Nonlinear quantum dynamics and its potential applications in quantum information

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THE UNREASONABLE EFFECTIVENSS OF MATHEMATICS IN THE NATURAL SCIENCES

Eugene Wigner

Mathematics, rightly viewed, possesses not only truth, but supreme beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

- BERTRAND RUSSELL, Study of Mathematics

There is a story about two friends, who were classmates in high school, talking about their jobs. One of them became a statistician and was working on population trends. He showed a reprint to his former classmate. The reprint started, as usual, with the Gaussian distribution and the statistician explained to his former classmate the meaning of the symbols for the actual population, for the average population, and so on. His classmate was a bit incredulous and was not quite sure whether the statistician was pulling his leg. "How can you know that?" was his query. "And what is this symbol here?" "Oh," said the statistician, "this is pi." "What is that?" "The ratio of the circumference of the circle to its diameter." "Well, now you are pushing your joke too far," said the classmate, "surely the population has nothing to do with the circumference of the circle."

Example: complex numbers

The complex numbers provide a particularly striking example for the foregoing. Certainly, nothing in our experience suggests the introduction of these quantities. Indeed, if a mathematician is asked to justify his interest in complex numbers, he will point, with some indignation, to the many

beautiful theorems in the theory of equations, of power series, and of analytic functions in general, which owe their origin to the introduction of complex numbers. The mathematician is not willing to give up his interest in these most beautiful accomplishments of his genius. [*The reader may be*

It is true, of course, that physics chooses certain mathematical concepts for the formulation of the laws of nature, and surely only a fraction of all mathematical concepts is used in physics. It is true also that the concepts which were chosen were not selected arbitrarily from a listing of mathematical terms but were developed, in many if not most cases, independently by the physicist and recognized then as having been conceived before by the mathematician. It is not true, however, as is so often stated, that this had to happen because mathematics uses the simplest possible concepts and these were bound to occur in any formalism. As we saw before, the concepts of mathematics are not chosen for their conceptual simplicity - even sequences of pairs of numbers are far from being the simplest concepts - but for their amenability to clever manipulations and to striking, brilliant arguments. Let us not forget that the Hilbert space of quantum mechanics is the complex Hilbert space, with a Hermitean scalar product. Surely to the unpreoccupied mind, complex numbers are far from natural or simple and they cannot be suggested by physical observations. Furthermore, the use of complex numbers is in this case not a calculational trick of applied mathematics but comes close to being a necessity in the formulation of the laws of quantum mechanics. Finally, it now begins to appear that not only complex numbers but so-called analytic functions are destined to play a decisive role in the formulation of quantum theory. I am referring to the rapidly developing theory of dispersion relations.

Complex dynamics & chaos

- ☞ Iterated rational polynomials: $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad f^{\circ n} \to ?$
- ☞ One century of complex chaos:

1871 idea of iterated functions by Ernst Schröder

Ueber iterirte Functionen., Math. Ann.

1906 first weird example by P. Fatou: $z \mapsto z^2/(z^2+2)$

1920ies G. Julia, S. Lattès, & . . .

1970ies Computers help visualize: B. Mandelbrot & . . .

- ☞ A good book:
  J.W. Milnor *Dynamics in One Complex Variable*
  (Princeton University Press, 2006)

Quest for the most general quantum evolution

Closed systems - unitary operators - linear evolution

Quantum channels - completely positive maps - linear evolution

Linear maps

- Rules of standard, non-relativistic quantum mechanics
- Ancilla systems
- Many identical copies: ensemble

Quest for the most general quantum evolution

Ancilla from the same ensemble

- Operation on n identical systems
  — larger Hilbert space
- Measurement + post-selection on $n - 1$ systems
- Nonlinear evolution for the remaining system
  — same Hilbert space

Quest for the most general quantum evolution

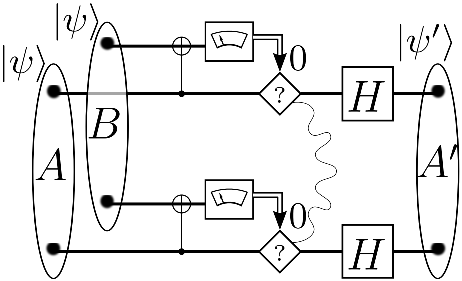
The resulting nonlinear map can be

- Deterministic
- Pure states → pure states
- Complex, deterministic chaos may emerge

T. Kiss, I. Jex, G. Alber, S. Vymětal, Phys. Rev. A **74**, 040301(R) (2006).

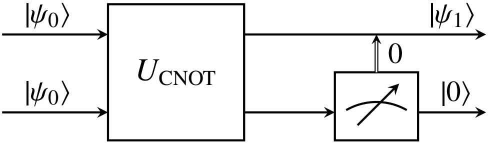
Basic scheme for a nonlinear state transformation

Motivation: entanglement distillation



Basic scheme for a nonlinear state transformation

$|\psi_0\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$



$|\psi_1\rangle = \alpha^2\,|0\rangle + \beta^2\,|1\rangle$

H. Bechmann-Pasquinucci, B. Huttner, & N. Gisin Phys. Lett. A **242**, 198 (1998).

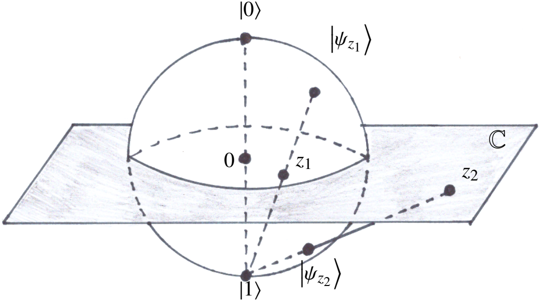
Basic scheme for a nonlinear state transformation

- Trick - parametrization:

$$|\psi\rangle = \alpha \left(|0\rangle + \frac{\beta}{\alpha}|1\rangle\right) = \frac{1}{\sqrt{1+|z|^2}}\left(|0\rangle + z|1\rangle\right) \quad \text{where} \quad z \in \mathbb{C}_\infty = \mathbb{C} \cup \infty$$

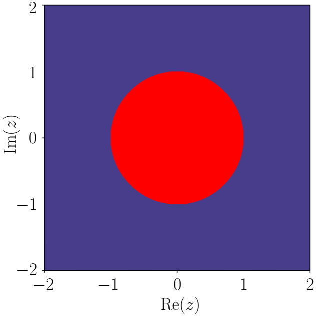
up to a global phase

- Visualization - Bloch sphere



stereographic projection

Iterations of the nonlinear map

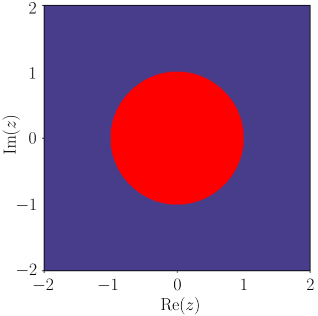


Iteration of $f(z) = z^2$ on the complex plane:

- $|z| < 1 \to 0$ (fixed point)
- $|z| > 1 \to \infty$ (fixed point)
- Julia set: $|z| = 1$ unit circle
  - chaotic dynamics
  - contains repelling cycles

  J. Milnor, *Dynamics in One Complex Variable* (Princeton University Press, 2006)

Iterations of the nonlinear map



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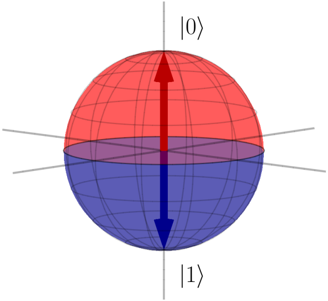
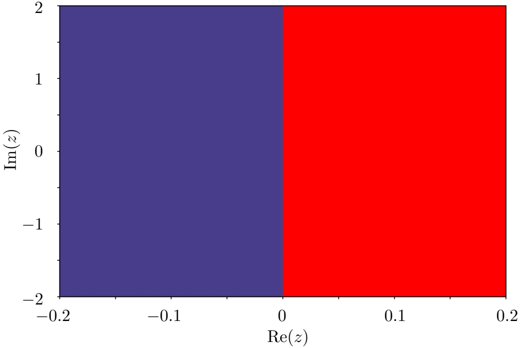


Iteration of $f(z) = z^2$ on the Bloch sphere:

- if $|z| < 1$ states converge to $|0\rangle$
- if $|z| > 1$ states converge to $|1\rangle$
- Julia set: equator
  - equally separates regions of convergence

An application for quantum state discrimination

- if instead of CNOT, we apply $U = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$
- nonlinear transformation: $f = \dfrac{2z}{1+z^2}$



- two superattractive fixed points: 1 and -1
- Julia set: imaginary axis

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

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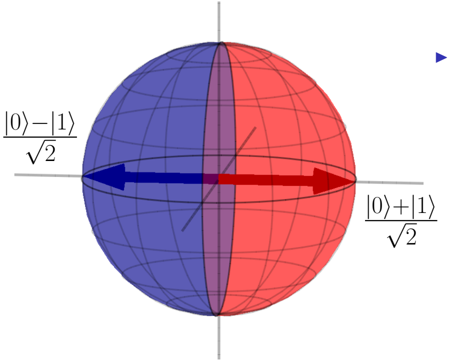
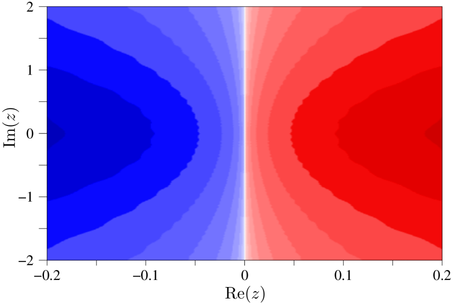


- Julia set: longitudinal great circle through y axis
  - equally separates regions of convergence

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

An application for quantum state discrimination

- if instead of CNOT, we apply $U = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$
- nonlinear transformation: $f = \dfrac{2z}{1+z^2}$
- From highly overlapping to almost orthogonal in only 3 steps



$$|\Psi_0\rangle_I = \frac{|0\rangle + 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

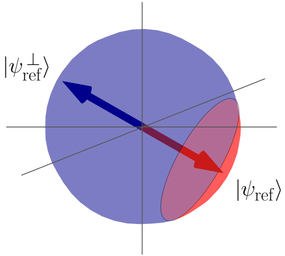
$$|\Psi_0\rangle_{II} = \frac{|0\rangle - 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

$$_I\langle\Psi_0\,|\,\Psi_0\rangle_{II} \approx 0.92$$

$$_I\langle\Psi_3\,|\,\Psi_3\rangle_{II} \approx 0.08$$

G. Zhu, O. Kálmán, K. Wang, L. Xiao, D. Qu, X. Zhan, Z. Bian, T. Kiss, P. Xue, PRA, **100**, 052307 (2019)

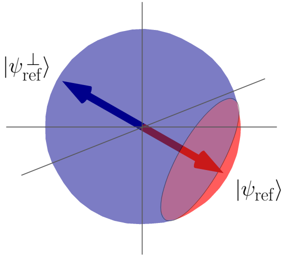
Quantum state matching



- define a reference state: $|\psi_{\mathrm{ref}}\rangle$
- define a neighborhood: $\varepsilon = |\langle\psi \mid \psi_{\mathrm{ref}}\rangle|$
- find which f corresponds to it
- find implementation of f
  - 2-qubit unitary+post-selection

O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)

Quantum state matching



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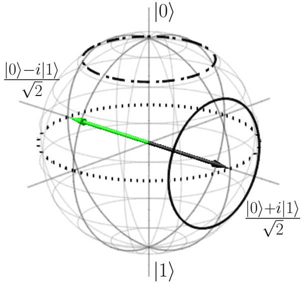
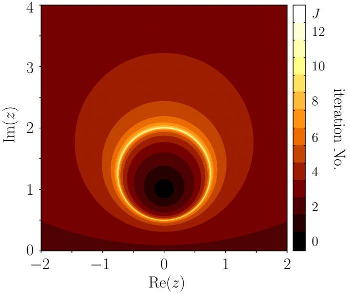




$$|\psi_{\mathrm{ref}}\rangle = \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}$$

$$\varepsilon^2 = 0.9$$

O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)

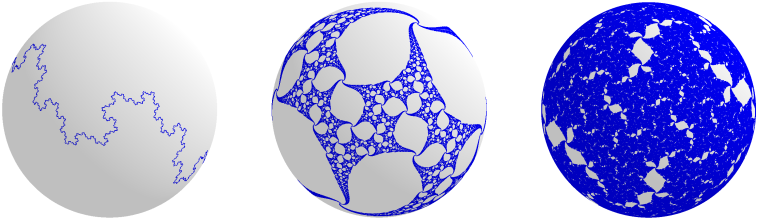
Iterative nonlinear quantum protocols

- Ensemble of qubits in *pure state* $|\psi_0\rangle \sim |0\rangle + z\,|1\rangle \quad (z \in \mathbb{C})$
  1. Take them pairwise: $|\Psi_0\rangle = |\psi_0\rangle_A \otimes |\psi_0\rangle_B$
  2. Apply an **entangling** two-qubit operation U
  3. Measure the state of qubit B — keep A only for result 0
- Smaller ensemble in *pure state* $|\psi_1\rangle \sim |0\rangle + f(z)\,|1\rangle$
- Quantum magnification bound: exponential downscaling of the ensemble

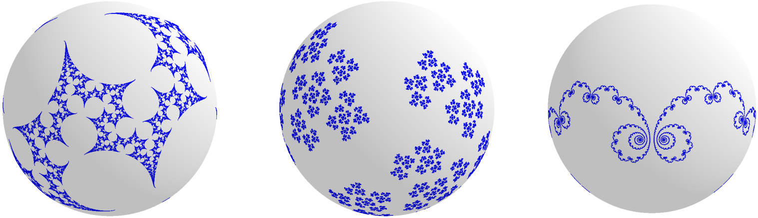
$$U \leftrightarrow f(z) = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2}$$

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016)

Iterative dynamics - Julia sets on the Bloch sphere



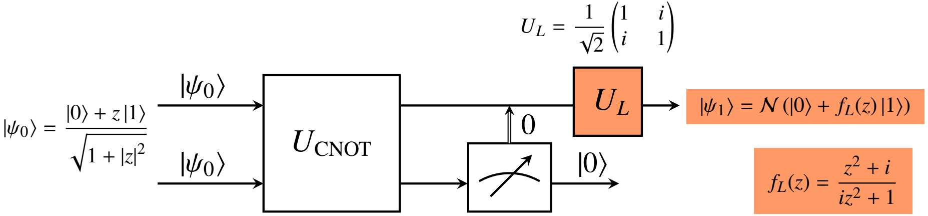
(a) $\theta = 0.4, \varphi = \frac{\pi}{2}$ **(b)** $\theta = 0.55, \varphi = \frac{\pi}{2}$ **(c)** $\theta = 0.633, \varphi = \frac{\pi}{2}$



(d) $\theta = 1.05, \varphi = \frac{\pi}{2}$ **(e)** $\theta = 0.5, \varphi = 0.5$ **(f)** $\theta = 0.232, \varphi = 0$

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

Scheme resulting in an ergodic (Lattès) map



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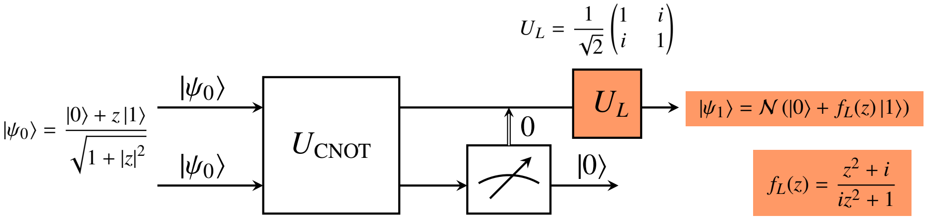
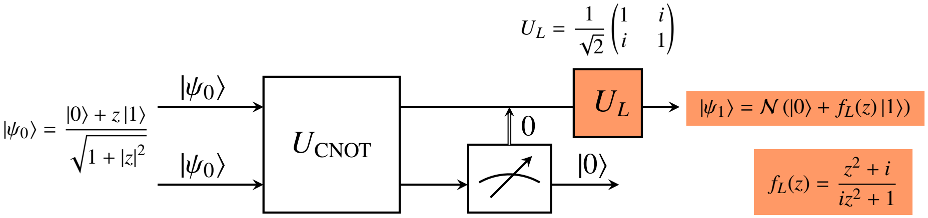


Iterative dynamics of pure initial states

- no attractive fixed cycles
- all pure initial states $\in$ Julia set
- every initial state is chaotic

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

Scheme resulting in an ergodic (Lattès) map



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A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

Does noise destroy this property?

Lattès map with noisy initial states

Dynamics represented by $\mathbb{R}^3 \to \mathbb{R}^3$ functions:

$$u' = \frac{u^2 - v^2}{1 + w^2}, \quad v' = \frac{2w}{1 + w^2}, \quad w' = -\frac{2uv}{1 + w^2}$$

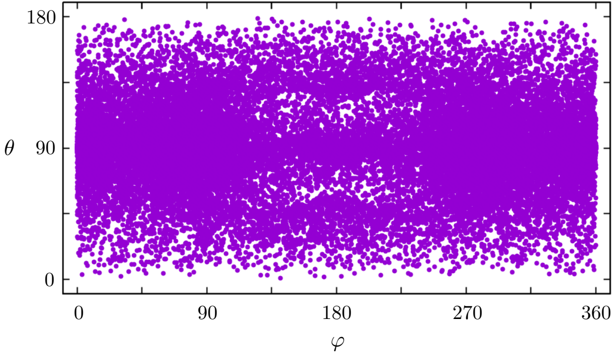
No book by Milnor! :-(

Asymptotics: all mixed initial states → completely mixed state

O. Kálmán, T. Kiss and I. Jex, J Russ Laser Res 39: 382 (2018)

Is there an ergodic regime for noisy initial states?

Evolution of slightly perturbed initial states

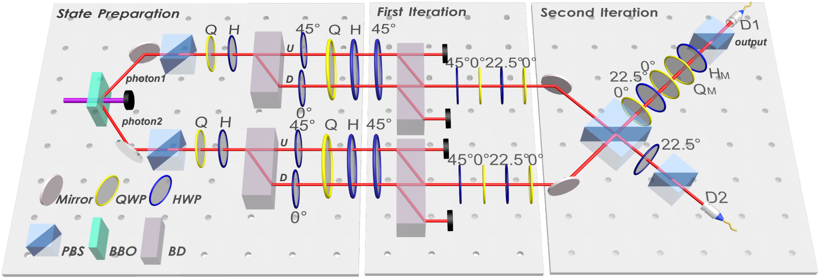


9 iterative steps

$r_0 = 0.99,\ \Delta(\theta_0, \varphi_0) \sim 10^{-3},\ r_{min} \sim 0.5$

D. Qu, O. Kálmán, G. Zhu, L. Xiao, K. Wang, T. Kiss, and P. Xue, New J Phys. **23** 083008 (2021)

Optical realization of the ergodic protocol



- 2 true steps of the scheme were realized
- down-converted photons from BBO crystal
- initial qubit states are encoded into:
  - the polarization degree of freedom and
  - the path degree of freedom of photons

D. Qu, O. Kálmán, G. Zhu, L. Xiao, K. Wang, T. Kiss, and P. Xue, New J Phys. **23** 083008 (2021)

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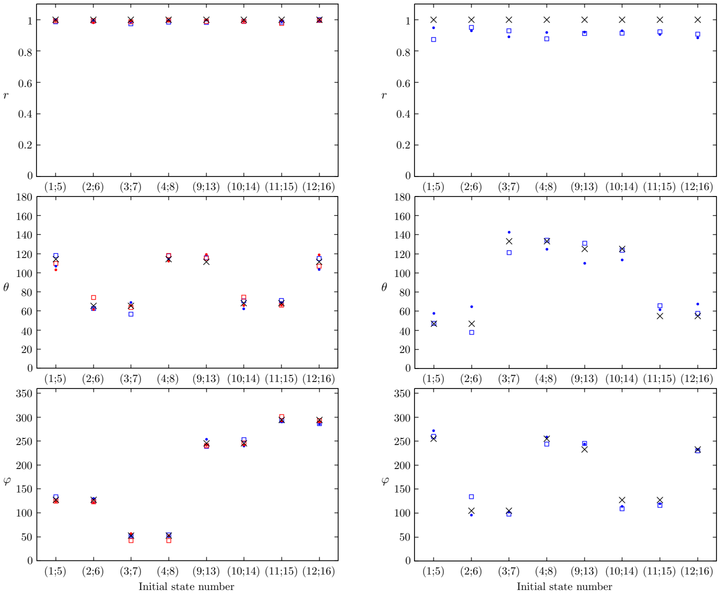
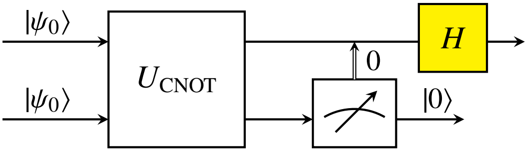


D. Qu, O. Kálmán, G. Zhu, L. Xiao, K. Wang, T. Kiss, and P. Xue, New J Phys. **23** 083008 (2021)

Scheme with CNOT & Hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$|\psi_0\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}}$$



$$|\psi_1\rangle = \mathcal{N}\left(|0\rangle + f_H(z)\,|1\rangle\right)$$

$$f_H(z) = \frac{1-z^2}{1+z^2}$$

Scheme with CNOT & Hadamard gate

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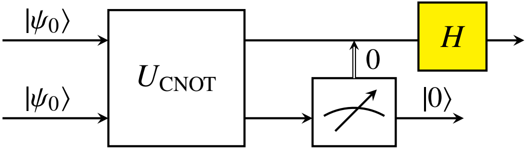

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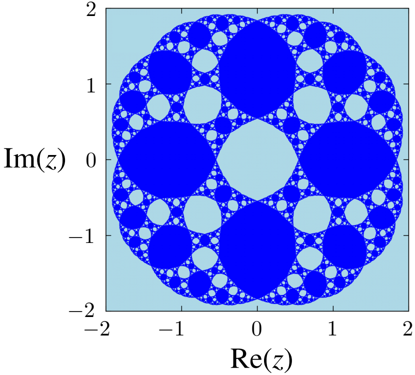
Iterative dynamics (pure initial states)



- attractive length-2 cycle:

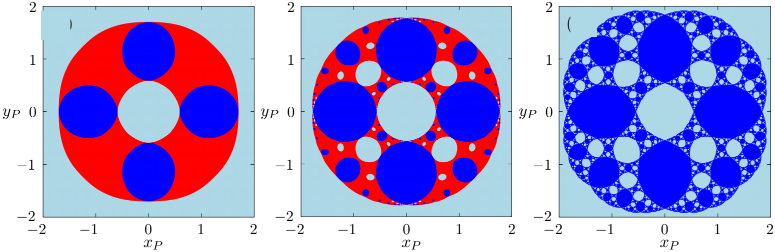
$$\left|\psi^{(1)}\right\rangle = |0\rangle \quad (z = 0)$$

$$\updownarrow$$

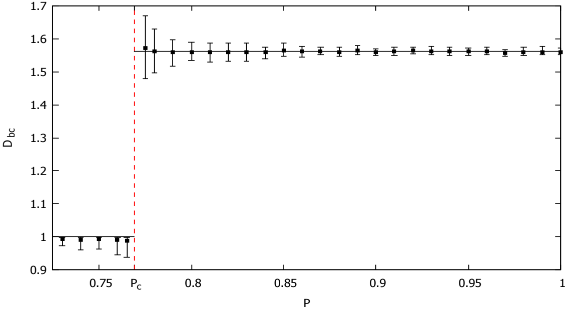
$$\left|\psi^{(2)}\right\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad (z = 1)$$

- border: **Julia set**

Fractal dimension D_{bc} as a function of purity P



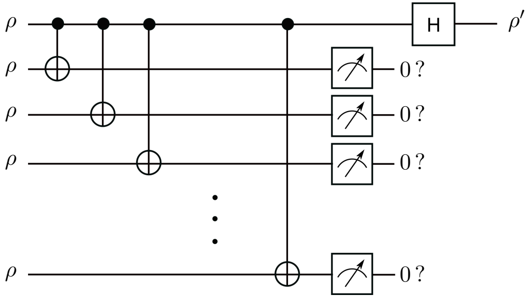
$P = 0.75$ $P = 0.87$ $P = 1$



P_c = purity of C_1

M. Malachov, I. Jex, O. Kálmán, T. Kiss, Chaos **29**, 033107 (2019).

Nonlinear (Hadamard) protocols of order $n > 2$



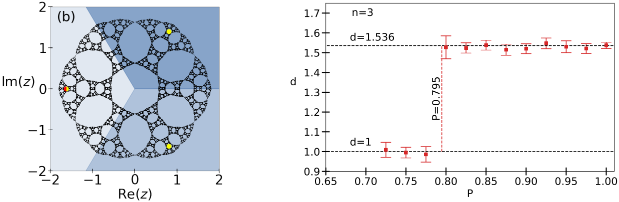
- Map of the pure case: $f_n(z) = \dfrac{1 - z^n}{1 + z^n}$
- Map for noisy states:

$$u' = \frac{(1+w)^n - (1-w)^n}{(1+w)^n + (1-w)^n}, \quad v' = -\frac{2\mathrm{Im}\left[(u+iv)^n\right]}{(1+w)^n + (1-w)^n}, \quad w' = \frac{2\mathrm{Re}\left[(u+iv)^n\right]}{(1+w)^n + (1-w)^n}$$

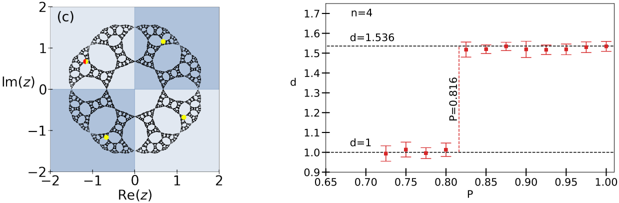
A. Portik, O. Kálmán, I. Jex, T. Kiss, Phys. Lett. A **431**, 127999 (2022).

Nonlinear (Hadamard) protocols of order $n > 2$

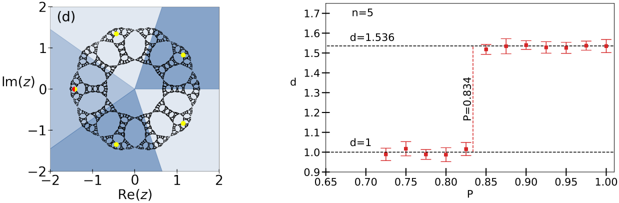
$n = 3$



$n = 4$

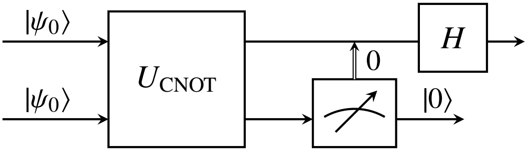


$n = 5$



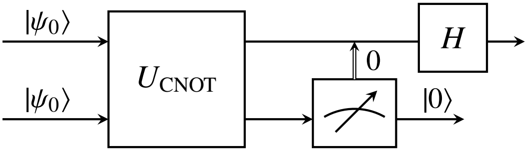
A. Portik, O. Kálmán, I. Jex, T. Kiss, Phys. Lett. A **431**, 127999 (2022).

Noisy evolution

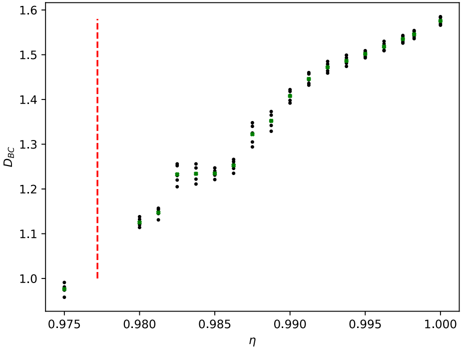


Detector errors: $\rho' = \eta_0 \rho'_{|0\rangle} + (1 - \eta_1)\rho'_{|1\rangle}$

Noisy evolution

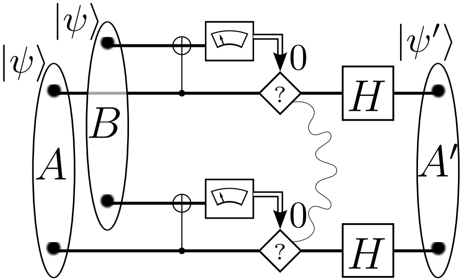


Detector errors: $\rho' = \eta_0 \rho'_{|0\rangle} + (1 - \eta_1)\rho'_{|1\rangle}$



Fractal dimension as a function of η

LOCC scheme with 2 qubits



$$|\psi\rangle = c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle$$

$$|\psi'\rangle = U_H \otimes U_H \left[\mathcal{N}(c_1^2|00\rangle + c_2^2|01\rangle + c_3^2|10\rangle + c_4^2|11\rangle)\right]$$

2 qubits: chaotic entanglement

Asymptotic states

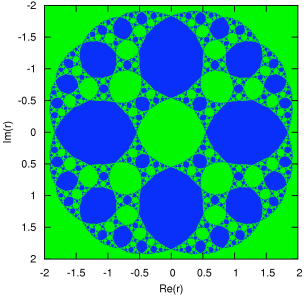
Green: Fully entangled:

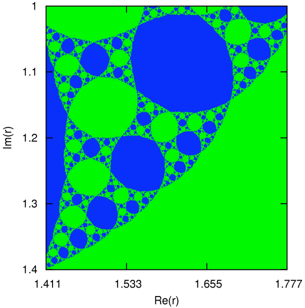
$$|\psi^{(\infty)}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Blue: Completely separable, oscillatory:

$$|\psi^{(\infty)}\rangle \rightarrow \left\{|00\rangle\,,\ \frac{1}{2}\left(|0\rangle + |1\rangle\right)\otimes\left(|0\rangle + |1\rangle\right)\right\}$$

T. Kiss, S. Vymětal, L. D. Tóth, A. Gábris,
I. Jex, G. Alber, PRL **107**, 100501 (2011)





Outlook

- Nonlinear protocols may be used for benchmarking QCs
  - First implementation of a nonlinear protocol (Lattès) for benchmarking by **András Gilyén** et al.

    A. Cornelissen, J. Bausch, and A. Gilyén, arXiv:2104.10698 (2021)
  - Matching an unknown qubit state to a reference qubit state
    - original scheme: O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)
- Measurement-induced dynamics in quantum communication
- Two-step distillation of Bell pairs

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Let me end on a more cheerful note. The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.