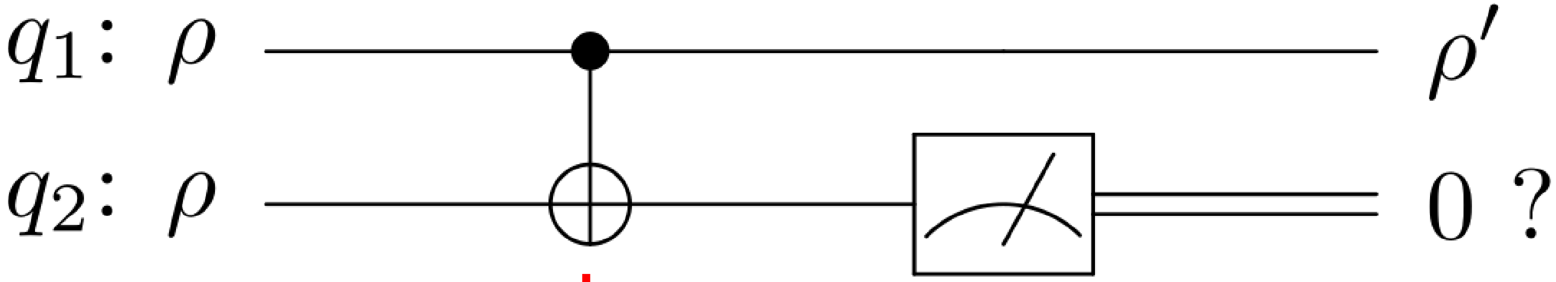
Iterated nth order nonlinear quantum dynamics

ELTE | FACULTY OF SCIENCE

WIGNER

GPU Day 2023

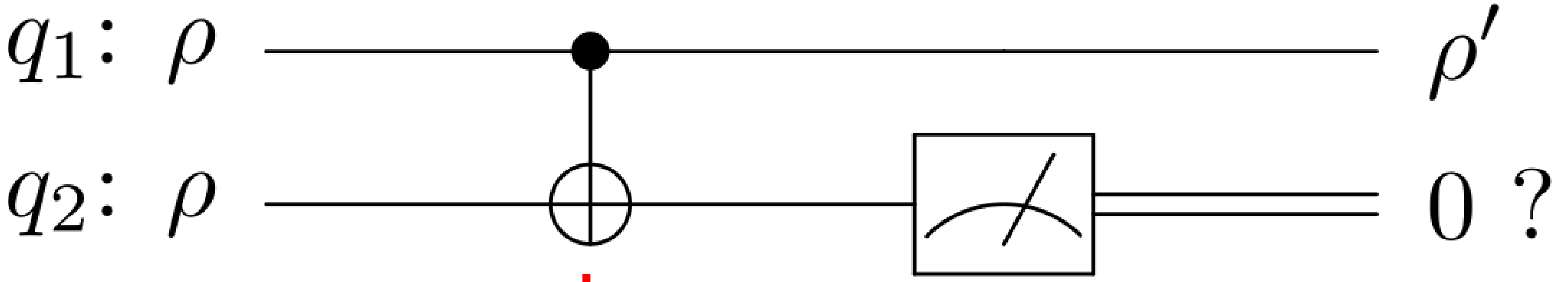
Nonlinear quantum protocol



$$CX = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

$$\rho = \frac{1}{\rho_{11} + \rho_{22}} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

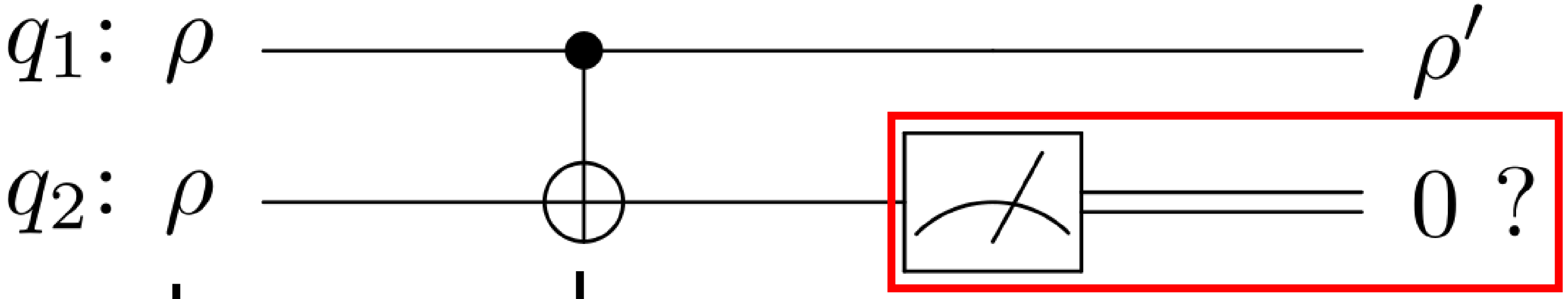
Nonlinear quantum protocol



$$CX\,\rho \otimes \rho\, CX^\dagger = \begin{bmatrix} \rho_{11}^2 & \rho_{11}\rho_{22} & \rho_{12}^2 & \rho_{12}\rho_{11} \\ \rho_{11}\rho_{21} & \rho_{11}\rho_{22} & \rho_{21}\rho_{11} & \rho_{12}\rho_{21} \\ \rho_{21}^2 & \rho_{21}\rho_{22} & \rho_{22}^2 & \rho_{22}\rho_{21} \\ \rho_{21}\rho_{11} & \rho_{21}\rho_{12} & \rho_{22}\rho_{12} & \rho_{22}\rho_{11} \end{bmatrix}$$

$$\rho = \frac{1}{\rho_{11} + \rho_{22}} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

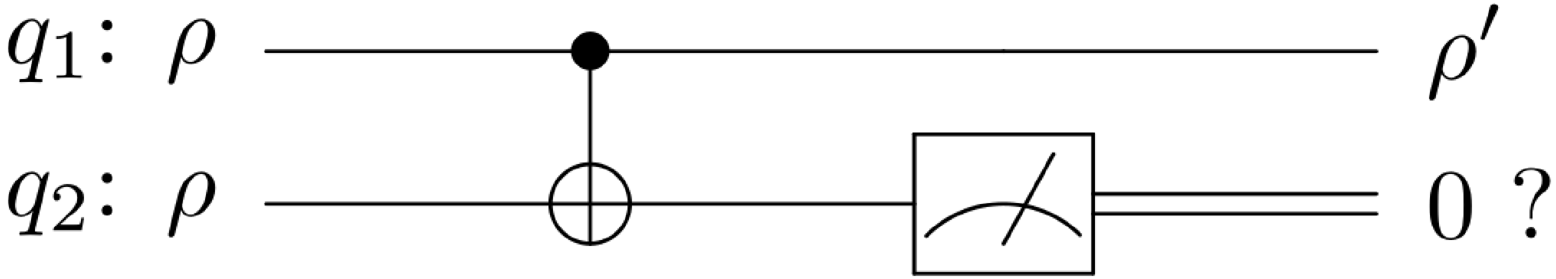
Nonlinear quantum protocol



$$CX\rho \otimes \rho CX^\dagger = \begin{array}{cc} \begin{bmatrix} \boxed{\rho_{11}^2} & \rho_{11}\rho_{22} & \boxed{\rho_{12}^2} & \rho_{12}\rho_{11} \\ \rho_{11}\rho_{21} & \rho_{11}\rho_{22} & \rho_{21}\rho_{11} & \rho_{12}\rho_{21} \\ \boxed{\rho_{21}^2} & \rho_{21}\rho_{22} & \boxed{\rho_{22}^2} & \rho_{22}\rho_{21} \\ \rho_{21}\rho_{11} & \rho_{21}\rho_{12} & \rho_{22}\rho_{12} & \rho_{22}\rho_{11} \end{bmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \\ \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \end{array}$$

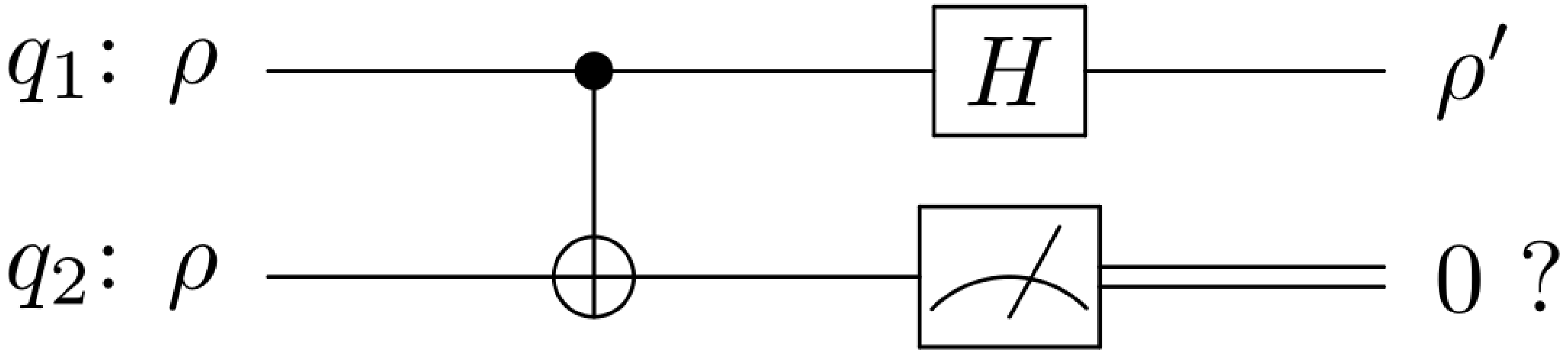
$$\rho = \frac{1}{\rho_{11} + \rho_{22}} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

Nonlinear quantum protocol



$$\rho = \frac{1}{\rho_{11} + \rho_{22}} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \longrightarrow \rho' = \frac{1}{\rho_{11}^2 + \rho_{22}^2} \begin{bmatrix} \rho_{11}^2 & \rho_{12}^2 \\ \rho_{21}^2 & \rho_{22}^2 \end{bmatrix}$$

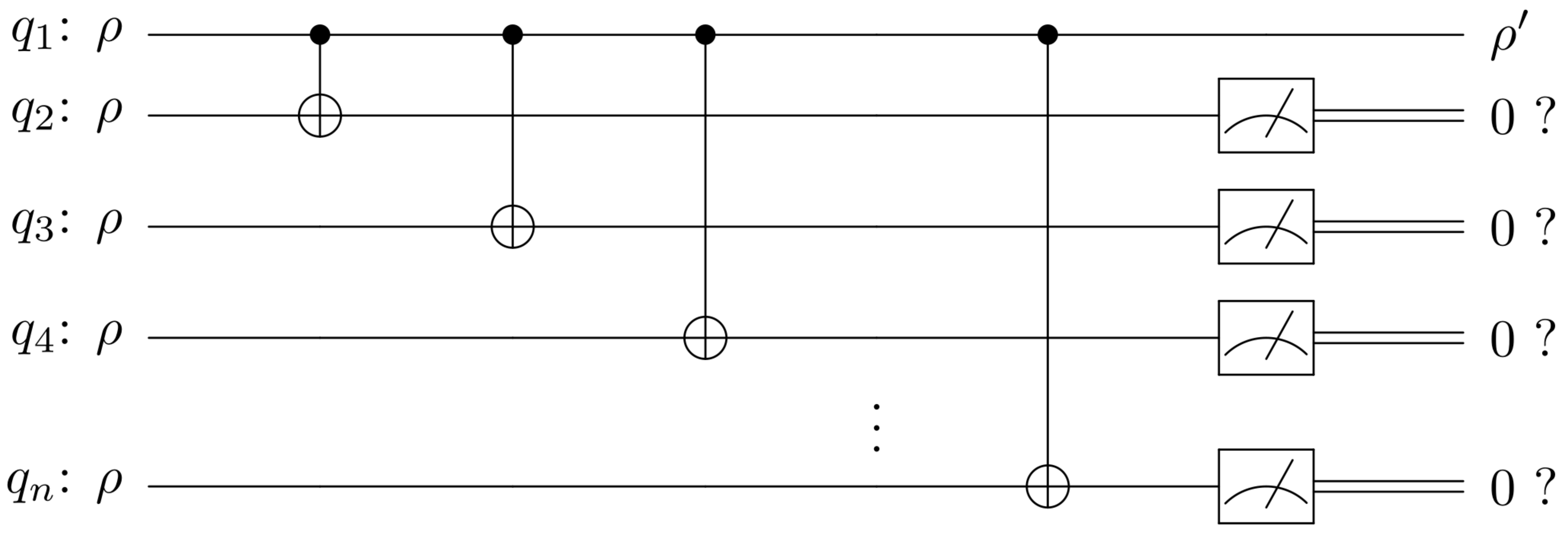
Nonlinear quantum protocol



$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

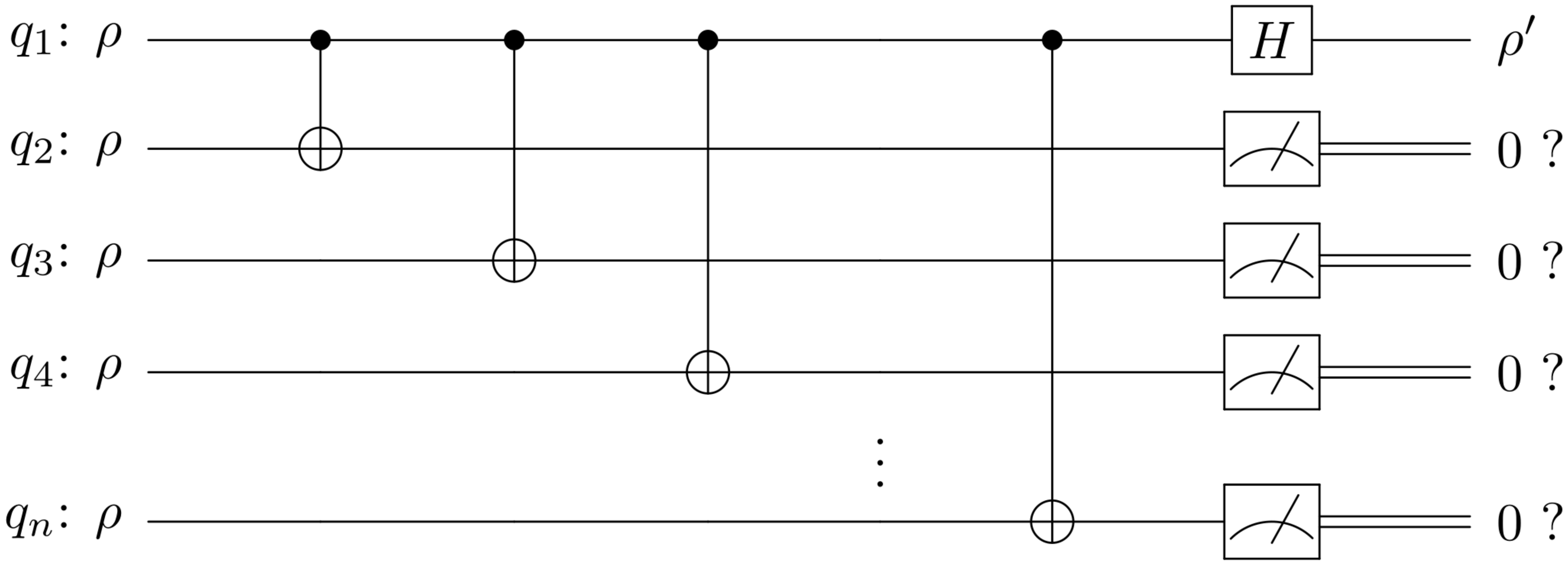
$$\rho = \frac{1}{\rho_{11} + \rho_{22}} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \longrightarrow \rho' = S(\rho) = HT(\rho)H^{\dagger}$$

The nth order nonlinear quantum protocol



$$\rho = \frac{1}{\rho_{11} + \rho_{22}} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \longrightarrow \rho' = \frac{1}{\rho_{11}^n + \rho_{22}^n} \begin{bmatrix} \rho_{11}^n & \rho_{12}^n \\ \rho_{21}^n & \rho_{22}^n \end{bmatrix}$$

The nth order nonlinear quantum protocol



$$\rho' = S(\rho) = HT(\rho)\,H^\dagger$$

Bloch coordinates

$$\rho' = S(\rho) = HT(\rho)\,H^{\dagger}$$

$$\rho = \frac{1}{2}\left(I + \vec{v}\cdot\vec{\sigma}\right) = \frac{1}{2}\begin{bmatrix} 1+w & u-iv \\ u+iv & 1-w \end{bmatrix}$$

$$\rightarrow \vec{v} = [u, v, w] \text{ and } \vec{\sigma} = [\sigma_x, \sigma_y, \sigma_z]$$

$$\rho' = \frac{1}{2}\begin{bmatrix} 1+w' & u'-iv' \\ u'+iv' & 1-w' \end{bmatrix}$$

Bloch coordinates

- second order protocol

$$\rho' = S(\rho) = HT(\rho)H^\dagger$$

$$\updownarrow$$

$$u' = \frac{2w}{1+w^2} \qquad v' = \frac{-2uv}{1+w^2} \qquad w' = \frac{u^2 - v^2}{1+w^2}$$

Bloch coordinates

- *n*th order protocol

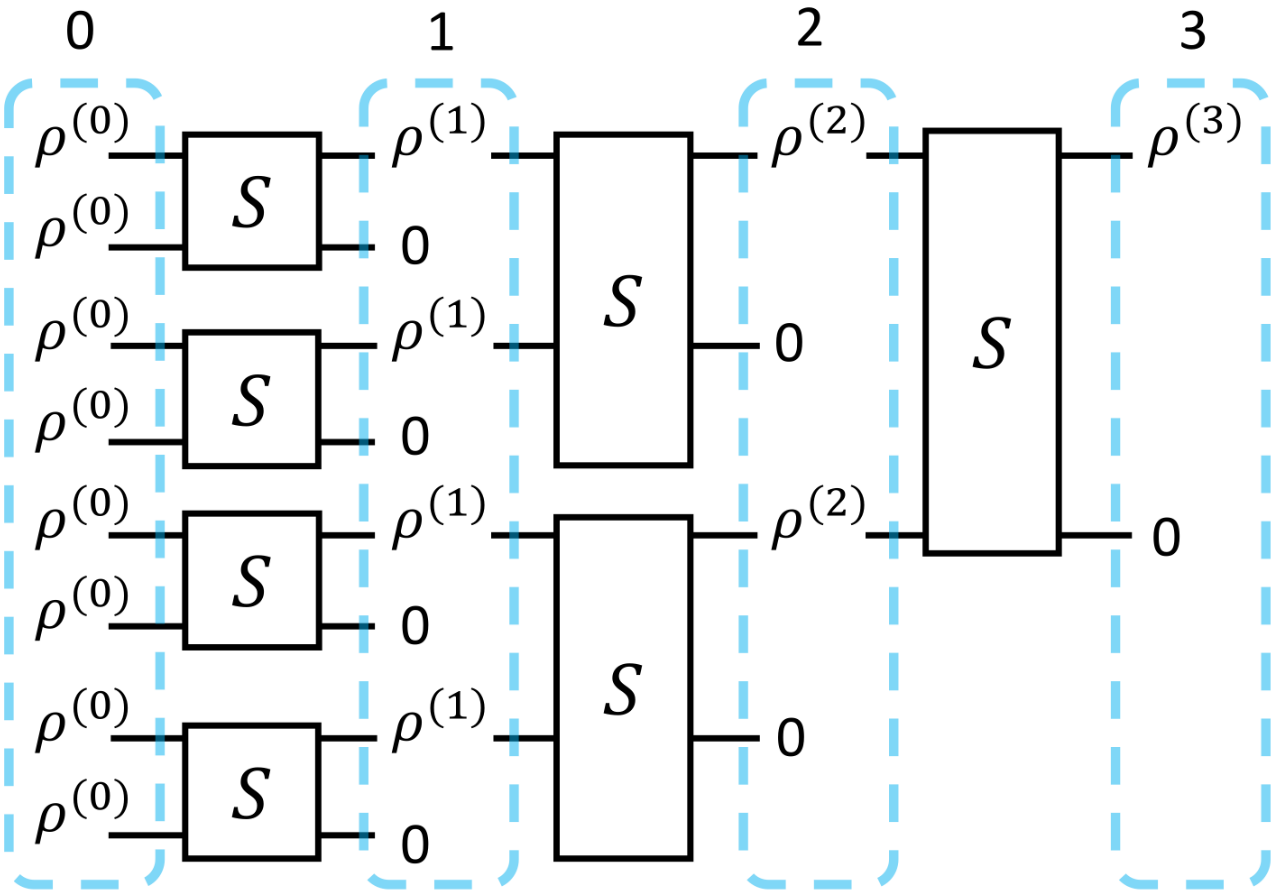
$$u' = \frac{(1+w)^n - (1-w)^n}{(1+w)^n + (1-w)^n}$$

$$v' = -\frac{2\mathrm{Im}\left[(u+iv)^n\right]}{(1+w)^n + (1-w)^n}$$

$$w' = \frac{2\mathrm{Re}\left[(u+iv)^n\right]}{(1+w)^n + (1-w)^n}$$

Iterated quantum protocol

- Second order protocol



1. $\rho^{(1)} = S\left(\rho^{0}\right)$

2. $\rho^{(2)} = S\left(S\left(\rho^{(0)}\right)\right)$

3. $\rho^{(3)} = S\left(S\left(S\left(\rho^{(0)}\right)\right)\right)$

n. $\rho^{(n)} = S^{\circ n}\left(\rho^{(0)}\right)$

Difference equation for the Bloch coordinates

$$
\begin{cases}
u_{k+1} = \dfrac{(1+w_k)^n - (1-w_k)^n}{(1+w_k)^n + (1-w_k)^n} \\[2ex]
v_{k+1} = -\dfrac{2\mathrm{Im}\left[(u_k + iv_k)^n\right]}{(1+w_k)^n + (1-w_k)^n} \\[2ex]
w_{k+1} = \dfrac{2\mathrm{Re}\left[(u_k + iv_k)^n\right]}{(1+w_k)^n + (1-w_k)^n}
\end{cases}
$$

characteristics of dynamics $\longleftrightarrow$ properties of the quantum protocol

Invariant sets of the dynamics

The invariant plane (u, v = 0, w)

$$u_{k+1}\big|_{v_k=0} = \frac{\left(1+w_k\right)^n - \left(1-w_k\right)^n}{\left(1+w_k\right)^n + \left(1-w_k\right)^n}$$

$$v_{k+1}\big|_{v_k=0} = -\frac{2\mathrm{Im}\left[u_k^n\right]}{\left(1+w_k\right)^n + \left(1-w_k\right)^n} = 0 \longrightarrow$$ 2 independent variables

$$w_{k+1}\big|_{v_k=0} = \frac{2u_k^n}{\left(1+w_k\right)^n + \left(1-w_k\right)^n}$$

Invariant sets of the dynamics

Pure states – The surface of the Bloch sphere $(u^2 + v^2 + w^2 = 1)$

$$R_{k+1}^2 = u_{k+1}^2 + v_{k+1}^2 + w_{k+1}^2$$

$$= \frac{\left[(1+w_k)^n - (1-w_k)^n\right]^2 + 4\left(u_k^2 + v_k^2\right)^n}{\left[(1+w_k)^n + (1-w_k)^n\right]^2}$$

$$= \frac{\left[(1+w_k)^n - (1-w_k)^n\right]^2 + 4\left(1 - w_k^2\right)^n}{\left[(1+w_k)^n + (1-w_k)^n\right]^2} = 1$$

Invariant sets of the dynamics

Pure states – The surface of the Bloch sphere

$$R_k^2 = 1 \Rightarrow R_{k+1}^2 = u_{k+1}^2 + v_{k+1}^2 + w_{k+1}^2 = 1$$

$\rightarrow$ $u^2 + v^2 + w^2 = 1$ $\rightarrow$ 2 independent variables

$$|z|^2 = \frac{(1-w)}{(1+w)} \text{ and } \arg(z) = \arctan\left(\frac{u}{v}\right) \rightarrow \quad N\left(|0\rangle + z\,|1\rangle\right)$$

Time evolution of the pure states

$$f_n(z) = \frac{1-z^n}{1+z^n}$$

Fixed points

Pure fixed points

$$f_n\left(z\right) = z$$

$$z^{n+1} + z^n + z - 1 = 0$$

$$\lambda = \left| \left[\frac{df_n(z)}{dz} \right]_{z_0} \right| = |f_n'(z_0)|$$

order (n)	Pure fixed points	
2	C_4	0.544
	–	$-0.772 + 1.115i$
	–	$-0.772 - 1.115i$
3	C_4	0.618
	C_6	-1.618
	–	i
	–	$-i$
4	C_4	0.668
	–	$-1.161 + 0.676i$
	–	$-1.161 - 0.676i$
	–	$0.327 + 0.85i$
	–	$0.327 - 0.85i$
5	C_4	0.704
	C_6	-1.42
	–	$-0.639 + 0.938i$
	–	$-0.639 - 0.938i$
	–	$0.496 + 0.729i$
	–	$0.496 - 0.729i$

Fixed points

Fixed points on the invariant plane

$$u = \frac{(1+w)^n - (1-w)^n}{(1+w)^n + (1-w)^n}$$

$$w = \frac{2u^n}{(1+w)^n + (1-w)^n}$$

order (n)	Fixed points on the $(u, 0, w)$ plane		P
2	C_3	$(0.639, 0, 0.361)$	0.769
	C_4	$(0.839, 0, 0.544)$	1
3	C_3	$(0.711, 0, 0.288)$	0.794
	C_4	$(0.894, 0.444)$	1
	C_5	$(-0.711, 0, -0.288)$	0.794
	C_6	$(-0.894, 0, -0.444)$	1
4	C_3	$(0.757, 0, 0.243)$	0.816
	C_4	$(0.924, 0, 0.383)$	1
5	C_3	$(0.789, 0, 0.211)$	0.834
	C_4	$(0.942, 0, 0.337)$	1
	C_5	$(-0.789, 0, -0.210707)$	0.834
	C_6	$(-0.942, 0, -0.337)$	1

Limit cycles

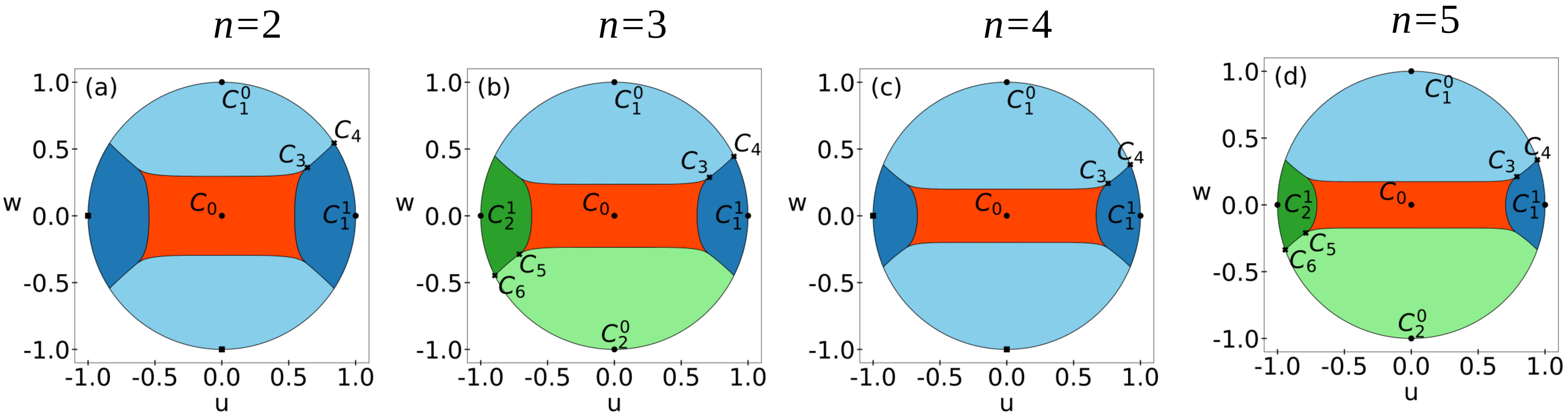
$$C_0 = (0,0,0)$$

$$(u_k, v_k, w_k) \longrightarrow C_i \quad \xrightarrow[\text{simulations}]{\text{numerical}} \quad C_1 = (0,0,1) \leftrightarrow (1,0,0) \;\; (0 \leftrightarrow 1)$$

$$C_2 = (0,0,-1) \leftrightarrow (-1,0,0) \;\; (\infty \leftrightarrow -1)$$

$$\left.\begin{aligned} \lambda^{(1)} &= |f_n'(0)\, f_n'(1)| = 0 \\ \lambda^{(2)} &= |f_n'(\infty)\, f_n'(-1)| = 0. \end{aligned}\right\} \text{super attractive cycles}$$

The convergence regions invariant plane



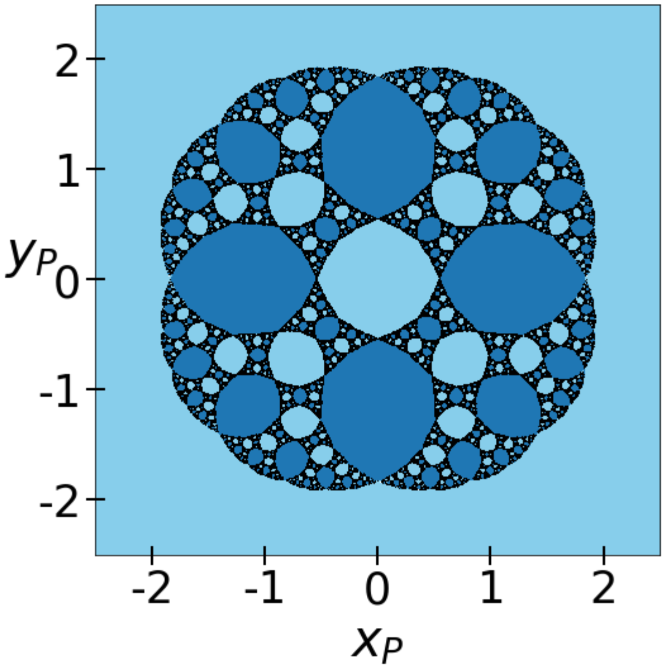
C_0 — maximally mixed state; attractive fixed point

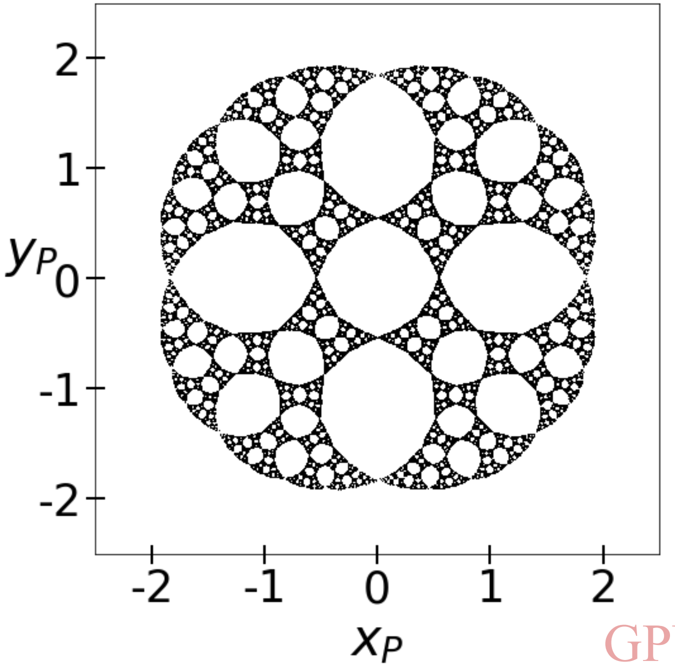
C_1, C_2 — superattractive cycles

C_3 — repelling fixed point

Julia set and Quasi-Julia set

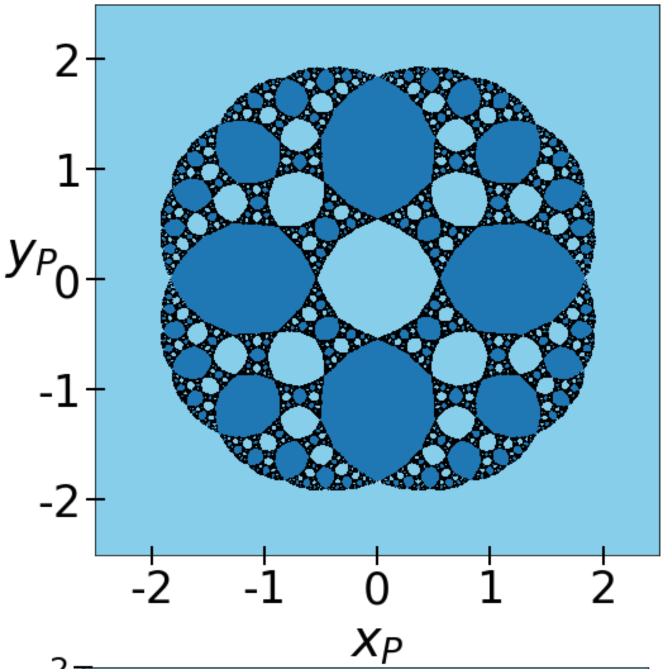
- pure convergence regions
- Julia set: border of pure basins of attraction

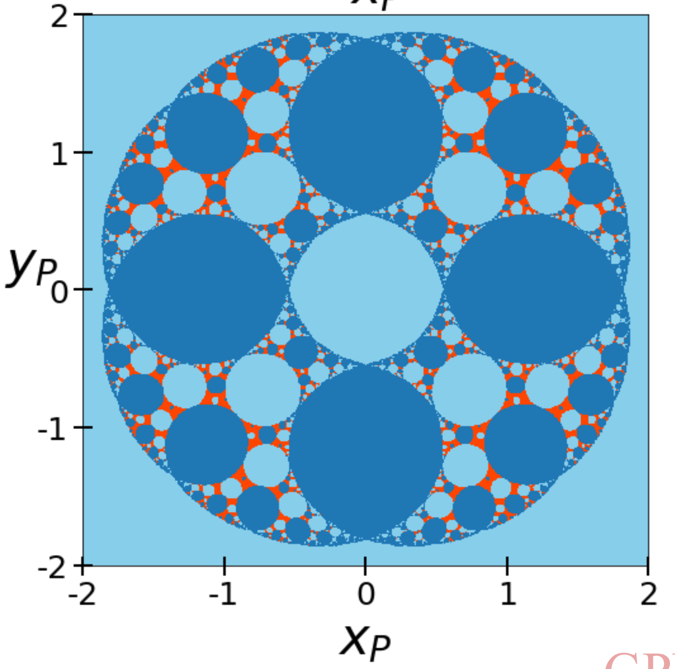




Julia set and Quasi-Julia set

- pure convergence regions
- Julia set: border of pure basins of attraction
- Quasi-Julia set: all mixed points situated at the boundary of the pure attraction regions



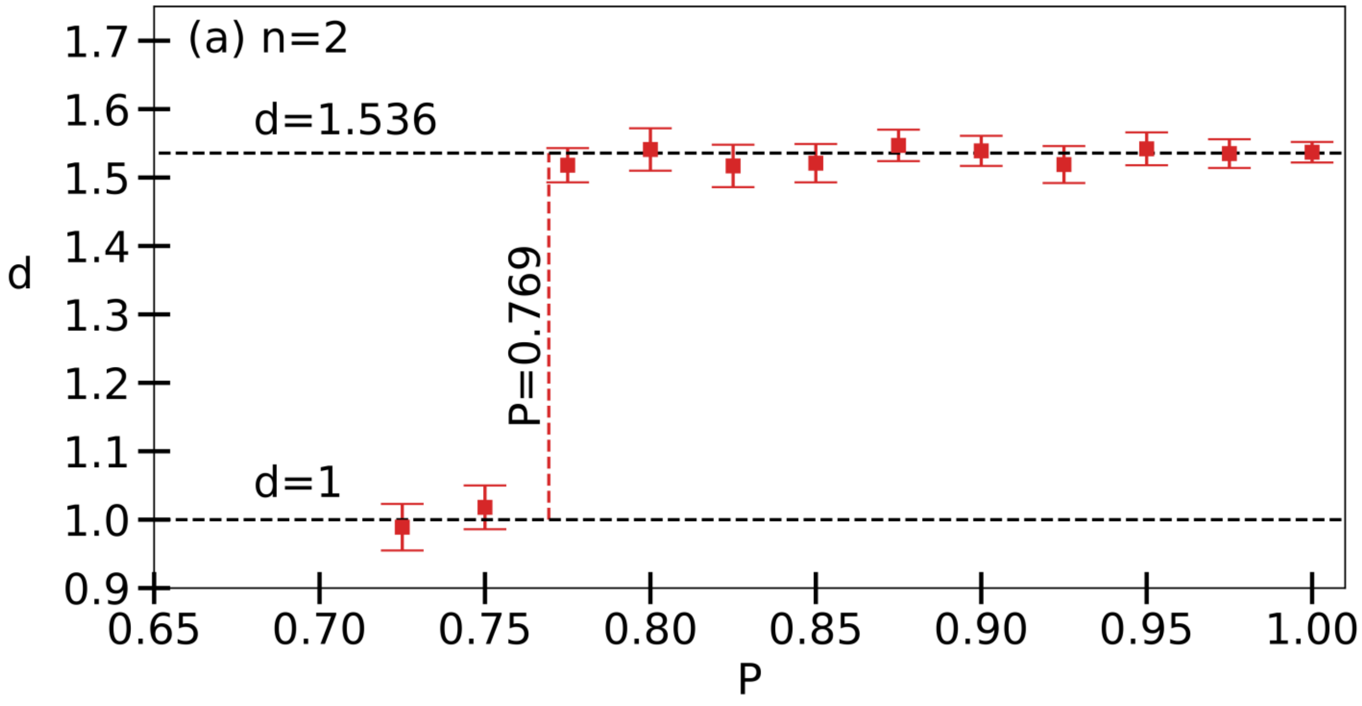


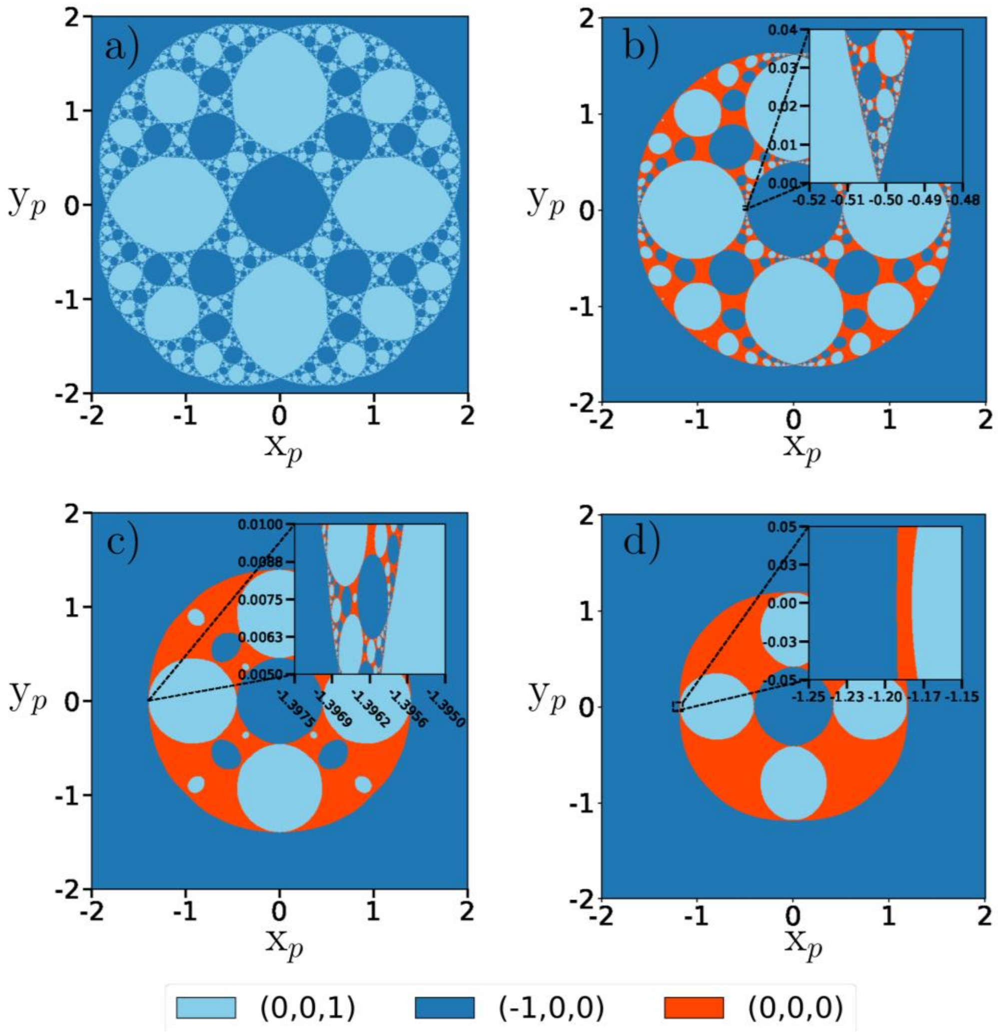
P=0.95

Phase transition

- the fractal dimension of the border is constant as a function of the purity of the initial state

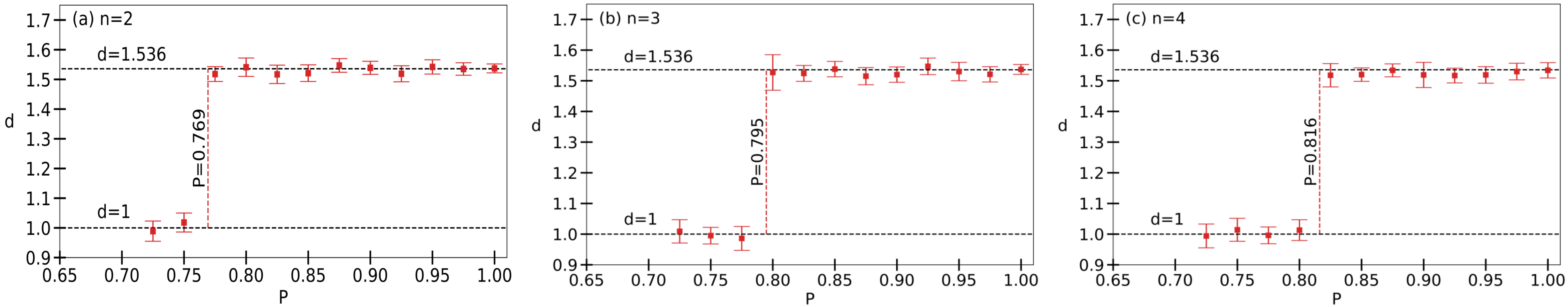
- below a critical purity value, the fractal disappears



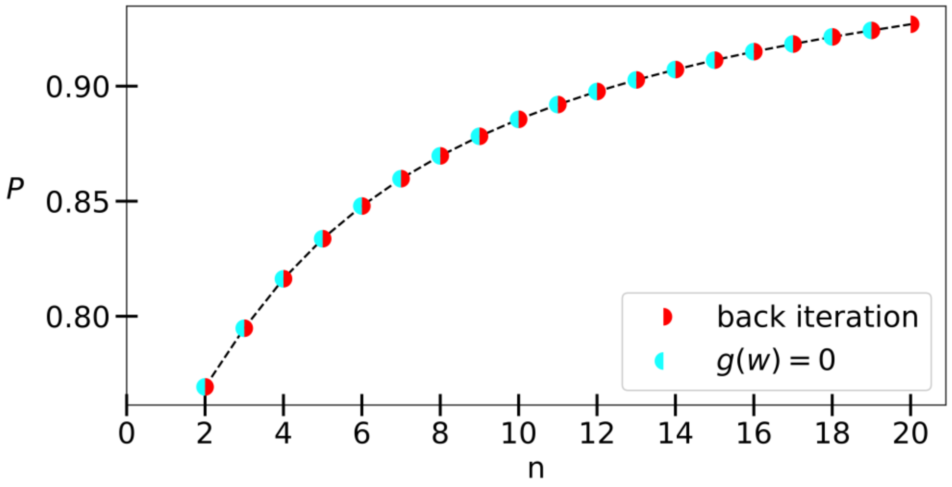


Critical purity

- the point of transition between the regions depends on the degree of the nonlinearity



- the critical purity is equal to the purity of the repelling fixed point located on the invariant plane



Thank you!

Portik, A., Kálmán, O., Jex, I., Kiss, T., 2022. Iterated nth order nonlinear quantum dynamics with mixed initial states. Physics Letters A 431, 127999. https://doi.org/10.1016/j.physleta.2022.127999

Wu, J., Jin, X., Mi, S., Tang, J., 2020. An effective method to compute the box-counting dimension based on the mathematical definition and intervals. Results in Engineering 6, 100106. https://doi.org/10.1016/j.rineng.2020.100106