Symmetries of spacetimes with a compact Cauchy horizon and the cosmic censor

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Einstein's theory is a metric theory of gravity

- **Eintein's equations:** $"lhs = rhs"$

$$[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}\,,$$

  - with matter fields satisfying their field equations
  - with energy-momentum tensor T_{ab} and with cosmological constant Λ

- in local coordinates $x^1, ..., x^4$ **the Ricci tensor** reads as

$$R_{\alpha\beta} = \partial_\varepsilon \Gamma^\varepsilon{}_{\alpha\beta} - \partial_\alpha \Gamma^\varepsilon{}_{\varepsilon\beta} + \Gamma^f{}_{\alpha\beta}\Gamma^\varepsilon{}_{\varepsilon\varphi} - \Gamma^\varphi{}_{\varepsilon\beta}\Gamma^\varepsilon{}_{\alpha\varphi}\,, \quad \text{where}$$

$$\Gamma^\gamma{}_{ab} = \tfrac{1}{2}\, g^{\gamma\varepsilon}\, \{\partial_\alpha g_{\varepsilon\beta} + \partial_\beta g_{\alpha\varepsilon} - \partial_\varepsilon g_{\alpha\beta}\}$$

  - **nonlinear:** the $\Gamma\,\Gamma$ terms are quotients of 8^{th}-order polynomials of g and ∂g

- was so difficult to solve these nonlinear equations without assuming *algebraic specialness of the curvature* or/and several types of *spacetime symmetries*
- $\Longrightarrow$ the corresponding **exact solutions** could only represent **some isolated points in the space of solutions** of Einstein's equations.

Spacetimes as Cauchy developments:

- **Yvonne Choquet-Bruhat (1952):** Einstein's equations as a set of coupled quasi-linear wave equations: **existence & uniqueness of local solutions**
- another celebrated result of **Choquet-Bruhat and Geroch (1969)**: They proved **the existence of unique** (up to the diffeomorphism invariance of the theory) **maximal Cauchy developments**.

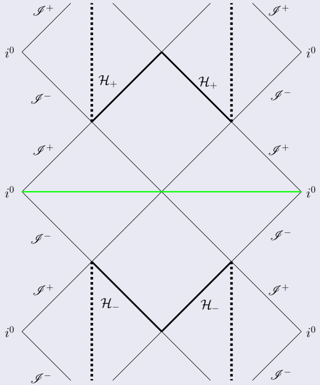
- In short, we can say that **Einstein's theory has a well-posed initial value formulation:** $\exists$ a map that is "one-to-one" & continuous & causal
- **(!)** linear PDEs with regular coefficients on Minkowski spacetime are guaranteed to have "global in time" solutions
- the singularity theorems of Penrose, Hawking, and Geroch taught us lessons that even the "maximal Cauchy developments" may guarantee only "local in time" existence
- since there is **no fixed background** in GR $\Longrightarrow$ the topology of M need not to be $\mathbb{R}^n$, Cauchy problem: M is constructed together with the metric

Could the predictive power of Einstein's theory be limited ?

- $\exists$ exact solutions in which the maximal Cauchy development of the data, induced on some otherwise maximal initial data surface, is a proper subset of the complete solution.

Kerr spacetime

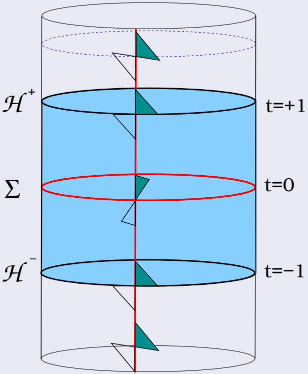
$M \approx \mathbb{R}^2 \times \mathbb{S}^2$



Taub-NUT spacetime

$M \approx \mathbb{R} \times \mathbb{S}^3 \quad [SU(2) \times U(1)]$

$ds^2 = 4\, dt\sigma_1 + 4\, \frac{1-t^2}{1+t^2}\, \sigma_1^2 + (1+t^2)\,(\sigma_2^2 + \sigma_3^2)$



The strong cosmic censor hypothesis by Penrose

- **Penrose's SCCH:** Sufficiently generic spacetimes are maximal globally hyperbolic developments, so they are never part of a larger spacetime.

A conjecture by Vince Moncrief

- **Moncrief** made extensive study of closed cosmological spacetimes in the early 80s'
- He concluded that spacetimes with a compact Cauchy horizon always admit a Killing vector field which is null on the horizon $\mathcal{H}$ and spacelike on $D(\Sigma)$
- Since spacetimes with symmetries are always special (not generic) - if Moncrief's conjecture is true - we have an indirect verification of Penrose's SCCH for spacetimes admitting a compact Cauchy horizon.

Moncrief's conjecture turned out to be correct, but the progress took some time

- (1983) Moncrief, Isenberg: *Symmetries of cosmological Cauchy horizons*, Commun. Math. Phys. 89, 387-413 C^{ω}**, with closed generators**
- (1999) Friedrich, Rácz, Wald: *On the Rigidity Theorem for Spacetimes with a Stationary Event Horizon or a Compact Cauchy Horizon* Commun. Math. Phys. 204, 691-707 C^{∞}**, with closed generators**
- (2000) Rácz: *On Further generalizations of the Rigidity Theorem for Spacetimes with a Stationary Event Horizon or a Compact Cauchy Horizon*, Classical Quant. Gravity 17, 153-178 C^{∞}**, with closed generators, involving Klein-Gordon, Maxvell, Yang-Mills-Higgs and dilaton fields**
- (2020) Moncrief, Isenberg: *Symmetries of cosmological Cauchy horizons with non-closed orbits*, Commun. Math. Phys. 374, 145-186 C^{ω}**, with "generic" generators in 4-dim**
- (2023) Petersen and Rácz: *Symmetries of Vacuum Spacetimes with a Compact Cauchy Horizon of Constant Nonzero Surface Gravity*, Ann. Henri Poincaré, https://doi.org/10.1007/s00023-023-01335-9 **generic vacuum case** (allowing ergodic generators)

Our results (covered by the last paper) apply to:

- any C^∞ connected time-oriented spacetime (M,g) of dim $n\,(\geq 2)$
- $\exists\ \Sigma$ a closed acausal topological hypersurface in M such that its **Cauchy development** $D(\Sigma)$ is a globally hyperbolic **proper submanifold in** (M,g)
- one of the Cauchy horizons, denoted by $\mathcal{H}$, [could be the future or the past one $\mathcal{H}^\pm := \overline{D^\pm(\Sigma)}\setminus D^\pm(\Sigma)$] was assumed to be **non-empty and compact**
- if the *null convergence condition* holds [$R_{ab}L^aL^b \geq 0$ for all lightlike vectors L^a] $\mathcal{H}$ is a **smooth totally geodesic** submanifold in M. [$R_{ab}=0$ in vaccum]
- also $\exists$ a C^∞ fn. κ and a C^∞ non-vanishing vector field V, tangent to the generators of $\mathcal{H}$, such that

$$\boxed{\nabla_V V = \kappa\, V}$$

- **assumed:** by rescaling V, κ can be guaranteed to be a **non-zero constant** (proven by Bustamente, Reiris; Gurriaran, Minguzzi 2021-22, "$\neq 0$" in all known examples)
- **We also have then that:**
  - $\exists$ a nowhere vanishing one-form ω on $\mathcal{H}$: tangent bundle $T\mathcal{H} = \mathbb{R}V \oplus ker(\omega)$
  - a time function $t : [0,\varepsilon) \times \mathcal{H} \to \mathbb{R}$ in a neighborhood of $\mathcal{H}$: its gradient $\partial_t|_{\mathcal{H}}$ is everywhere transverse to $\mathcal{H}$

Our main result read then as:

Theorem: [Existence of a Killing vector field]

Assume that $R_{ab} = 0$ and that $\mathcal{H}$ is a compact Cauchy horizon in (M, g) such that its surface gravity can be normalised to a non-zero constant. Then (*regardless if the generators of $\mathcal{H}$ are closed or ergodic*) **there exists a smooth non-trivial Killing vector field** W^a

$$\boxed{\mathcal{L}_W g_{ab} = 0}$$

on $\mathcal{H} \sqcup D(\Sigma)$, such that W^a is **lightlike on** $\mathcal{H}$ and **spacelike in** $D(\Sigma)$ **near** $\mathcal{H}$, and **any smooth extension of** W^a across $\mathcal{H}$ to the complement of $\overline{D(\Sigma)}$ is **timelike near the horizon** $\mathcal{H}$.

- The proof we applied was based on two key steps:

We used a pair of coupled wave equations:

Lemma:

relating an arbitrary smooth vector field V^a to the Lie derivatives of the metric g_{ab} and the Ricci tensor R_{ab} with respect to V^a

$$\nabla^e\nabla_e V^a - \nabla^f\big[\mathcal{L}_V g_{fh} + (\nabla_e V^e)\, g_{fh}\big]g^{ha} = -\, R^a{}_f V^f$$

$$\nabla^e\nabla_e(\mathcal{L}_V g_{ab}) + 2\, R_a{}^e{}_b{}^f(\mathcal{L}_V g_{ef}) + \mathcal{L}_{[-\nabla^e\nabla_e V]}g_{ab} = -\,2\,\mathcal{L}_V R_{ab} - \mathcal{L}_{[R^e{}_f V^f]}g_{ab} + 2\, R_{(a|}{}^f\mathcal{L}_V g_{f|b)}$$

(1) We proved that solutions to these wave equations, with initial data on a compact Cauchy horizon, $\mathcal{H}$, **extend** into the globally hyperbolic domain, $D(\Sigma)$.

Remarks:

- no use of Einstein's equations or any other filed equation
- hold for any sufficiently regular vector field V^a
- the signature of g_{ab} does not matter, so it could be arbitrary
- for a metric of Lorentzian signature on the horizon $\mathcal{H}$
  $\nabla^e\nabla_e \ldots = -2\,\nabla_V\nabla_t \ldots + c\,\nabla_t \ldots + \{\text{l.o.lin.diff.op. acting along } \mathcal{H}\}$

Our main trick used the "maximum principle":

- (2) We showed that not only the Lie derivative of the metric, $\mathcal{L}_V g_{ab}$, with respect to a "candidate" Killing vector field V, but also its transverse derivatives $(\nabla_t)^k\,[\,\mathcal{L}_V g_{ab}\,]$, up to any order, vanish on $\mathcal{H}$.
- take any of the C^∞ symmetric 2-tensor fields

$$\boxed{\mathfrak{a}_{ab} = (\nabla_t)^k\,[\,\mathcal{L}_V g_{ab}\,] \ \ (\in E^* \otimes_{sym} E^*)} \text{ on } \mathcal{H}$$

- the metric $\mathfrak{g}$, defined on the space of such smooth symmetric 2-tensor fields,

$$\boxed{\mathfrak{g}(\mathfrak{a},\mathfrak{a}) := g^{ik}g^{jl}\,\mathfrak{a}_{ij}\mathfrak{a}_{kl}}$$

  is positive definite on $E^* \otimes_{sym} E^*$ since g is such on $E = ker(\omega)$
- using $\nabla^e\nabla_e \dots = -2\,\nabla_V\nabla_t \dots + c\,\nabla_t \dots + \{\text{l.o.lin.diff.op. acting along } \mathcal{H}\}$ and its time-derivatives, we showed that the equation holds, with $\beta \neq 0$, on $\mathcal{H}$

$$\boxed{\mathrm{d}_V\,\mathfrak{g}(\mathfrak{a},\mathfrak{a}) + \beta\,\mathfrak{g}(\mathfrak{a},\mathfrak{a}) = 0}$$

- $\mathcal{H}$ is compact, the function $\mathfrak{g}(\mathfrak{a},\mathfrak{a})$ must attain its maximum and minimum. whence at these locations $\mathrm{d}_V\,\mathfrak{g}(\mathfrak{a},\mathfrak{a}) = 0$ which $\&\ \beta \neq 0 \Longrightarrow \mathfrak{g}(\mathfrak{a},\mathfrak{a}) \equiv 0$ **!!!** no need to refer to the individual generators **!!!**

Summary:

- We proved that **any smooth** vacuum spacetime containing a compact Cauchy horizon, with surface gravity that can be normalized to a non-zero constant, **admits a Killing vector field**
  - Our result is generic. It holds in any dimension and is independent of the structure of the space of generators of the Cauchy horizon.
- This result, together with the proof that surface gravity can be normalized to a non-zero constant, proves Moncrief's conjecture
  - $\Longrightarrow$The maximal globally hyperbolic vacuum development of generic initial data cannot have a compact Cauchy horizon.
- As an important by-product, our result **supports the validity of the strong cosmic censorship conjecture of Penrose** in the case of closed cosmological spacetimes.