Detection of typical bipartite entanglement by local generalized measurements

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Entanglement and quantum measurements - introductory remarks

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entanglement - valuable resource for quantum information processing

- bipartite scenario as simplest case
  separable bipartite quantum states ρ form convex set

$$\rho = \sum_m p_m \, \rho_m^A \otimes \rho_m^B, \quad \sum_m p_m = 1, \quad p_m \geq 0$$

  entangled quantum states $\equiv$ not separable quantum states
- Peres-Horodecki condition for bipartite entanglement
  negative eigenvalues of partial transpose of $\rho \longrightarrow$ entanglement
  sufficient and necessary for $2 \otimes 3$ quantum systems
  sufficient for arbitrary dimensional bipartite quantum systems

$\longrightarrow$ problem: partial transposition is not a quantum operation

Entanglement and quantum measurements

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quantum key distribution and quantum communication require
local measurements for provable entanglement
$\longrightarrow$ dependence of measurement-based local entanglement detection on

- nature of local quantum measurements
- dimensionality of bipartite quantum systems for typical quantum states?

here:

- compare known sufficient conditions for local bipartite entanglement
  resulting from different classes of local measurements:
  positive operator valued measures ((N, M) POVMs) and
  local hermitian operator bases (LOOs)
- quantitative study of efficiencies of bipartite entanglement detection:
  statistical exploration of quantum state space for different dimensions

Entanglement and quantum measurements

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two main results:

- symmetry properties of (N, M) POVMs $\longrightarrow$ scaling properties
  relate equally efficient bipartite local entanglement detection scenarios
  - all correlation-matrix-based sufficient entanglement conditions are equivalent
  - joint-probability-based sufficient conditions exhibit characteristic scaling properties
- study of entanglement detection efficiences for typical bipartite quantum states
  statistical exploration of state space by Monte-Carlo methods
  $\longrightarrow$ lower bounds on Euclidean volume ratios between
  locally detectable entangled states and all bipartite quantum states

(M.Schumacher, G.A., arXiv:2305.14226 (2023))

Contents

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- (N, M) POVMS as unified description of generalized quantum measurements
  - basic properties
  - (N, M) POVMs and orthonormal hermitian operator bases
    $\longrightarrow$ characteristic scaling properties
- known sufficient conditions for bipartite entanglement
  - violations of inequalities for
    correlation matrices and joint probability distributions of (N, M) POVMs and LOOs
  - scaling properties $\longrightarrow$ equivalent sufficient bipartite entanglement conditions
- statistical properties of typical bipartite entanglement detection

Generalized measurements with positive operator valued measures ((N, M) POVMs)

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(N, M) POVMs yield unified description of quantum measurements, such as

- projective measurements with complete mutually unbiased bases (MUBs)
- mutually unbiased measurements (MUMs)
- symmetric informationally complete POVMs (SIC-POVMs) and generalizations, i.e. GSIC-POVMs

(K. Siudzinska, PRA **105**, 042209 (2022))

positive operators valued measures (POVMs):

$$\{\Pi_a \geq 0 \mid a = 1, \cdots, M\} \quad \text{with} \quad \sum_{a=1}^{M} \Pi_a = \mathbf{1}_d$$

$p_a = \mathrm{Tr}\{\rho \Pi_a\} \cdots$ probability of measurement result $a \in \{1, \cdots, M\}$
special case: projective measurements with $\Pi_a \Pi_{a'} = \delta_{aa'} \Pi_a$

(N, M) POVMs for d-dimensional quantum systems - basic properties

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(N, M) POVM: N POVMs each with M measurement results

$$\{\Pi_{i(\alpha,a)} \mid \alpha = 1, \cdots, N;\ a = 1, \cdots; M\},\quad i(\alpha, a) := (\alpha - 1)M + a \in \{1, \cdots, NM\}$$

defining relations (K. Siudzinska, PRA **105**, 042209 (2022))

$$\begin{aligned}
\mathrm{Tr}\{\Pi_{i(\alpha,a)}\} &= \frac{d}{M}, \\
\mathrm{Tr}\{\Pi_{i(\alpha,a)}\Pi_{i(\alpha,a')}\} &= x\delta_{a,a'} + (1 - \delta_{a,a'})\frac{d - Mx}{M(M-1)}, \\
\mathrm{Tr}\{\Pi_{i(\alpha,a)}\Pi_{i(\beta,a')}\} &= \frac{d}{M^2} \quad (\alpha \neq \beta)
\end{aligned}$$

parameters (d, N, M, x) constrained by

$$\frac{d}{M^2} < x \leq \min\left(\frac{d}{M}, \frac{d^2}{M^2}\right)$$

(N, M) POVMs for d-dimensional quantum systems - basic properties

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$$\frac{d}{M^2} < x \leq \min\left(\frac{d}{M}, \frac{d^2}{M^2}\right)$$

informational completeness (d^2 linear independent POVM elements)

$$(M-1)N + 1 = d^2 \qquad \left(\sum_{a=1}^{M} \Pi_{i(\alpha,a)} = \mathbf{1}_d\right)$$

possible solutions:

- $(N, M) = (1, d^2)$ maximal value $x = 1/d^2$ SIC-POVM
  $1/d^3 < x \leq 1/d^2$ GSIC-POVM
- $(N, M) = (d+1, d)$ maximal value $x = 1$ complete set of MUBs
  $1/d < x \leq 1$ MUMs

construction: expansion using orthonormal hermitian operator basis

- defining relations
- positive semidefiniteness $\Pi_{i(\alpha,a)} \geq 0 \longrightarrow$ possible complications

Informationally complete (N, M) POVMs and orthonormal hermitian operator bases

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expansion of informationally complete (N, M) POVM

$$\Pi = \{\Pi_{i(\alpha,a)} \mid \alpha = 1, \cdots, N; a = 1, \cdots, M\}, \quad (M-1)N + 1 = d^2$$

by orthonormal hermitian operator basis $G = (G_1, \cdots, G_{d^2})^T$

$\longrightarrow$ Hilbert space $\mathcal{H}_{d^2} = \text{span}(G)$ with

Hilbert-Schmidt (HS) scalar product $\langle G_i \mid G_j \rangle_{HS} := \text{Tr}\{G_i^\dagger G_j\}$

$$\longrightarrow \quad \Pi = G^T S \quad \text{with} \quad S : \mathcal{H}_{d^2} \to \mathcal{H}_{NM}$$

most general form of S irrespective of positive semidefiniteness?

necessary relations fullfilled by all (N, M) POVMs

Informationally complete (N, M) POVMs and orthonormal hermitian operator bases

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starting point $\Pi = G^T S$, and $S : \mathcal{H}_{d^2} \to \mathcal{H}_{NM}$

main idea: defining relations of informationally complete (N, M) POVM $\longrightarrow S$

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$$\mathrm{Tr}\{\Pi_{i(\alpha,a)}\} = \frac{d}{M},$$

$$\mathrm{Tr}\{\Pi_{i(\alpha,a)}\Pi_{i(\alpha,a')}\} = x\delta_{a,a'} + (1-\delta_{a,a'})\frac{d-Mx}{M(M-1)}, \quad \longrightarrow \quad (S^T S)_{i,j} = \sum_{\mu=1}^{d^2} X_{i,\mu}\Lambda_\mu X^T_{\mu,j}$$

$$\mathrm{Tr}\{\Pi_{i(\alpha,a)}\Pi_{i(\beta,a')}\} = \frac{d}{M^2} \quad (\alpha \neq \beta)$$

$$i \in \{1, \cdots, NM\}, \ \mu \in \{1, \cdots, d^2\}$$

Informationally complete (N, M) POVMs and orthonormal hermitian operator bases

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$$(S^T S)_{i,j} = \sum_{\mu=1}^{d^2} X_{i,\mu} \Lambda_\mu X^T_{\mu,j}$$

- eigenvalues $\Lambda_\mu > 0, \quad \mu = \{1, \cdots, d^2\}$ and eigenvectors $X_{i(\alpha,a),\mu}, \quad i \in \{1, \cdots, NM\}$
  - $\Lambda_1 = dN/M, \; X_{i,1} = 1/\sqrt{NM}$ (non-degenerate eigenvalue)
  - $\Lambda_\mu \equiv \Gamma = (xM^2 - d)/(M(M-1))$ $(d^2 - 1)$-fold degenerate
    $$\sum_{a=1}^{M} X_{i(\alpha,a),\mu} = 0 \text{ for } \mu = \{2, \cdots, d^2\}$$
  - orthonormality of eigenvectors $\sum_{i=1}^{NM} \left(X^T\right)_{\mu,i} X_{i,\nu} = \delta_{\mu,\nu}$
- polar decomposition$\longrightarrow$ $S_{\mu,i} = \sum_{\nu=1}^{d^2} O^T_{\mu,\nu} \sqrt{\Lambda_\nu} X^T_{\nu,i}$,
  $O \cdots$ orthogonal $d^2 \times d^2$-matrix

Informationally complete (N, M) POVMs and orthonormal hermitian operator bases

- POVM condition for each $\alpha \in \{1, \cdots, N\}$ $\sum_{a=1}^{M} \Pi_i(\alpha, a) = \mathbf{1}_d = \sqrt{d} \sum_{\mu=1}^{d^2} G_\mu O^T_{\mu,1}$
  $\longrightarrow$ new basis $\tilde{G} = OG$ with

$$\tilde{G}_1 = \frac{\mathbf{1}_d}{\sqrt{d}}, \quad \mathrm{Tr}\{\tilde{G}_\mu\} = 0 \text{ for } \mu = \{2, \cdots, d^2\}$$

  implies $\Pi = G^T S = \tilde{G}^T \tilde{S}$ with $\tilde{S}_{\mu,i} = \sqrt{\Lambda_\mu} X^T_{\mu,i}$

$$\Lambda_1 = dN/M > 0, \; \Lambda_\mu = \Gamma = (xM^2 - d)/(M(M-1)) > 0$$

consequences:

- all $(d^2 - 1)$ traceless orthonormal hermitian basis vectors of $\tilde{G}$ are scaled by the same factor $\sqrt{\Gamma}$
- only the basis vector $\tilde{G}_1$ is scaled by the different factor $\sqrt{\Lambda_1}$

$\longrightarrow$ characteristic scaling property of all (N, M) POVMs
influence on sufficient conditions of bipartite entanglement?

Sufficient conditions for bipartite entanglement

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local measurements described by local hermitian operators

$\mathcal{A} = \{A_1, \cdots, A_{N_A}\}$ and $\mathcal{B} = \{B_1, \cdots, B_{N_B}\}$

correlation matrix of quantum state ρ with $\rho^A = \mathrm{Tr}_B\{\rho\}, \ \rho^B = \mathrm{Tr}_A\{\rho\}$

$$C\left(\mathcal{A}, \mathcal{B} \mid \rho\right)_{ij} = \mathrm{Tr}_{AB}\{A_i \otimes B_j \left(\rho - \rho^A \otimes \rho^B\right)\}$$

sufficient bipartite entanglement condition: (S.Lai and S. Luo, PRA **106**, 042402 (2022))

ρ separable implies inequality ($||A||_1 := \mathrm{Tr}\{\sqrt{A^\dagger A}\}$)

$$||C\left(\mathcal{A}, \mathcal{B} \mid \rho\right)||_1 \quad \leq \quad \sqrt{\Sigma_A \Sigma_B}$$

$$\text{with } \Sigma_A = \max_{\sigma^A} \sum_{i=1}^{N_A} \left(\left(\mathrm{Tr}\{A_i \sigma^A\}\right)^2 - \left(\mathrm{Tr}\{A_i \rho^A\}\right)^2\right)$$

$\longrightarrow$ violation of this inequality sufficient for bipartite entanglement

valid for arbitrary

- dimensions of local quantum systems
- local measurements

Measurements by local orthonormal hermitian operators (LOOs)

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sufficient bipartite entanglement condition: violation of inequality

$$||C\left(\mathcal{A},\mathcal{B} \mid \rho\right)||_1 \quad \leq \quad \sqrt{\Sigma_A \Sigma_B}$$

$\mathcal{A}$ and $\mathcal{B}$ local orthonormal hermitian bases (LOOs)

$$\longrightarrow \quad \Sigma_A = \max_{\sigma^A} \sum_{i=1}^{N_A} \left(\left(\mathrm{Tr}\{A_i \sigma^A\}\right)^2 - \left(\mathrm{Tr}\{A_i \rho^A\}\right)^2 \right) = 1 - \mathrm{Tr}\{(\rho^A)^2\} \text{ etc.}$$

$$||C\left(\mathcal{A},\mathcal{B} \mid \rho\right)||_1^2 \quad \leq \quad \left(1 - \mathrm{Tr}\{(\rho^A)^2\}\right)\left(1 - \mathrm{Tr}\{(\rho^B)^2\}\right)$$

(C.-J. Zhang et al., PRA **77**, 060301 R (2008))

- left hand side independent of chosen LOOs
- right hand side only depends on local purities of bipartite quantum state ρ

Measurements by informationally complete (N, M) POVMs

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sufficient bipartite entanglement condition: violation of inequality

$$||C\left(\mathcal{A}, \mathcal{B} \mid \rho\right)||_1 \quad \leq \quad \sqrt{\Sigma_A \Sigma_B}$$

$\mathcal{A} = \Pi^A$ and $\mathcal{B} = \Pi^B$ local informationally complete (N, M) POVMs

$$\longrightarrow \quad \Sigma_A = \max_{\sigma^A} \sum_{i=1}^{N_A M_A} \left(\left(\mathrm{Tr}\{\Pi_{i(\alpha,a)} \sigma^A\}\right)^2 - \left(\mathrm{Tr}\{\Pi_{i(\alpha,a)} \rho^A\}\right)^2 \right) =$$

$$\Gamma_A \left(1 - \mathrm{Tr}\{(\rho^A)^2\}\right) \text{ with } \Gamma_A = (x_A M_A^2 - d_A)/(M_A(M_A - 1)) \text{ etc.}$$

$$\longrightarrow \quad ||C\left(\Pi^A, \Pi^B \mid \rho\right)||_1^2 \quad \leq \quad \Gamma_A \Gamma_B \left(1 - \mathrm{Tr}\{(\rho^A)^2\}\right)\left(1 - \mathrm{Tr}\{(\rho^B)^2\}\right)$$

(L. Lai and S. Luo, Commun.Theor.Phys. **75**, 065101 (2023))

- right hand side depends on scaling parameters Γ_A, Γ_B and local purities
- left hand side has complicated dependence on parameters of Π^A and Π^B

$\longrightarrow$ comparison between different (N, M) POVMs?

Local entanglement detection by informationally complete (N, M) POVMs and LOOs

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sufficient condition for entanglement detection by local (N, M) POVMs

$$||C\left(\Pi^A, \Pi^B \mid \rho\right)||_1^2 \quad \leq \quad \Gamma_A \Gamma_B \left(1 - \mathrm{Tr}\{(\rho^A)^2\}\right)\left(1 - \mathrm{Tr}\{(\rho^B)^2\}\right)$$

scaling properties of (N, M) POVMs $\longrightarrow$ relation between correlation matrices

$$||C\left(\Pi^A, \Pi^B \mid \rho\right)||_1 = ||\sqrt{\Lambda^A} C\left(\tilde{G}^A, \tilde{G}^B \mid \rho\right)\sqrt{\Lambda^B}||_1$$

use $C\left(\tilde{G}_1^A, \tilde{G}^B \mid \rho\right) = C\left(\tilde{G}^A, \tilde{G}_1^B \mid \rho\right) = 0$ and scaling property of (N, M) POVMs

$$\longrightarrow \quad ||C\left(\Pi^A, \Pi^B \mid \rho\right)||_1 \quad = \quad \sqrt{\Gamma_A \Gamma_B}||C\left(\tilde{G}^A, \tilde{G}^B \mid \rho\right)||_1$$

$$\longrightarrow \quad ||C\left(\tilde{G}^A, \tilde{G}^B \mid \rho\right)||_1^2 \quad = \quad \frac{1}{\Gamma_A \Gamma_B}||C\left(\Pi^A, \Pi^B \mid \rho\right)||_1^2 \leq \left(1 - \mathrm{Tr}\{(\rho^A)^2\}\right)\left(1 - \mathrm{Tr}\{(\rho^B)^2\}\right)$$

all (N, M) POVMs are equally powerful as LOOs in detecting entanglement locally

(M.Schumacher, G.A., arXiv:2305.14226 (2023))

Local entanglement detection by joint probability distributions

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joint probability distribution $P(\Pi^A, \Pi^B \mid \rho) = \mathrm{Tr}\{(\Pi^A)^T \otimes \Pi^B \rho\}$

correlation matrix $C\left(\Pi^A, \Pi^B \mid \rho\right) = \mathrm{Tr}\{(\Pi^A)^T \otimes \Pi^B \left(\rho - \rho^A \otimes \rho^B\right)\}$

triangular inequality $||P(\Pi^A, \Pi^B \mid \rho)||_1 \leq ||C(\Pi^A, \Pi^B \mid \rho)||_1 + ||\,\mathrm{Tr}\{(\Pi^A)^T \otimes \Pi^B \rho^A \otimes \rho^B\}||_1$

$$||P(\Pi^A, \Pi^B \mid \rho)||_1^2 \quad \leq \quad (U_A U_B)^2, \quad U_A = \frac{d_A - 1}{d_A} \frac{d_A^2 + M_A^2 x_A}{M_A(M_A - 1)} \text{ etc.}$$

left hand side has complicated dependence on parameters of Π^A and Π^B

scaling property of (N, M) POVMs $\longrightarrow$ $||P(\Pi^A, \Pi^B \mid \rho)||_1 = ||\sqrt{\Lambda^A} P(\tilde{G}^A, \tilde{G}^B \mid \rho) \sqrt{\Lambda^B}||_1$

$$\tilde{x}_A = x_A M_A^2 / d_A^2, \quad \gamma_A = d_A(d_A - 1)/(M_A(M_A - 1)) \text{ etc.}$$

$$\tilde{P}(\tilde{x}_A, \tilde{x}_B \mid \rho) := \sqrt{\Lambda^A/\gamma_A} P(\tilde{G}^A, \tilde{G}^B \mid \rho) \sqrt{\Lambda^B/\gamma_B}$$

$$\longrightarrow \quad ||\tilde{P}(\tilde{x}_A, \tilde{x}_B \mid \rho)||_1^2 \quad \leq \quad (1 + \tilde{x}_A)(1 + \tilde{x}_B)$$

given dimension d_A, d_B and equal values of scaled parameters $\tilde{x}_A$, $\tilde{x}_B$ yield the same sufficient conditions for entanglement

Detection of typical bipartite entanglement by local measurements

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Hilbert space of hermitian operators of $d_A \times d_B$-dimensional bipartite quantum system

- Euclidean vector space
- Euclidean volume measure of convex set of quantum states

$\longrightarrow$ determination of volumes by Monte Carlo methods

here:

- use hit-and-run Monte Carlo algorithm
  random walk inside a convex set
  efficient generation of uniform distributions over convex sets
- sample 10^8 randomly distributed bipartite quantum states
- vary d_A and d_B
- determine lower bounds on relative volumes R of entangles states
  by sufficient conditions of bipartite entanglement

Lower bounds on typical bipartite entanglement

R_{NPT} (Peres-Horodecki), R_{Cor} (correlation matrix), R_{SIC} (joint probabilities of SIC POVM)

(d_A, d_B)	R_{NPT}	R_{SIC}	R_{Cor}
$(2, 2)$	$0,75784 \pm 1,7(4)$	$0,67060 \pm 2,2(4)$	$0,68860 \pm 2,1(4)$
$(2, 3)$	$0,97303 \pm 7(5)$	$0,39732 \pm 5,6(4)$	$0,43853 \pm 5,5(4)$
$(2, 4)$	$0,998696 \pm 1,6(5)$	$0,02710 \pm 2,7(4)$	$0,04504 \pm 3,5(4)$
$(3, 3)$	$0,999895 \pm 4(6)$	$0,75680 \pm 8,2(4)$	$0,76364 \pm 8,1(4)$
$(3, 4)$	1	$0,3605 \pm 1,8(3)$	$0,3795 \pm 1,8(3)$
$(4, 4)$	1	$0,6378 \pm 7,7(3)$	$0,6419 \pm 7,7(3)$

SIC-POVM: optimal $(1, d^2)$ POVM
$x_A = 1/d_A^2, x_B = 1/d_B^2$

local measurements underestimate
lower bounds on
bipartite entanglement R_{NPT}

$R_{NPT} > R_{Cor} \geq R_{SIC}$

(M.Schumacher, G.A.,
arXiv:2305.14226 (2023))

Typical bipartite entanglement detection - joint probabilty distributions for $d_A = 2, \quad d_B = 3$

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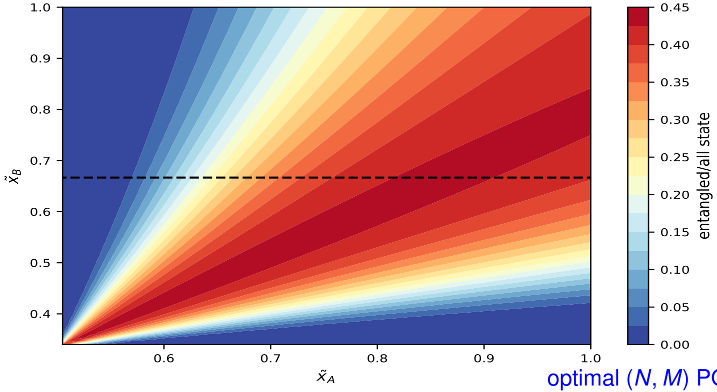
scaled parameters $\tilde{x}_A = x_A M_A^2/d_A^2, \quad \tilde{x}_B = x_B M_B^2/d_B^2$ with $1/d < \tilde{x} \leq \min(1, M/d)$



dashed line: upper bounds for $M_B = 2$

all (N, M) POVMs with $(\tilde{x}_A, \tilde{x}_B)$ yield same result

optimal cases achievable by $\tilde{x}$ close to $1/d$ - always physically realizable

optimal (N, M) POVMs not necessary!

Conclusions

main aims:

- known sufficient conditions for local bipartite entanglement
  compare different classes of local measurements ((N, M) POVMs)
- quantitative study of efficiencies of typical bipartite entanglement detection
  statistical exploration of quantum state space for different dimensions

main results:

- symmetry properties of (N, M) POVMs $\longrightarrow$ scaling properties
  - all correlation-matrix-based sufficient entanglement conditions are equivalent
  - scaling properties of joint-probability-based sufficient conditions
- typical detectable bipartite entanglement
  - local measurements underestimate bipartite entanglement
  - optimal local bipartite entanglement detection possible by (N, M) POVMs with x-values close to lower possible bounds - always realizable
    optimal (N, M) POVMs not necessary!

  $\longrightarrow$ interesting perspectives for applications in quantum communication