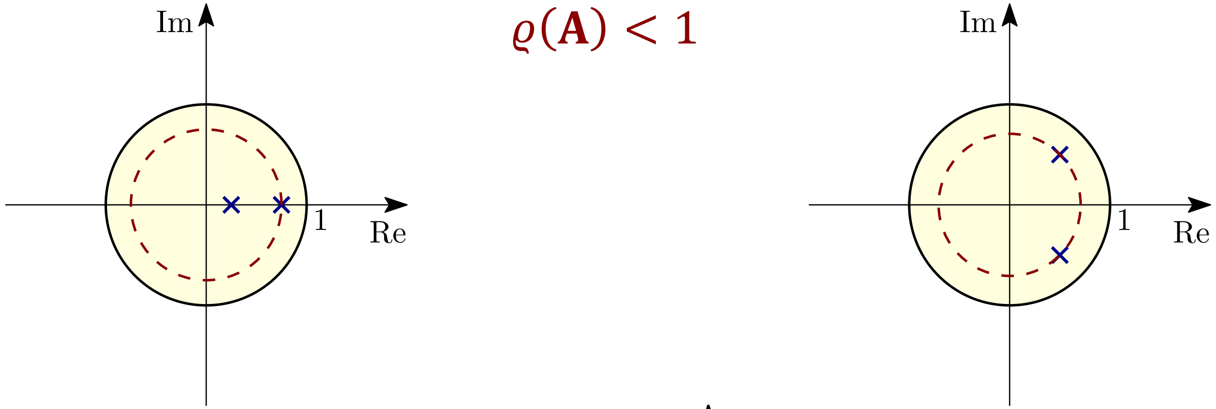
$$\begin{bmatrix} x_{k+1} \\ v_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1-\frac{p}{2} & 1-\frac{d}{2} \\ -p & 1-d \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_k \\ v_k \end{bmatrix} - \operatorname{sgn}(v_k) \underbrace{\begin{bmatrix} \frac{\sigma}{2} \\ \sigma \end{bmatrix}}_{\mathbf{a}}$$

$$\mathbf{x}_{k+1} = \begin{cases} \mathbf{A}\,\mathbf{x}_k - \mathbf{a} & \text{if} \quad v_k > 0 \\ \mathbf{A}\,\mathbf{x}_k + \mathbf{a} & \phantom{\text{if}} \quad v_k < 0 \end{cases}$$

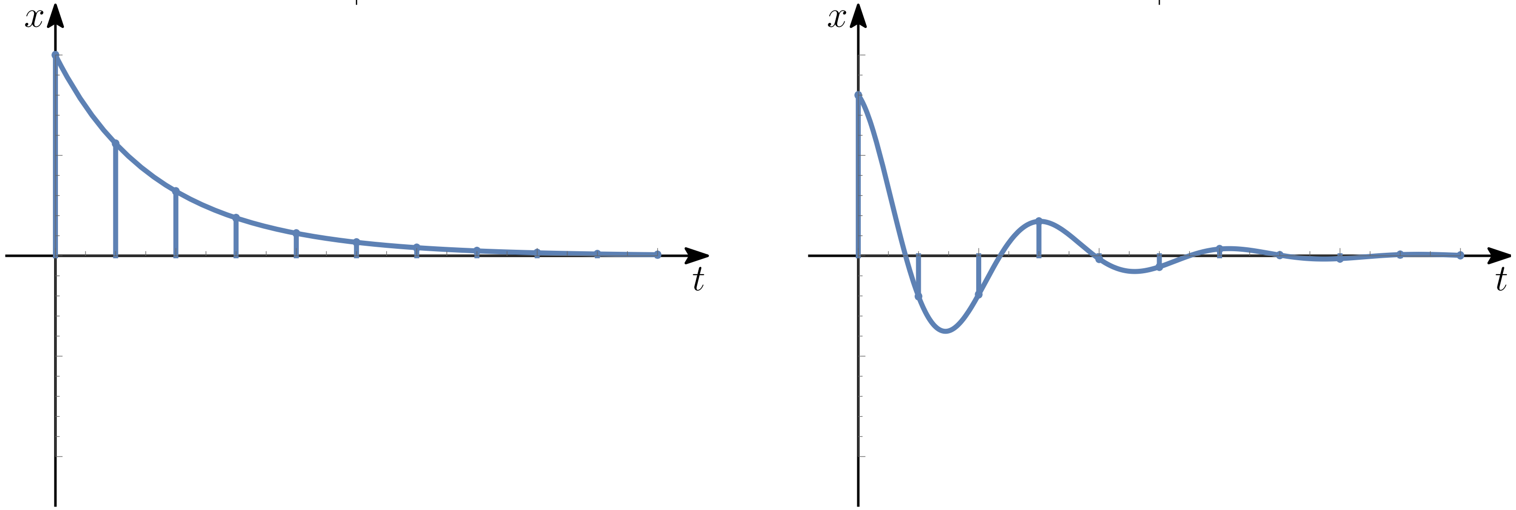
- Special case – frictionless system: $\mathbf{x}_{k+1} = \mathbf{A}\,\mathbf{x}_k$

Stability of the frictionless system

- Eigenvalues and spectral radius of $\mathbf{A}$:

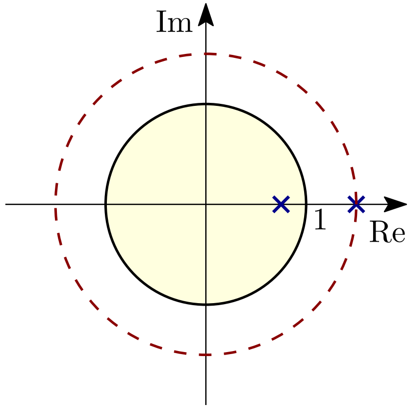


- Position vs time:

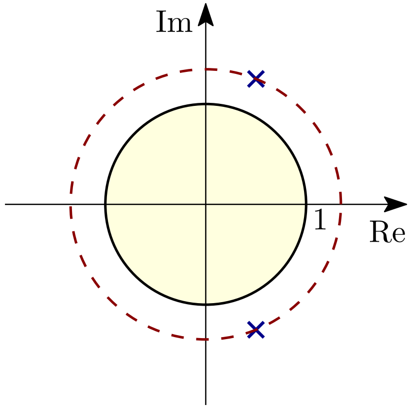


Stability of the frictionless system

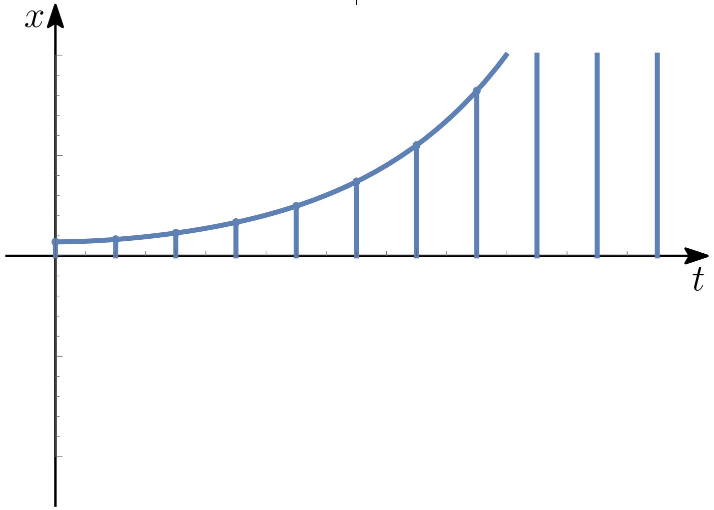
- Eigenvalues and spectral radius of $\mathbf{A}$:

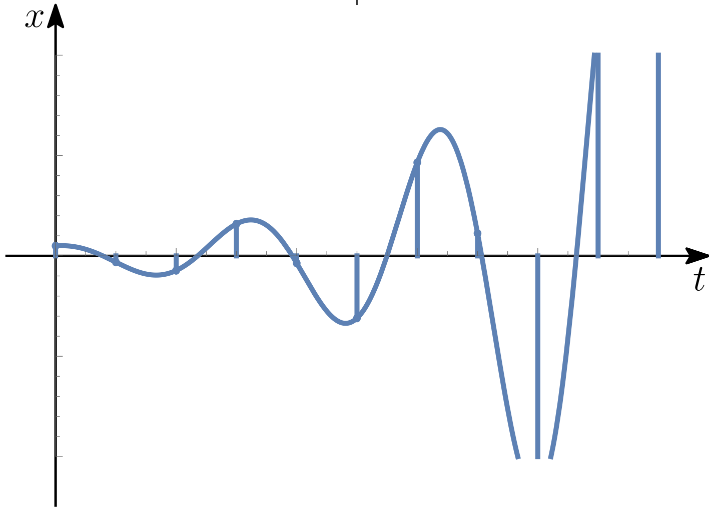


$\varrho(\mathbf{A}) > 1$



- Position vs time:





Iterated function systems

- A set of contraction mappings $\{f_i\}$ on a metric space
- S is a fixed set with the property

$$S = \bigcup_{i=1}^{N} f_i(S)$$

- Or it can be expressed with the generator function

$$F(A) = \bigcup_{i=1}^{N} f_i(A)$$

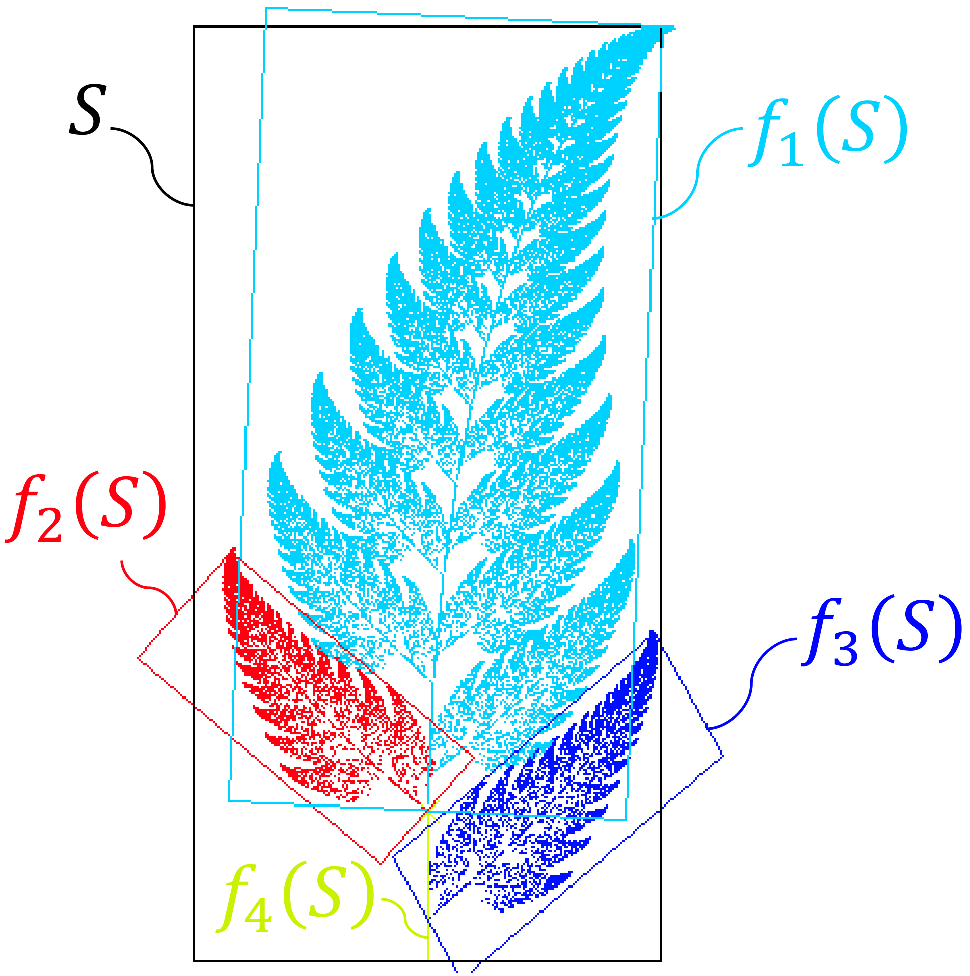
as the following limit on any initial set A:

$$S = \lim_{n\to\infty} F^n(A)$$

Example: Barnsley's Fern

$$S = \bigcup_{i=1}^{N} f_i(S)$$

- $f_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$
- $f_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$
- $f_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}$
- $f_4\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$



Position control task as an IFS

- Discrete-time model describing the original system:

$$\mathbf{x}_{k+1} = \begin{cases} \mathbf{A}\,\mathbf{x}_k - \mathbf{a} & \text{if} \quad v_k > 0 \\ \mathbf{A}\,\mathbf{x}_k + \mathbf{a} & \qquad v_k < 0 \end{cases}$$

- IFS with $\{f_1, f_2\}$ where

$$f_1(\mathbf{x}) = \mathbf{Ax} - \mathbf{a}$$
$$f_2(\mathbf{x}) = \mathbf{Ax} + \mathbf{a}$$

Calculating S with generator function

- Generator function of the IFS

$$F(A) = \bigcup_{i=1}^{N} f_i(A) = f_1(A) \cup f_2(A)$$

where $A = \{\mathbf{x}\}$ is a set of $\mathbf{x} = \begin{bmatrix} x \\ v \end{bmatrix}$ state vectors

$$F\colon \{\mathbf{x}\} \mapsto \{\mathbf{A}\,\mathbf{x} - \mathbf{a}\,\mathrm{sgn}(v)\}$$

- Goal is to calculate S

$$S = \lim_{n\to\infty} F^n(A)$$

Generator function implemented as OpenCL kernel

```
__kernel void generator_function(float p, float d, float sigma, __global float *x, __global float *v, unsigned long n){
    // check global id to prevent overindexing
    size_t gid = get_global_id(0);
    if(gid>=n)
        return;

    // copy state values
    float x0 = x[gid];
    float v0 = v[gid];

    // calculate matrix coefficients
    float A11=1-p/2, A12=1-d/2;
    float A21=-p,    A22=1-d;
    float a1 = sigma/2;
    float a2 = sigma;

    // apply mapping
    x[gid] = A11*x0 + A12*v0 - a1*sign(v0);
    v[gid] = A21*x0 + A22*v0 - a2*sign(v0);
}
```

$$\mathbf{A} = \begin{bmatrix} 1-\frac{p}{2} & 1-\frac{d}{2} \\ -p & 1-d \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} \frac{\sigma}{2} \\ \sigma \end{bmatrix}$$

$$\mathbf{x} \mapsto \mathbf{A}\,\mathbf{x} - \mathbf{a}\,\mathrm{sgn}(v)$$

Simulation results

- Test parameters:

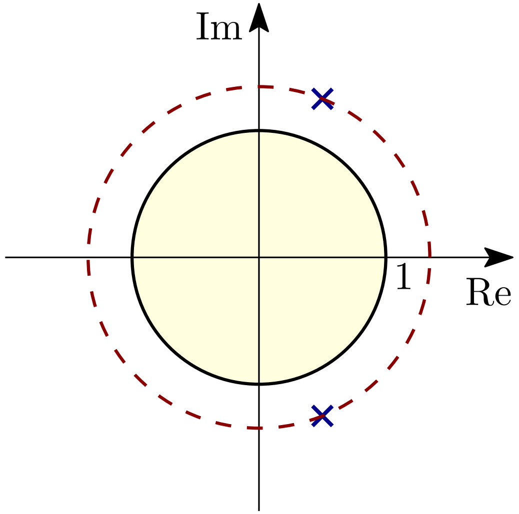
$$p = 0.430163$$
$$d = 0.00508131$$
$$\sigma = 1$$

- Matrices of the mappings:

$$\mathbf{A} = \begin{bmatrix} 0.7849 & 0.9975 \\ -0.4302 & 0.9949 \end{bmatrix}$$

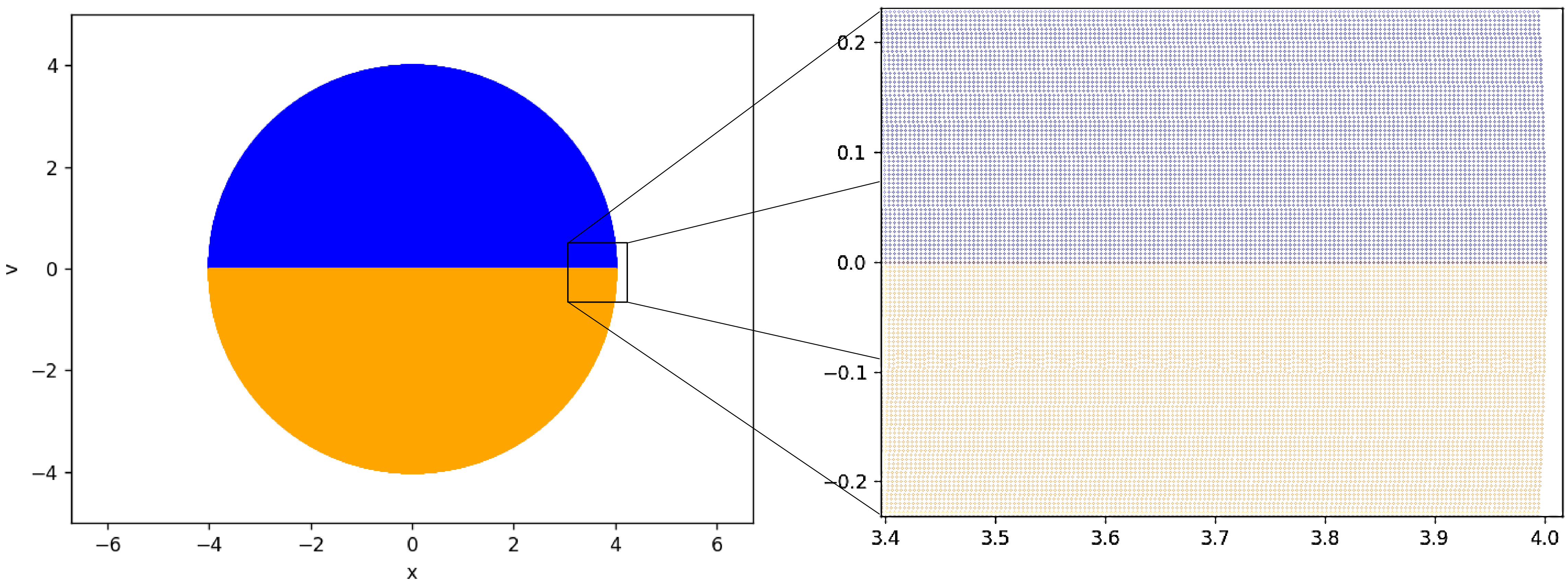
$$\mathbf{a} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

- Spectral radius > 1 $\Rightarrow$ non contractive mapping

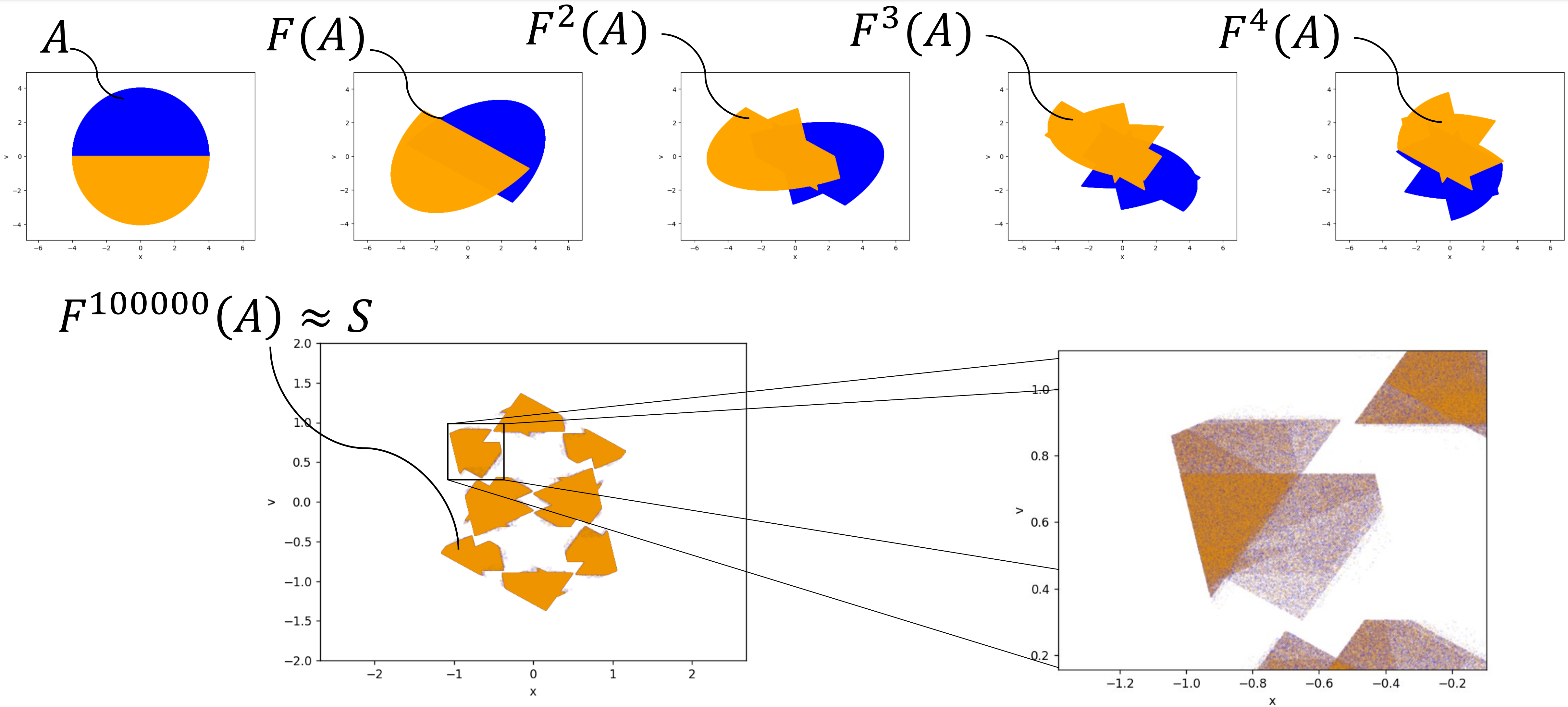


Simulation results

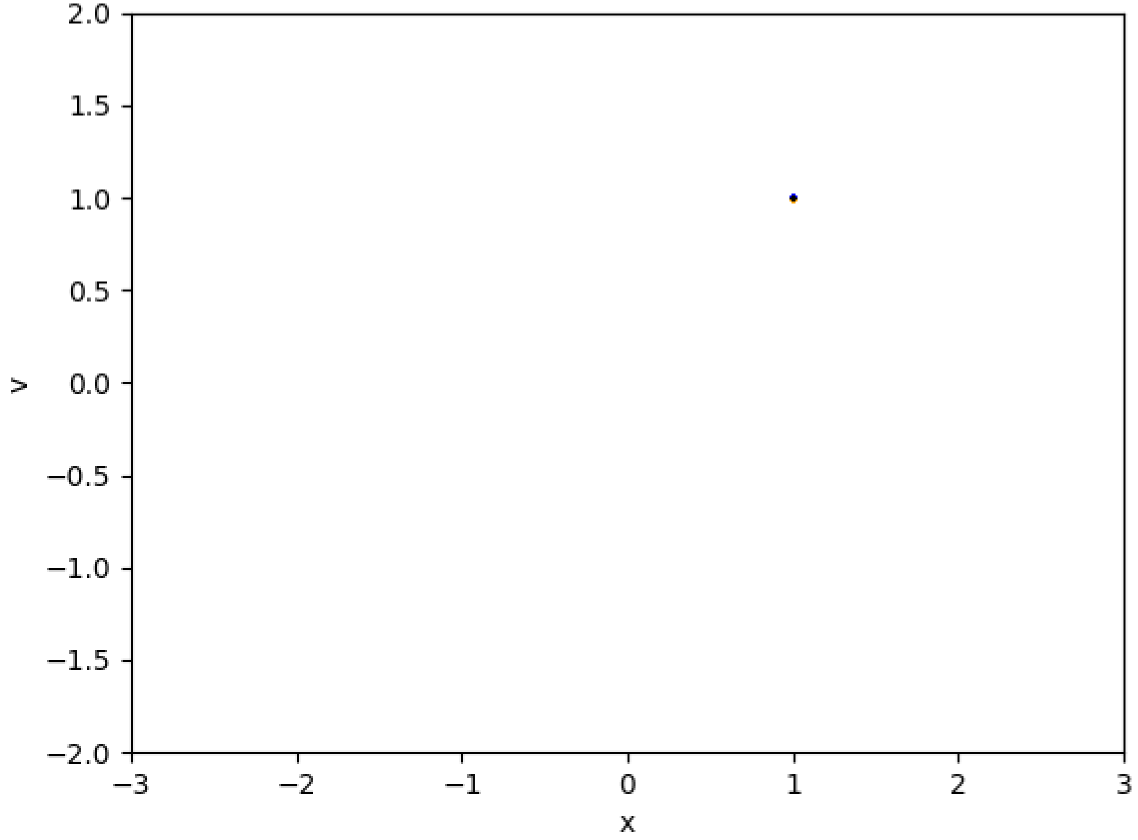
- 1573079 separate points in the state space

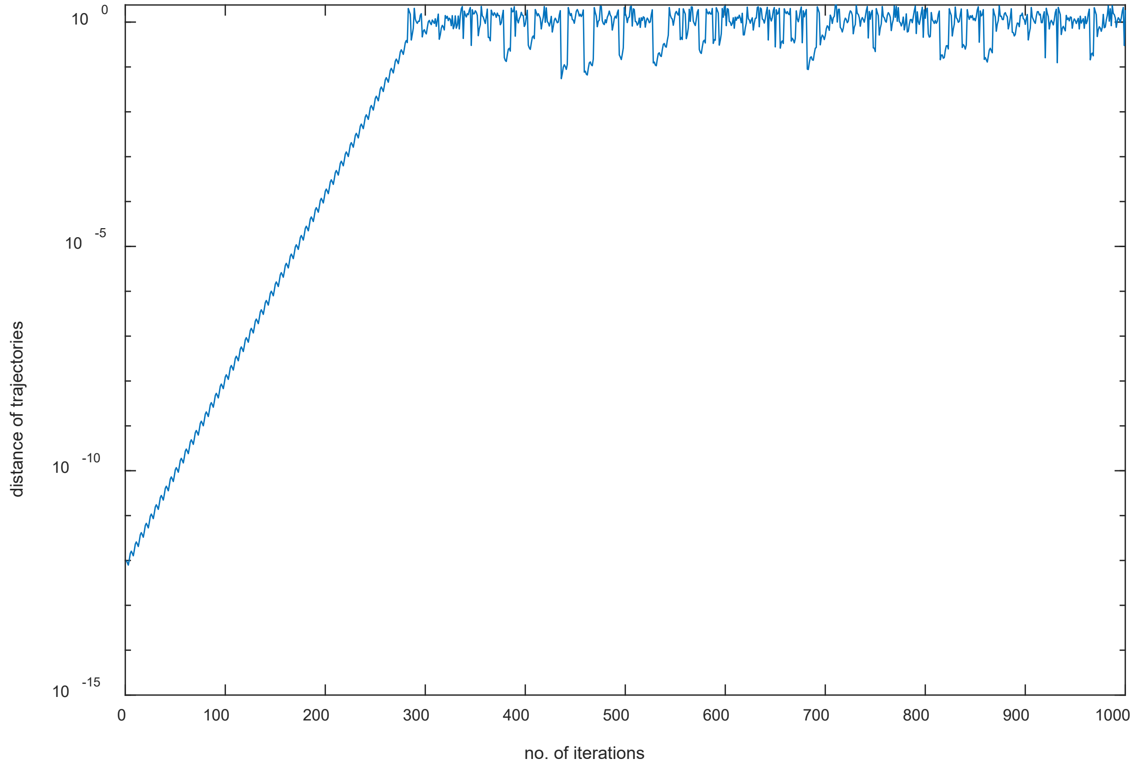


Simulation results



Separation of trajectories





Estimated domindant Lyapunov-exponent:
$\lambda \approx 0.0952 > 0 \Rightarrow$ chaotic behavior

Summary

- Analysis of a position control task with dry friction
- Considering system dynamics as an iterated function systems (IFS)
- IFS evaluation on GPU
- Simulation results: chaotic behavior can occur

Thank you for your attention!

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