# Wigner functions in quantum optics


## W.P. Schleich




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**Texas A&M AgriLife Research**
**Institute for Quantum Science and Engineering (IQSE) and**
**Department of Physics and Astronomy**
**Texas A&M University, College Station**

# Classical Liouville equation

initial distribution: $$f_0 = f_0(z, v)$$

propagation: $$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} - \frac{1}{m_i} \frac{\partial V}{\partial z} \frac{\partial}{\partial v} \right) f(z, v; t) = 0$$


*JUNE 1, 1932* *PHYSICAL REVIEW* *VOLUME 40*

# On the Quantum Correction For Thermodynamic Equilibrium

*By* E. Wigner
*Department of Physics, Princeton University*
(Received March 14, 1932)

If a wave function $\psi(x_1 \cdots x_n)$ is given one may build the following expression[2]

$$P(x_1, \cdots, x_n; p_1, \cdots, p_n) = \left(\frac{1}{h\pi}\right)^n \int_{-\infty}^{\infty} \cdots \int dy_1 \cdots dy_n \psi(x_1 + y_1 \cdots x_n + y_n)^* \psi(x_1 - y_1 \cdots x_n - y_n) e^{2i(p_1y_1 + \cdots + p_ny_n)/h} \quad (5)$$

and call it the probability-function of the simultaneous values of $x_1 \cdots x_n$ for the coordinates and $p_1 \cdots p_n$ for the momenta.

[2] This expression was found by L. Szilard and the present author some years ago for another purpose.

# Overview

- equivalence principle in Wigner phase space
- inertial and gravitational mass in quantum mechanics
- Kasevich-Chu atom interferometer in Wigner phase space
- inverted harmonic oscillator, Hawking radiation, and Landau-Zener transitions

# Linear gravitational potential

$$V_l(z) \equiv m_g g z$$

$$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} - \underbrace{\frac{m_g}{m_i} g}\, \frac{\partial}{\partial v} \right) f(z, v; t) = 0$$

$$f(z, v; t) = f_0\left( z - vt - \frac{1}{2} \frac{m_g}{m_i} g\, t^2, v + \frac{m_g}{m_i} gt \right)$$

# Quantum Liouville equation: linear gravitational potential

$$\left(\hat{\mathscr{L}}_{\rm cl} - \hat{\mathscr{L}}_{\rm o}\right) W = 0$$

$$\hat{\mathscr{L}}_{\rm o} = \sum_{l=1}^{\infty} (\ldots)\ \hbar^{2l} \frac{\partial^{2l+1} V(z)}{\partial z^{2l+1}} (\ldots)$$

$$\uparrow$$

$$V \sim z$$

$$\hat{\mathscr{L}}_{\rm cl} = \left(\frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial z} - \frac{\partial V}{\partial z}\frac{\partial}{\partial p}\right)$$

$$\hat{\mathscr{L}}_{\rm cl} W = 0$$

# Quantum kinematics: linear gravitational potential

$$\left(-\frac{\hbar^2}{8m_i}\frac{\partial^2}{\partial z^2}+\frac{p^2}{2m_i}+V(z)+\hat{\mathscr{L}}_{\mathrm{e}}\right)W_E = E\,W_E$$

$$\uparrow$$

$$\sim z$$

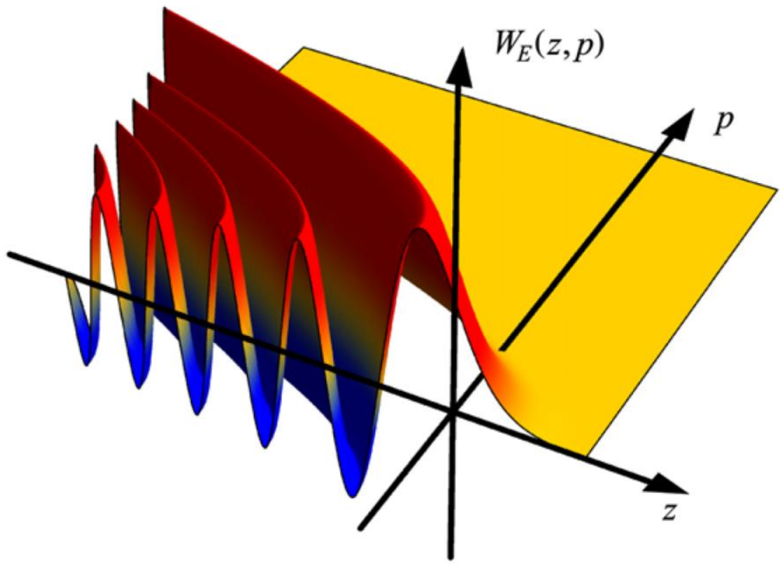
$$\hat{\mathscr{L}}_{\mathrm{e}} = \sum_{l=1}^{\infty}(\ldots)\,\hbar^{2l}\,\frac{\partial^{2l}V(z)}{\partial z^{2l}}(\ldots)$$

Appl Phys B (2010) 100: 43–60
DOI 10.1007/s00340-010-4085-8

**Applied Physics B**
Lasers and Optics

# Inertial and gravitational mass in quantum mechanics

**E. Kajari · N.L. Harshman · E.M. Rasel · S. Stenholm ·
G. Süßmann · W.P. Schleich**

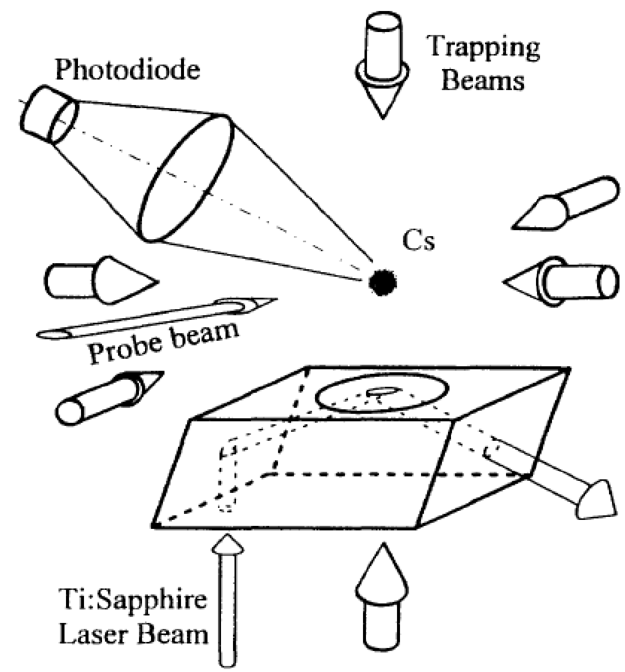


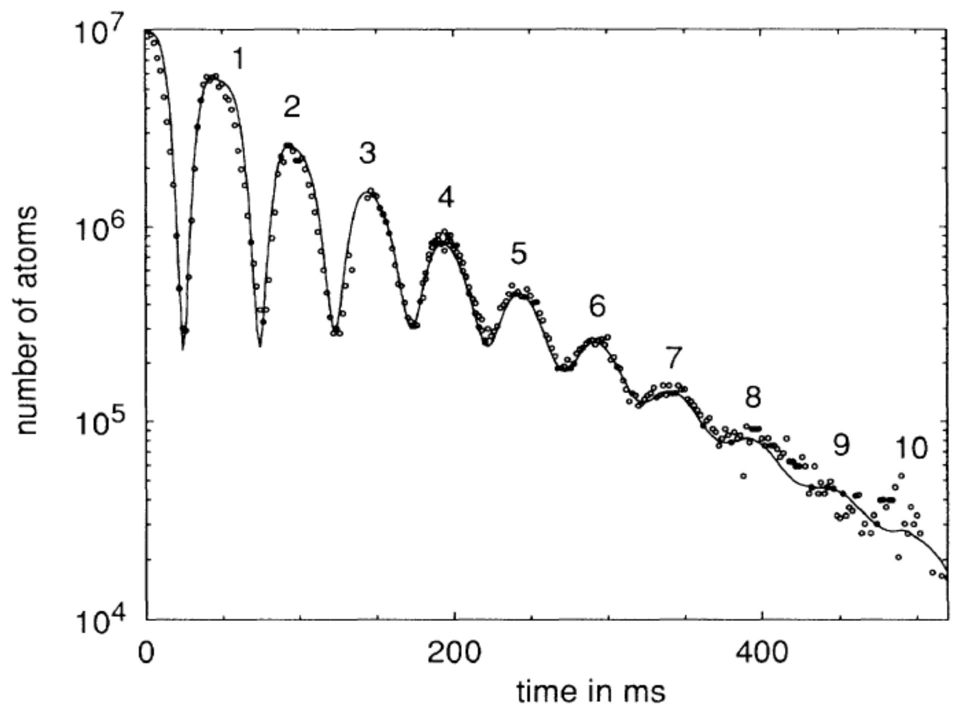

$$u_E(z) = \mathcal{N}_E \cdot \mathrm{Ai}(kz - \varepsilon)$$

$$k = \left(\frac{2m_i m_g g}{\hbar^2}\right)^{1/3}$$

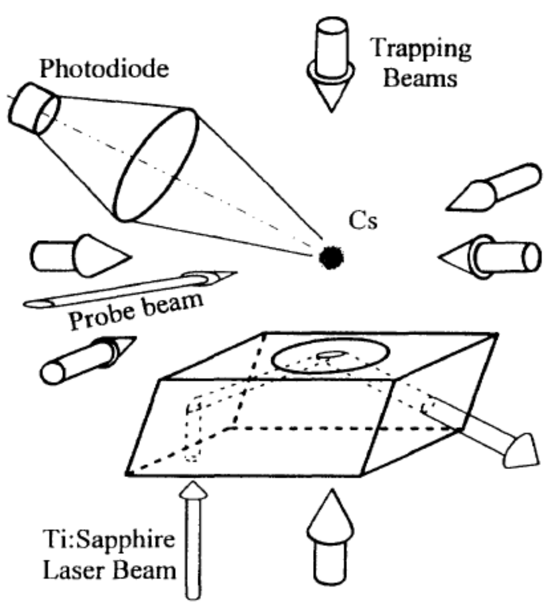
# Cesium Atoms Bouncing in a Stable Gravitational Cavity

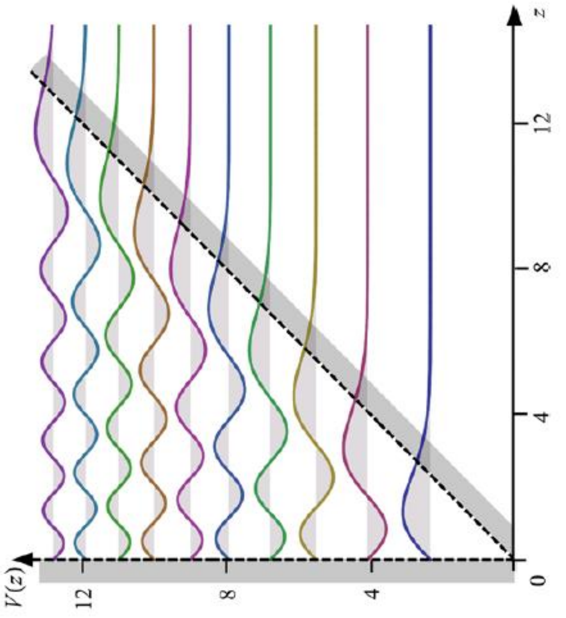
C. G. Aminoff,* A. M. Steane, P. Bouyer, P. Desbiolles, J. Dalibard, and C. Cohen-Tannoudji
*Laboratoire de Spectroscopie Hertzienne de l'Ecole Normale Supérieure and Collège de France, 24 rue Lhomond, F-75231 Paris CEDEX 05, France*



# Atom trampoline




$$E_n = \left(\frac{1}{2}\hbar^2 g^2\right)^{\frac{1}{3}} m_g^{\frac{2}{3}} m_i^{-\frac{1}{3}} a_{n+1}$$

# Quantum states of neutrons in the Earth's gravitational field

**Valery V. Nesvizhevsky*, Hans G. Börner*, Alexander K. Petukhov*, Hartmut Abele†, Stefan Baeßler†, Frank J. Rueß†, Thilo Stöferle†, Alexander Westphal†, Alexei M. Gagarski‡, Guennady A. Petrov‡ & Alexander V. Strelkov§**

** Institute Laue-Langevin, 6 rue Jules Horowitz, Grenoble F-38042, France*
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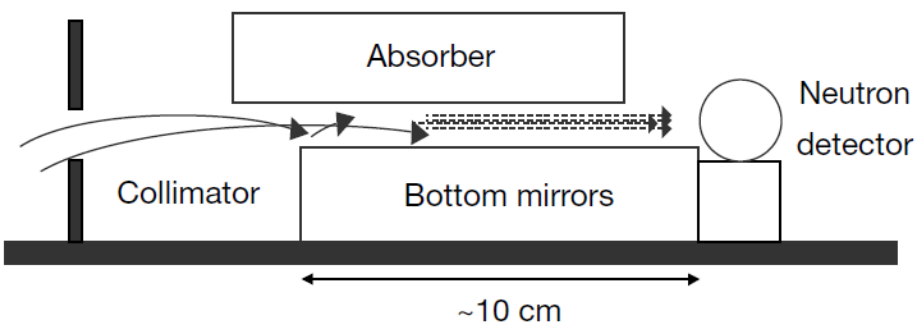
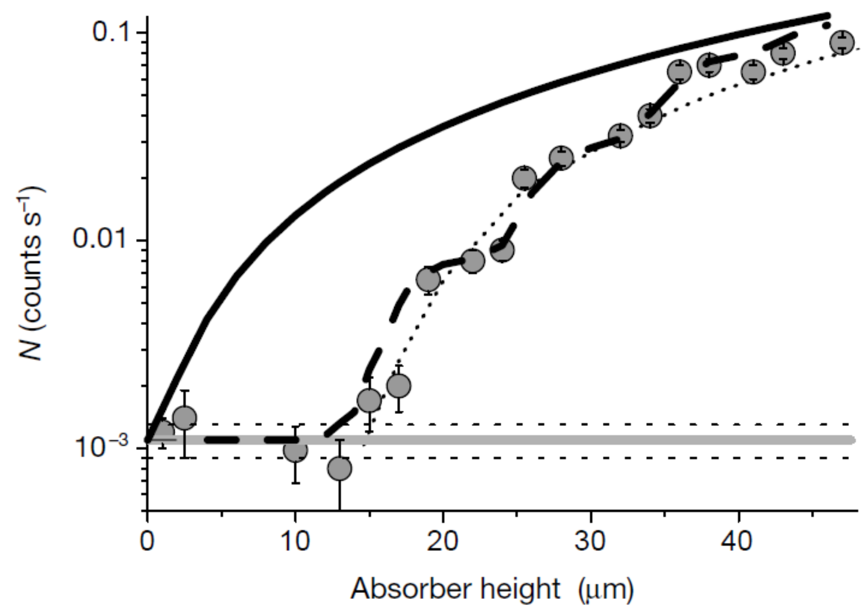
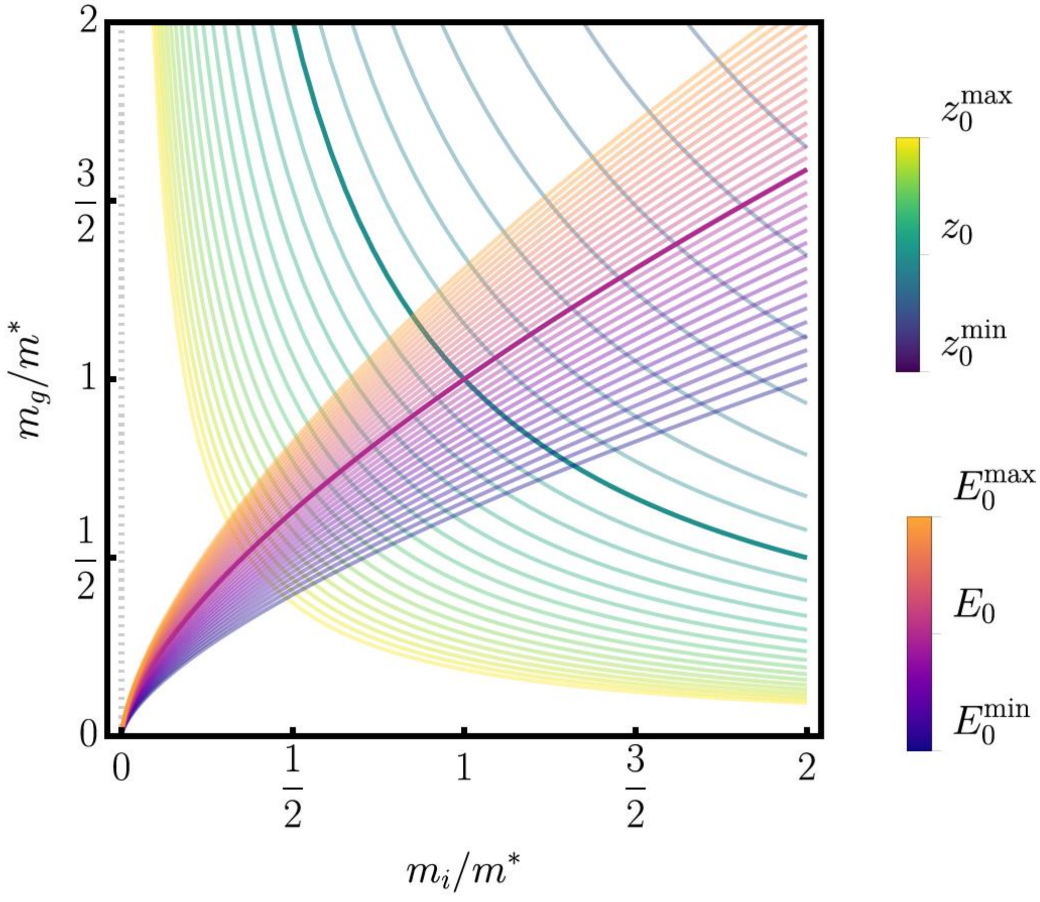
*‡ Petersburg Nuclear Physics Institute, Orlova Roscha, Gatchina, Leningrad reg. R-188350, Russia*
*§ Joint Institute for Nuclear Research, Dubna, Moscow reg. R-141980, Russia*



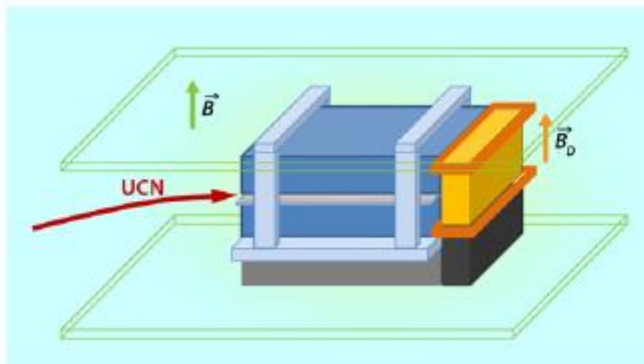
$$\mathrm{Ai}\left(\frac{z}{z_0} - \frac{E}{E_0}\right) \qquad E_0 = \left(\frac{\hbar^2 m_g^2 g^2}{2m_i}\right)^{1/3} \qquad z_0 = \left(\frac{\hbar^2}{2m_i m_g g}\right)^{1/3}$$

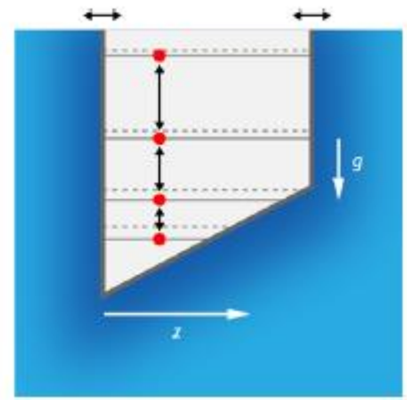


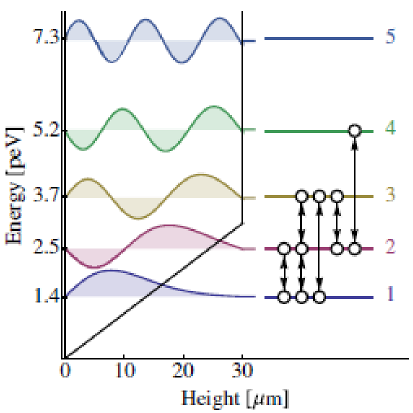
Selected for a Viewpoint in *Physics*
PRL **112,** 151105 (2014) PHYSICAL REVIEW LETTERS week ending 18 APRIL 2014

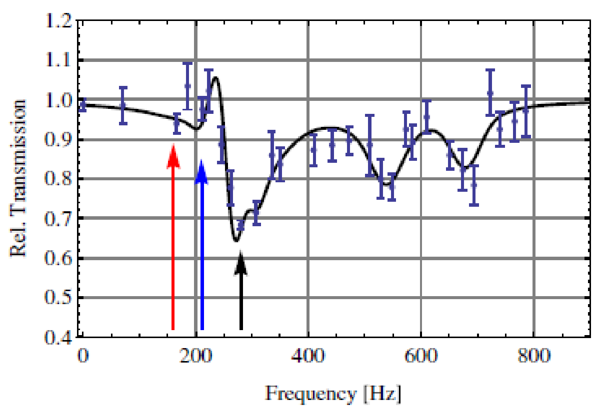
# Gravity Resonance Spectroscopy Constrains Dark Energy and Dark Matter Scenarios

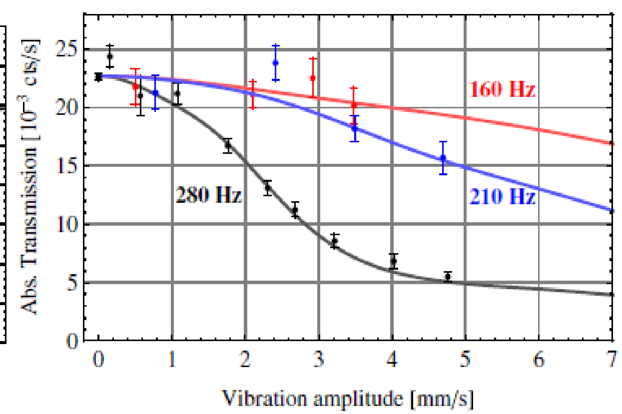
T. Jenke,[1,*] G. Cronenberg,[1] J. Burgdörfer,[2] L. A. Chizhova,[2] P. Geltenbort,[3] A. N. Ivanov,[1] T. Lauer,[4] T. Lins,[1,†] S. Rotter,[2] H. Saul,[1,‡] U. Schmidt,[5] and H. Abele[1,§]



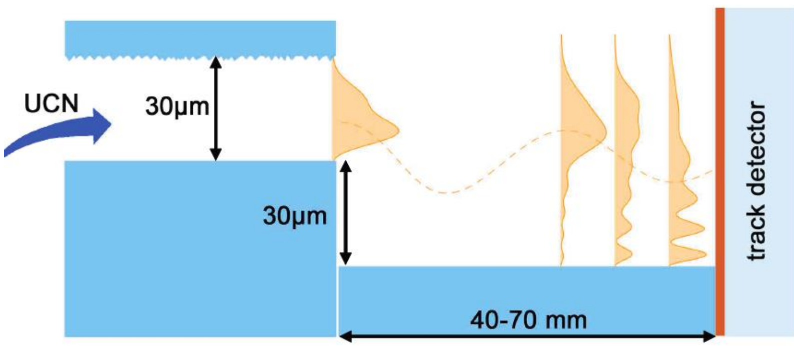



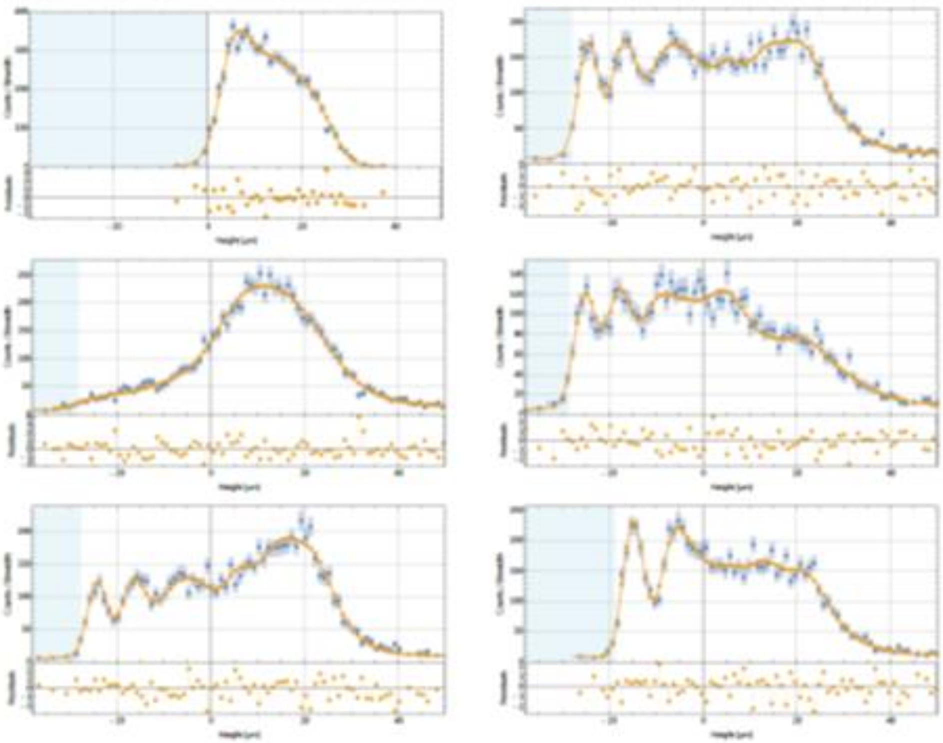



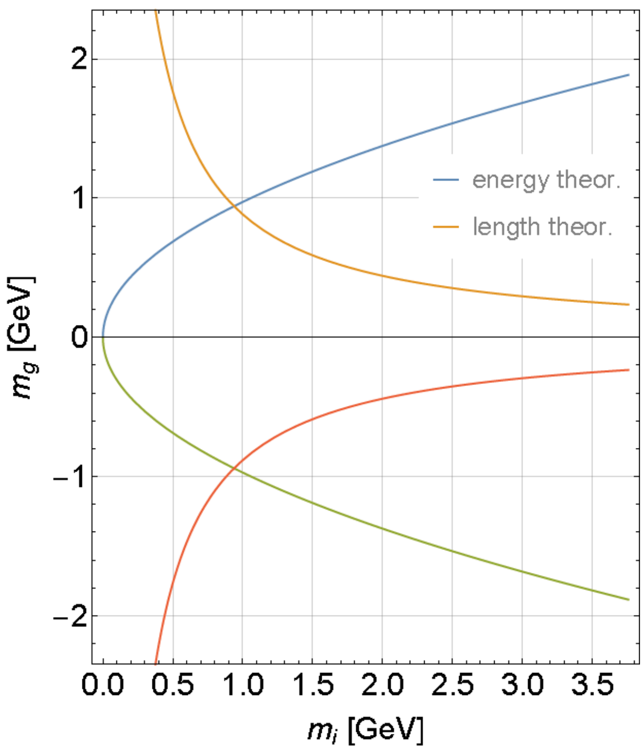
# Vienna experiment (H. Abele)



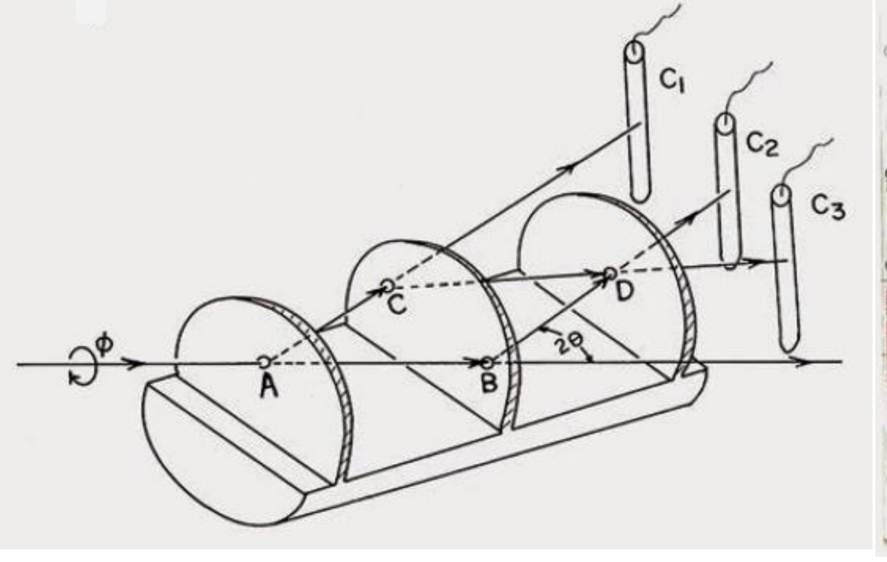
# Vienna experiment (H. Abele)

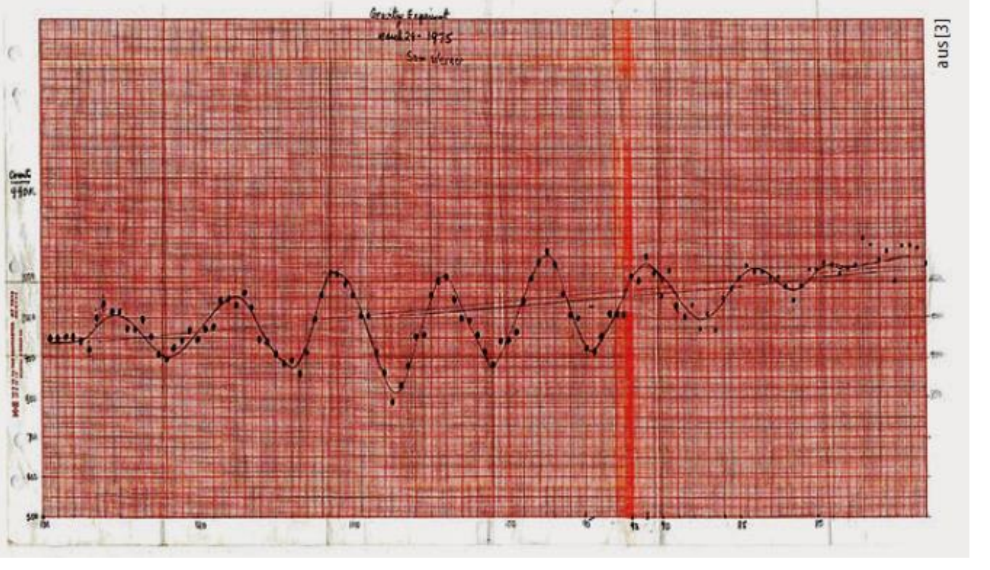
$$z_0 = \left(\frac{\hbar^2}{2m_i m_g g}\right)^{1/3} = 5.87\,\mu m \qquad E_0 = -\left(\frac{\hbar^2 m_g^2 g^2}{2m_i}\right)^{1/3} = -0.602\,peV$$



# A. Overhauser, R. Colella, S. Werner

# COW Experiment



# Atomic Interferometry Using Stimulated Raman Transitions

Mark Kasevich and Steven Chu
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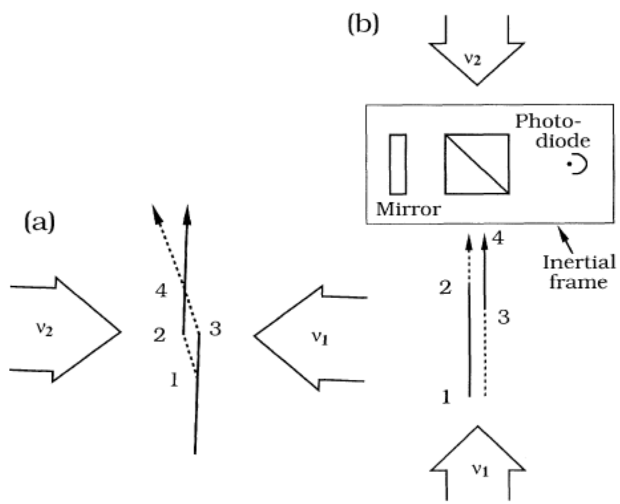
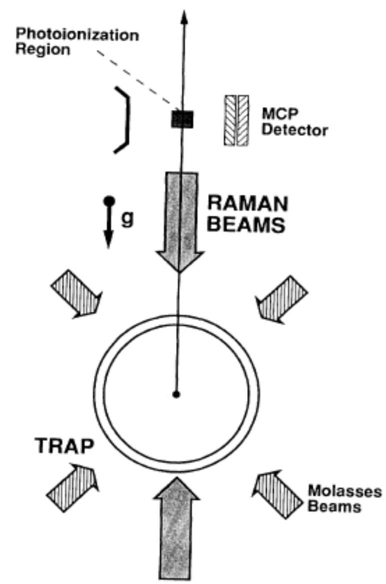
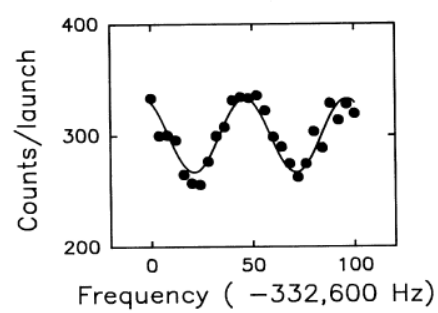
(Received 23 April 1991)



Article

# Observation of Bose–Einstein condensates in an Earth-orbiting research lab

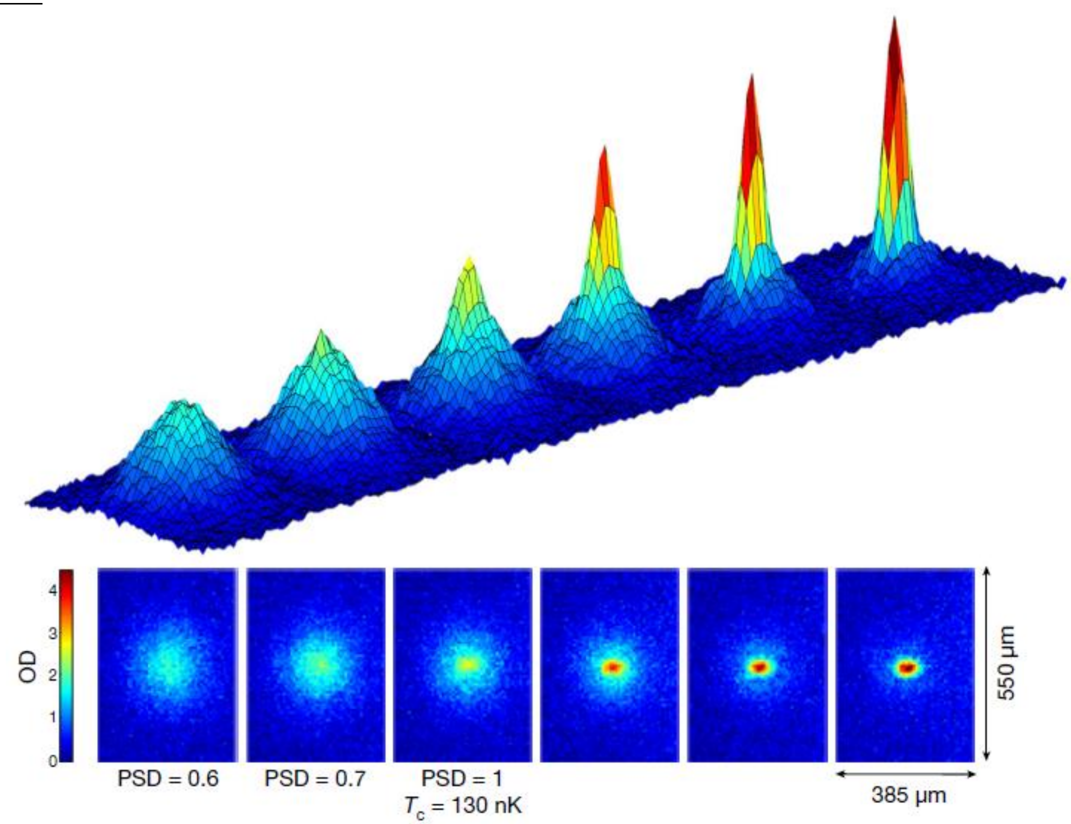
https://doi.org/10.1038/s41586-020-2346-1

Received: 30 October 2019

Accepted: 26 March 2020

Published online: 11 June 2020

David C. Aveline[1,2]✉, Jason R. Williams[1,2], Ethan R. Elliott[1,2], Chelsea Dutenhoffer[1], James R. Kellogg[1], James M. Kohel[1], Norman E. Lay[1], Kamal Oudrhiri[1], Robert F. Shotwell[1], Nan Yu[1] & Robert J. Thompson[1]✉



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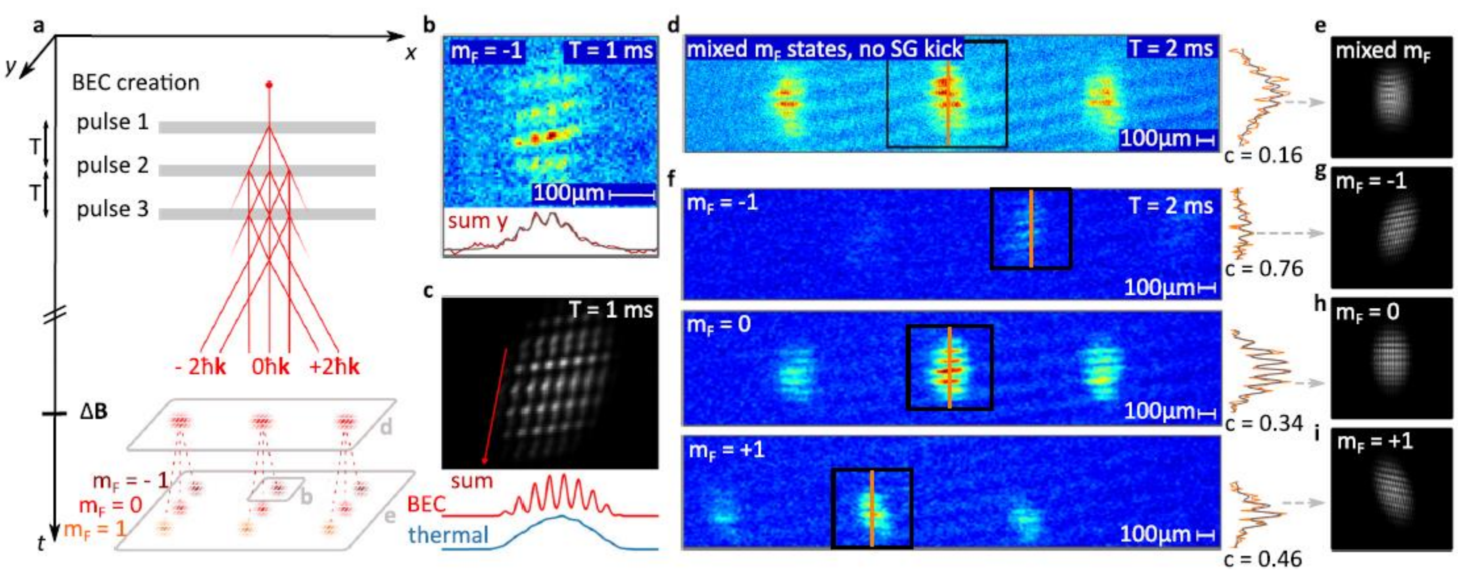
https://doi.org/10.1038/s41467-021-21628-z OPEN

# Ultracold atom interferometry in space

Maike D. Lachmann[1,16], Holger Ahlers[1,16], Dennis Becker[1], Aline N. Dinkelaker[2,12], Jens Grosse[3,4], Ortwin Hellmig[5], Hauke Müntinga[3,13], Vladimir Schkolnik[2], Stephan T. Seidel[1,14], Thijs Wendrich[1], André Wenzlawski[6], Benjamin Carrick[7,15], Naceur Gaaloul[1], Daniel Lüdtke[7], Claus Braxmaier[3,4], Wolfgang Ertmer[1], Markus Krutzik[2], Claus Lämmerzahl[3], Achim Peters[2], Wolfgang P. Schleich[8,9,10], Klaus Sengstock[5], Andreas Wicht[11], Patrick Windpassinger[6] & Ernst M. Rasel[1]✉

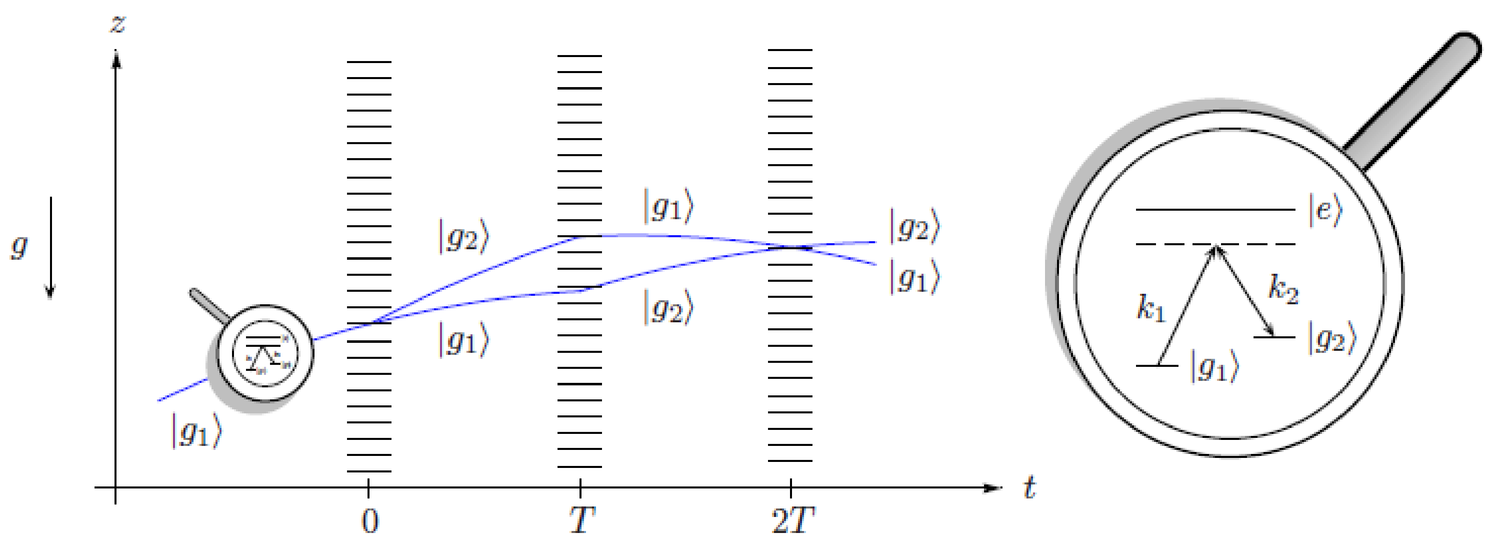


New Journal of Physics
The open access journal for physics

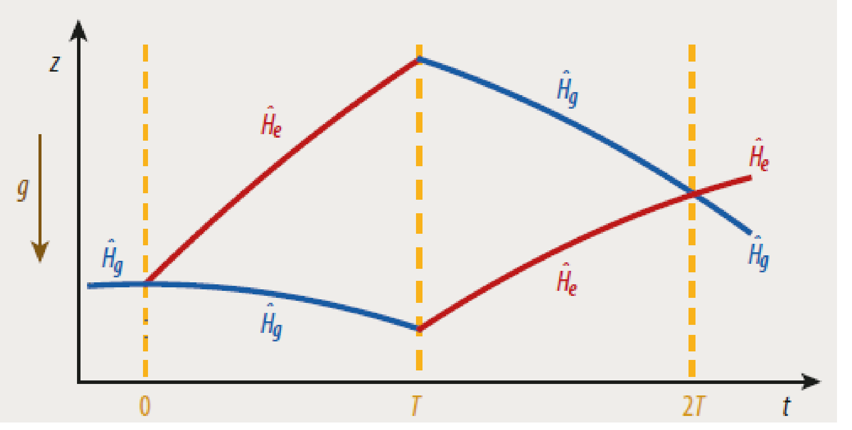
# A representation-free description of the Kasevich–Chu interferometer: a resolution of the redshift controversy

Wolfgang P Schleich[1], Daniel M Greenberger[2]
and Ernst M Rasel[3]

*New Journal of Physics* **15** (2013) 013007 (48pp)



# Phase difference in Kasevich-Chu interferometer




$$\hat{H}_g \equiv \frac{\hat{p}^2}{2m} + mg\hat{z} \qquad \hat{H}_e \equiv \frac{(\hat{p} + \hbar k)^2}{2m} + mg\hat{z}$$

$$[\hat{H}_g, \hat{H}_e] = \frac{\hbar k}{m}[\hat{H}_g, \hat{p}]$$

# Phase of the interferometer

Proceedings of the International School of Physics "Enrico Fermi"
Course 188 "Atom Interferometry", edited by G. M. Tino and M. A. Kasevich
(IOS, Amsterdam; SIF, Bologna) 2014
DOI 10.3254/978-1-61499-448-0-171

## The interface of gravity and quantum mechanics illuminated by Wigner phase space

E. Giese, W. Zeller, S. Kleinert, M. Meister, V. Tamma and A. Roura

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W. P. Schleich

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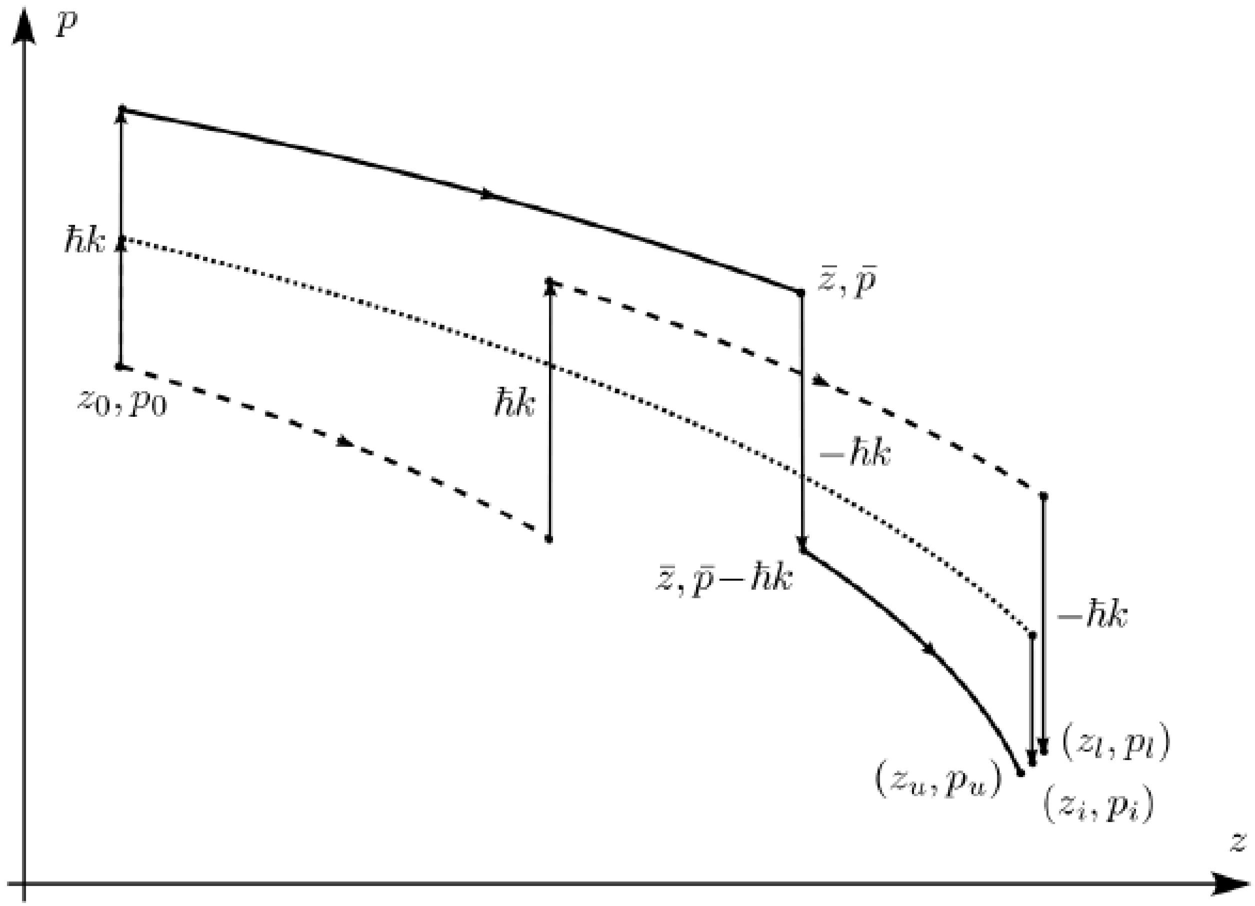
*Texas A&M University Institute for Advanced Study, Institute for Quantum Science and Engineering (IQSE) and Department of Physics and Astronomy*
*Texas A&M University - College Station, Tx 77843-4242, USA*

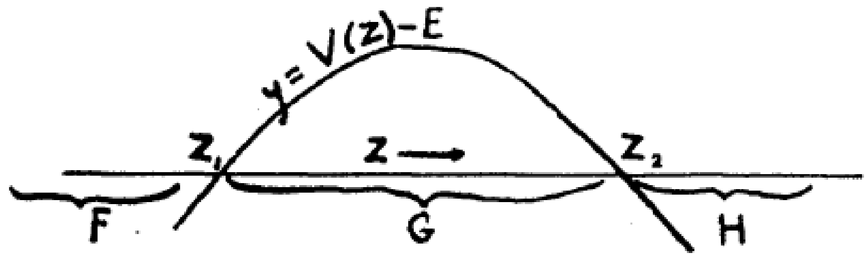
$$\left[\frac{\partial}{\partial t} + \frac{1}{m}\left(p + \frac{1}{2}\hbar k\right)\frac{\partial}{\partial z} - \frac{\partial V}{\partial z}\frac{\partial}{\partial p}\right] W_{21} = -\mathrm{i}\frac{k}{m}\left(p + \frac{1}{2}\hbar k\right) W_{21}$$

# Kasevich-Chu interferometer in phase space



# A Contribution to the Theory of the B. W. K. Method

EDWIN C. KEMBLE, *Research Laboratory of Physics, Harvard University*




$$T = \frac{1}{1 + e^{-2\pi\varepsilon}}$$

$$\varepsilon \equiv E/(\hbar\Omega)$$

Physics Letters A 377 (2013) 1822–1825

Contents lists available at SciVerse ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

# Tunneling of an energy eigenstate through a parabolic barrier viewed from Wigner phase space

D.M. Heim [a,*], W.P. Schleich [a], P.M. Alsing [b], J.P. Dahl [c], S. Varro [d]

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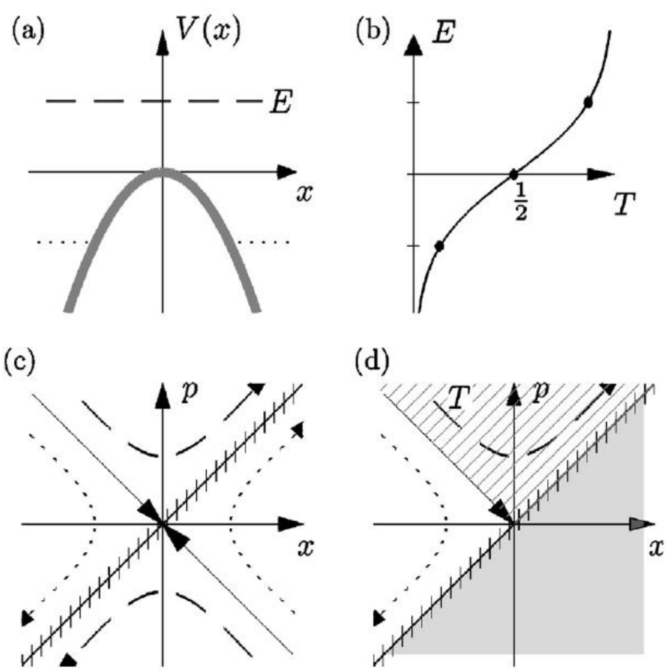
[b] Information Directorate, Air Force Research Laboratory, Rome, NY 13441, USA
[c] Chemical Physics, Department of Chemistry, Technical University of Denmark, DTU 207, DK-2800 Kgs. Lyngby, Denmark
[d] Wigner Research Centre for Physics, Hungarian Academy of Sciences, Institute for Solid State Physics and Optics, 1525 Budapest, Hungary



# Phase space equations for Wigner function

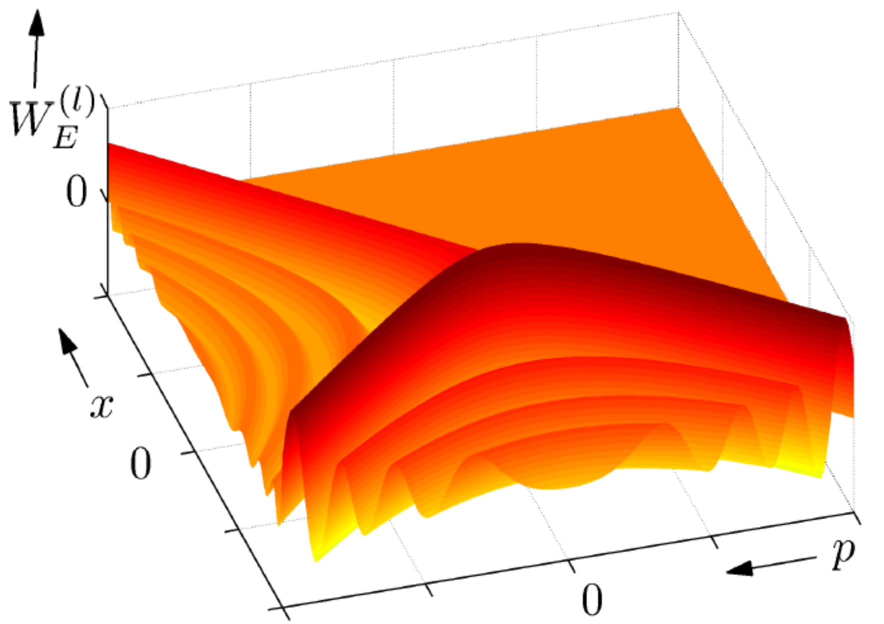
$$\left[\frac{p}{M}\frac{\partial}{\partial x} + M\Omega^2 x \frac{\partial}{\partial p}\right] W_E(x, p) = 0$$

$$\left\{\left[\frac{p^2}{2M} - \frac{1}{2}M\Omega^2 x^2\right] - \frac{\hbar^2}{8}\left[\frac{1}{M}\frac{\partial^2}{\partial x^2} - M\Omega^2 \frac{\partial^2}{\partial p^2}\right]\right\}$$

$$\times W_E(x, p) = E W_E(x, p)$$

# Wigner function of energy eigenstate

$$\mathcal{W}_\varepsilon(\eta) = \frac{1}{\pi}\int_{-2}^{+2} \mathrm{d}\tau\, \frac{\exp[-\mathrm{i}\varepsilon \ln(\frac{2+\tau}{2-\tau})]}{\sqrt{4-\tau^2}} \mathrm{e}^{\mathrm{i}\eta\tau}$$



AVS Quantum Science ARTICLE scitation.org/journal/aqs

# The logarithmic phase singularity in the inverted harmonic oscillator

Cite as: AVS Quantum Sci. 4, 024402 (2022); doi: 10.1116/5.0074429
Submitted: 8 October 2021 · Accepted: 14 March 2022 ·
Published Online: 7 April 2022

View Online Export Citation CrossMark

Freyja Ullinger,[1,2] Matthias Zimmermann,[2] and Wolfgang P. Schleich[1,3]

$$\hat{H} \equiv \frac{\hat{p}^2}{2m} - \frac{m\omega^2}{2}\hat{x}^2$$

$$\hat{\xi} \equiv \frac{1}{\sqrt{2}}\left(\hat{x} - \frac{\hat{p}}{m\omega}\right)$$

$$\hat{\pi} \equiv \frac{1}{\sqrt{2}}\left(\hat{p} + m\omega\hat{x}\right)$$

$$\hat{H} = -\omega\hat{\xi}\hat{\pi} + \frac{i}{2}\hbar\omega$$

$$u(\xi) = \frac{N_E}{\sqrt{|\xi|}}\exp\left(-i\frac{E}{\hbar\omega}\ln|\xi|\right)$$

# Black hole explosions?

QUANTUM gravitational effects are usually ignored in calculations of the formation and evolution of black holes. The justification for this is that the radius of curvature of space-time outside the event horizon is very large compared to the Planck length $(G\hbar/c^3)^{1/2} \approx 10^{-33}$ cm, the length scale on which quantum fluctuations of the metric are expected to be of order unity.

From this it follows that the number of particles emitted in this wave packet mode is $(\exp(2\pi\omega/\kappa) - 1)^{-1}$ times the number of particles that would have been absorbed from a similar wave packet incident on the black hole from $I^-$.

In the expressions given above for $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ there is a logarithmic singularity in the factors $[-i(\omega - \omega')]^{-1+i\omega/\kappa}$ and $[-i(\omega + \omega')]^{-1+i\omega/\kappa}$.

S. W. HAWKING

*Department of Applied Mathematics and Theoretical Physics*
*and*
*Institute of Astronomy*
*University of Cambridge*

# Quantum optics approach to radiation from atoms falling into a black hole

Marlan O. Scully[a,b,c,1], Stephen Fulling[a,d], David M. Lee[a], Don N. Page[e], Wolfgang P. Schleich[a,f], and Anatoly A. Svidzinsky[a]

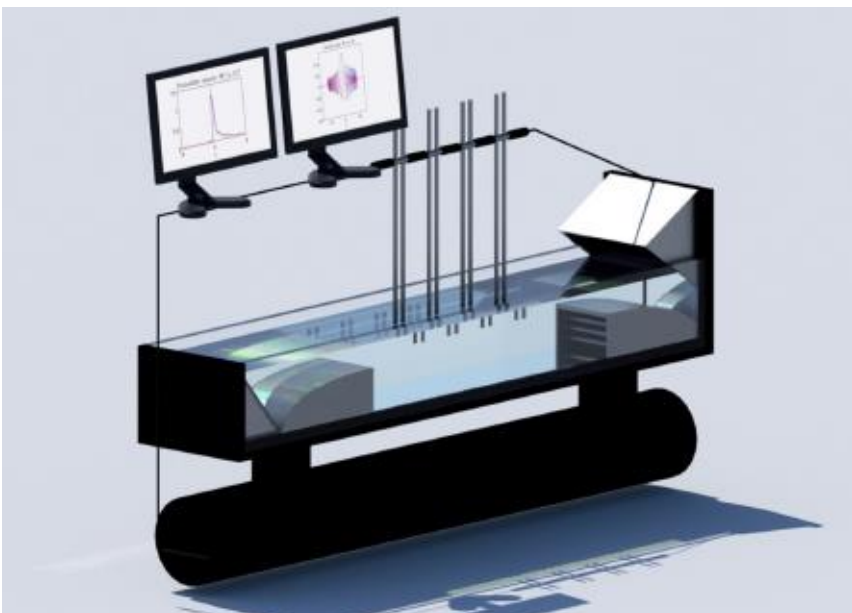
$$P_{exc} = \frac{g^2}{\omega^2}\left|\int_0^\infty dx e^{-2i\nu \ln x} e^{-ix\left(1+\frac{2\nu}{\omega}\right)}\right|^2$$

$$P_{exc} = \frac{4\pi g^2 \nu}{\omega^2 \left(1+\frac{2\nu}{\omega}\right)^2} \frac{1}{e^{4\pi\nu}-1}$$

PNAS | August 7, 2018 | vol. 115 | no. 32 | 8131–8136

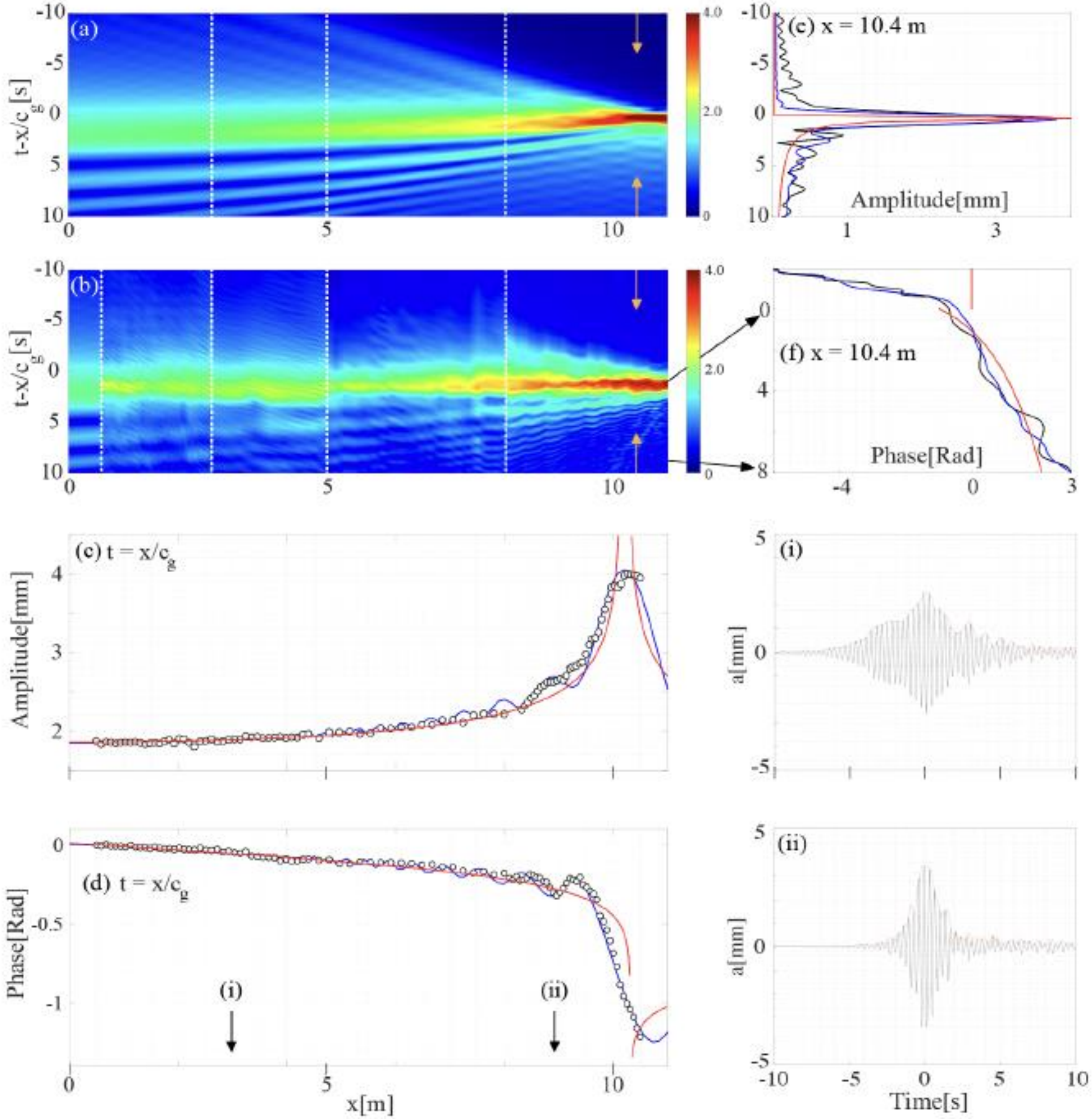
# Observation of a phase space horizon with surface gravity water waves

Georgi Gary Rozenman [1,2†,*], Freyja Ullinger[3,4†], Matthias Zimmermann[3†], Maxim A. Efremov[3,4], Lev Shemer[5], Wolfgang P. Schleich[4,6], and Ady Arie[2,*]



$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}$$

$$i\frac{\partial A}{\partial \xi} = \frac{\partial^2 A}{\partial \tau^2}$$



(a)
(b)
(c) x = 10.4 m
(f) x = 10.4 m
(c) t = x/c_g
(d) t = x/c_g
(i)
(ii)
t-x/c_g[s]
Amplitude[mm]
Phase[Rad]
a[mm]
x[m]
Time[s]

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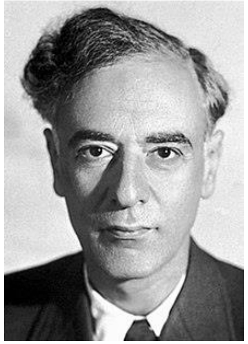
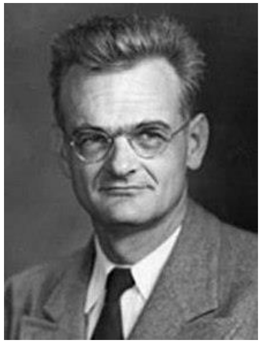
Journal of Physics B: Atomic, Molecular and Optical Physics
J. Phys. B: At. Mol. Opt. Phys. **56** (2023) 104001 (6pp)
https://doi.org/10.1088/1361-6455/acc774

# The Landau–Zener formula made simple

**Eric P Glasbrenner**[1,*] **and Wolfgang P Schleich**[1,2]



$$H \equiv \hbar \begin{pmatrix} -\alpha t & g \\ g & \alpha t \end{pmatrix}$$



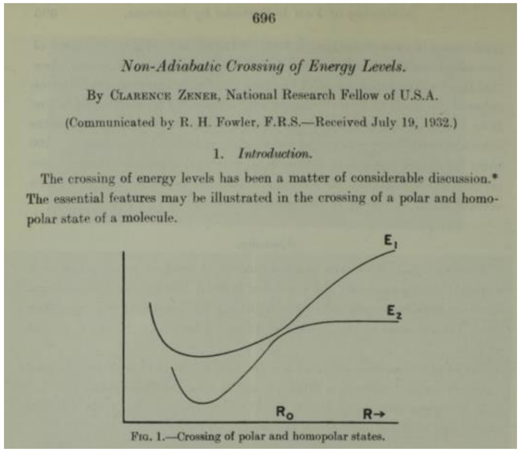
9. A THEORY OF ENERGY TRANSFER. II

The results of I are applied to: 1, predissociation; 2, excitation of oscillations during an optical transition; 3, the case of closely approaching potential curves. The question of the cross-section is discussed.

The results obtained in I[1] can be applied to a number of effects. It must be added, however, that during the collision of similar atoms it is necessary to add to the selection rules developed in I the postulate of the change in parity which is evident from the fact that the electron-spin components which are normal to the line connecting the nuclei change their sign when mirrored.

1. The most direct generalisation is the predissociation. For the dissociation probability during an oscillation we can use here expression (21) directly. This can also be written in the following form:

$$\omega = \frac{4\pi \hbar D^2}{(F_1 - F_2) m^2 r^4} \cdot \frac{j^2}{v}. \quad (1)$$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*Non-Adiabatic Crossing of Energy Levels.*

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(Communicated by R. H. Fowler, F.R.S.—Received July 19, 1932.)

1. *Introduction.*

The crossing of energy levels has been a matter of considerable discussion.* The essential features may be illustrated in the crossing of a polar and homopolar state of a molecule.



Fig. 1.—Crossing of polar and homopolar states.

# Alternative approach

$$\dot{a}(t) = -\mathrm{i}g\,\mathrm{e}^{-\mathrm{i}\alpha t^2/2}\,b(t)$$

$$\dot{b}(t) = -\mathrm{i}g\,\mathrm{e}^{\mathrm{i}\alpha t^2/2}\,a(t)$$

$$\ddot{a}(t) + \mathrm{i}\alpha t\dot{a}(t) + g^2 a(t) = 0$$

$$\dot{a}_S(t) = -g^2\frac{1}{\varepsilon + \mathrm{i}\alpha t}a_S(t) = -g^2\frac{\varepsilon - \mathrm{i}\alpha t}{\varepsilon^2 + (\alpha t)^2}a_S(t)$$

# Summary

- equivalence principle in Wigner phase space
- inertial and gravitational mass in quantum mechanics
- Kasevich-Chu atom interferometer in Wigner phase space
- inverted harmonic oscillator, Hawking radiation, and Landau-Zener transitions