The Semiclassical Einstein Equations and the Stability of Linearized Solutions

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Plan of the talk

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References

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- P. Meda, N. Pinamonti, and D. Siemssen. Existence and Uniqueness of Solutions of the Semiclassical Einstein Equation in Cosmological Models. In: *Ann. Henri Poincaré* **22** (2021), 3965–4015. arXiv: 2007.14665. DOI: 10.1007/s00023-021-01067-8.
- P. Meda, S. Murro, and N. Pinamonti. "Linear Stability of Minkowski Spacetime in Semiclassical Gravity". **In preparation** (expected preprint on November 2023).

Introduction

QFT vs. GR

How to describe quantum matter and gravity **interplay**?

- **QFT in curved spacetimes**: quantum matter field ϕ on a physical state ω propagating over classical Lorentzian spacetimes $(\mathcal{M}, g)$
- **Semiclassical gravity** studies backreaction on the spacetime geometry

Semiclassical Einstein Equations

$$G_{ab}[g] = 8\pi G \left\langle :T_{ab}: \right\rangle_{\omega} [\phi, g] \qquad G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

- **Solutions** $(\mathcal{M}, g)$ provide physical predictions about this interplay.
- **Validity**:
  - *Quantum gravity effects* are negligible.
  - *Fluctuations* of the quantum stress-energy tensor are small.
- **Physical applications**:
  1. *Black Hole Physics*: Hawking radiation, evaporation.
  2. *Cosmology*: Inflationary Universe.

The quantum stress-energy tensor $\langle :T_{ab}: \rangle_\omega$

- **Stress-energy tensor** of a scalar field

$$T_{ab} = \frac{1}{2}\nabla_a\nabla_b\phi^2 + \frac{1}{4}g_{ab}\Box\phi^2 - \phi\nabla_a\nabla_b\phi + \frac{1}{2}g_{ab}g^{cd}\phi\nabla_c\nabla_d\phi +$$
$$+ \xi\left(G_{ab} - \nabla_a\nabla_b - g_{ab}\Box\right)\phi^2 - \frac{1}{2}g_{ab}m^2\phi^2.$$

- **AQFT**: quantum scalar field $\phi(f)$ over *globally hyperbolic spacetimes* Details

$$\phi(Pf) = 0, \qquad P = -\Box + m^2 + \xi R, \qquad [\phi(f_1), \phi(f_2)] = i\Delta(f_1, f_2)\mathbb{1},$$

where $\Delta = \Delta_R - \Delta_A$ is the *causal propagator*.

- **Hadamard point-splitting** $:\phi^2:$, $:\phi\nabla_a\nabla_b\phi:$, $:T_{ab}:$ (normal ordering)

$$\omega_2 \doteq \mathcal{H}_{0^+} + \mathcal{W} \qquad \mathcal{H}_{0^+} = \frac{1}{8\pi^2}\lim_{\epsilon\to 0^+}\left(\frac{U}{\sigma_\varepsilon} + V\log\left(\frac{\sigma_\varepsilon}{\lambda^2}\right)\right) \qquad \textit{Hadamard state}$$

- **Locality and conservation**

$$\langle :T_{ab}: \rangle_\omega \doteq \lim_{x'\to x} D_{ab}\left(\omega_2(x,x') - \mathcal{H}_{0^+}(x,x')\right) \qquad \textit{local and covariant.}$$

$$\nabla^a \langle :T_{ab}: \rangle_\omega = 0 \qquad \textit{covariant conservation.}$$

- **Renormalization freedoms**

$$:\tilde{T}_{ab}: = :T_{ab}: + c_1 m^4 g_{ab} + c_2 m^2 G_{ab} + \alpha I_{ab} + \beta J_{ab}, \qquad c_1, c_2, \alpha, \beta \in \mathbb{R}.$$

Cosmology

Cosmological spacetimes

- **Friedmann-Lemaître-Robertson-Walker metric** $(\mathcal{M}, g)$, where $\mathcal{M} = I_{t,\tau} \times \Sigma$,

$$g = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\mathbf{x}^2 = a(\tau)^2 \left(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2\right)$$

  where

  - $\mathrm{d}t$ **cosmological time**, $\mathrm{d}\tau = a^{-1}\mathrm{d}t$ **conformal time**
  - $a(t)$ **scale factor** describes the history of the Universe

- **Λ-CDM model**: matter is described by a perfect fluid $T_a{}^b = \mathrm{diag}(-\varrho, p, p, p)$, where $p = w\varrho$, $w = 0, 1/3, -1$ (*ordinary matter*, *radiation*, *dark energy*).

- **Friedmann equations**

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varrho, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varrho + 3p)$$

- **Inflationary Cosmology**: quantum contributions drove the expansion of the Universe close to the Big Bang (*Starobinsky model of Inflation*).

Single-field model of Inflation

- **Quantum Scalar Field**

$$-\Box\phi + m^2\phi + \xi R\phi = 0.$$

- **Initial-value formulation** for the FLRW spacetime $(\mathcal{M}, g)$ and the quantum matter field (ϕ, ω)

$$\begin{cases} -R = 8\pi G \langle :T: \rangle_\omega , \\ G_{00}(\tau_0) = 8\pi G \langle :T_{00}: \rangle_\omega (\tau_0), \\ \nabla^a \langle :T_{ab}: \rangle_\omega = 0, \quad \checkmark \end{cases}$$

equipped with four initial data $\left(a_0, a_0', a_0'', a_0^{(3)}\right)$ and with initial conditions for ω.

- The scale factor $a(\tau)$ is the **unique degree of freedom** of the problem.
- The energy constraint fixes the quantum state ω for given $\left(a_0, a_0', a_0'', a_0^{(3)}\right)$.
- $\langle :T: \rangle_\omega$ contains **fourth-order derivatives** of $a(\tau)$ for **arbitrary couplings** $\xi \in \mathbb{R}$.

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$$\Downarrow$$

Proving existence and uniqueness of cosmological semiclassical solutions for arbitrary couplings ξ

Quantum states on FLRW spacetimes

Proposition (energy constraint)

Given four initial data $\left(a_0, a_0', a_0'', a_0^{(3)}\right)$, it is always possible to select a sufficiently regular quantum state ω such that the **energy constraint** is satisfied at $\tau = \tau_0$:

$$G_{00}(\tau_0) = 8\pi G \left\langle :T_{00}: \right\rangle_\omega (\tau_0)$$

- **"Vacuum-like" state**: quasi-free, pure, homogeneous and isotropic

$$\omega_2(x, y) = \lim_{\epsilon \to 0^+} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\bar{\zeta}_k(\tau_x)}{a(\tau_x)} \frac{\zeta_k(\tau_y)}{a(\tau_y)} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-\epsilon k} d\mathbf{k} \qquad k \doteq |\mathbf{k}|$$

- **Temporal modes** $\zeta_k(\tau)$

$$\zeta_k''(\tau) + \Omega_k^2(\tau)\zeta_k(\tau) = 0 \qquad \Omega_k^2 = k^2 + a^2 m^2 + (\xi - 1/6)a^2 R$$

  satisfying $\zeta_k' \bar{\zeta}_k - \zeta_k \bar{\zeta}_k' = i$.

- Modes ζ_k define a **sufficiently regular state** ω if at $\tau = \tau_0$ Details

$$\left\langle :\phi^2: \right\rangle_\omega \in C^2\left([\tau_0, \tau_1]\right), \qquad \left\langle :T_{00}: \right\rangle_\omega \in C^0\left([\tau_0, \tau_1]\right).$$

- Given $\left(a_0, a_0', a_0'', a_0^{(3)}\right)$, there is still freedom in the choice of ζ_k

Cosmological semiclassical equation

Solve the **traced semiclassical Einstein equations**: $-R = 8\pi G \langle :T: \rangle_\omega$

$$\langle :T: \rangle_\omega = \left(3\left(\xi - \frac{1}{6}\right)\Box - m^2 \right) \langle :\phi^2: \rangle_\omega + \langle :T: \rangle_\omega^{(an)} + c_1 m^4 + c_2 m^2 R + \gamma \Box R.$$

- **Trace anomaly**

$$\langle :T: \rangle_\omega^{(an)} = \frac{1}{4\pi^2} \left(\frac{(6\xi - 1)^2 R^2}{288} + \frac{R_{abcd}R^{abcd} - R_{ab}R^{ab}}{720} \right)$$

- **Renormalization constants**

$$\underbrace{c_1 \Rightarrow \Lambda,}_{\text{Cosmological constant}} \qquad \underbrace{c_2 \Rightarrow G,}_{\text{Newton constant}} \qquad \underbrace{\gamma.}_{\text{Quantum freedom}}$$

- **Non-classical dynamics** for non-conformal couplings $\xi \neq \frac{1}{6}$:
  1. $\Box \langle :\phi^2: \rangle_\omega$ and $\Box R$ contain **higher order derivatives** of $a(\tau)$ up to $a^{(4)}(\tau)$.
  2. $\Box \langle :\phi^2: \rangle_\omega$ is highly **non local functional** of $a(\tau)$.

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$$\Downarrow$$

Rewrite the equation in a new form to get an initial-value problem which admits a unique solution solution $a(\tau)$ in $[\tau_0, \tau_1]$ given $\left(a_0, a_0', a_0'', a_0^{(3)}\right)$

HowTo: a simple semiclassical model

- **Quantum scalar field** ϕ coupled with a **classical scalar field** ψ in flat spacetime

$$\psi = \Lambda + \langle :\phi^2: \rangle_0 , \qquad V = \int_{\mathcal{M}} \mathcal{L}_I g d^4x = -\frac{\lambda}{2} \int_{\mathcal{M}} \psi \phi^2 g d^4x, \qquad g \in \mathcal{D}(\mathcal{M}),$$

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- **Perturbative expansion** of $\langle :\phi^2: \rangle_0$ in λ in the vacuum state ▸ pAQFT

$$\langle :\phi^2: \rangle_0^{(\text{lin})} = \langle R_V(:\phi^2:) \rangle_0^{(\text{lin})} = -i\lambda \int_{\mathcal{M}} \left(\Delta_F^2(y-x) - \Delta_+^2(y-x) \right) \psi(y) d^4y.$$

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- Truncating at first order provides a **linearized** semiclassical equation for ψ

$$\psi = \Lambda + \lambda \mathcal{T}[\psi] + \dots$$

$$\mathcal{T}[f] \doteq - \int_{t_0}^{t} f'(s) \log(t-s) \mathrm{d}s, \qquad f \in \mathcal{C}^1 \left([t_0, t] \right).$$

1. **Unbounded (tame) retarded operator** which looses derivatives

$$\| \mathcal{T}[f] \|_\infty \leq C \left(\|f\|_\infty + \|\partial f\|_\infty \right), \qquad \| \mathcal{T}[f] \|_\infty \nleq \tilde{C} \|f\|_\infty .$$

2. Recursive procedures to obtain numerical solutions **fail to converge**
3. **Inverse** $\mathcal{T}^{-1}$ has nicer properties: $\| \mathcal{T}^{-1}[f] \|_\infty \leq C' \|f\|_\infty$

Inversion procedure

- Using the **inversion formula** for $\mathcal{T}^{-1}[f]$, the **new inverted equation**

$$\psi = \psi_0 + \mathcal{T}^{-1}\left[\psi - \Lambda - \ldots\right]$$

  can be treated by fixed point methods (*Banach fixed point theorem*), because recursive constructions of ψ now converge!

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- **Apply this idea to the cosmological equation**: roughly,

$$-R = 8\pi G \langle :T: \rangle_\omega \quad \overset{\not\Box}{\Longrightarrow} \quad \partial_\tau \left\langle :\phi^2: \right\rangle_\omega = \mathcal{S} \quad \xrightarrow{\left\langle :\phi^2: \right\rangle_\omega} \quad X' = \alpha_\xi \mathcal{T}\left[X'\right] + \ldots$$

$$\Big\Downarrow \ \textbf{inversion}$$

$$X' = X'_0 + \frac{1}{\alpha_\xi}\mathcal{T}^{-1}\left[X' - \ldots\right], \qquad X(\tau) = \frac{a''(\tau)}{a(\tau)}.$$

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$$\Big\Downarrow \textbf{ inversion}$$

$$X' = X'_0 + \frac{1}{\alpha_\xi}\mathcal{T}^{-1}[X' - \dots], \qquad X(\tau) = \frac{a''(\tau)}{a(\tau)}.$$

Main result: existence and uniqueness of local solutions

Given some initial data $\left(a_0, a'_0, a''_0, a_0^{(3)}\right)$ and a sufficiently regular state ω which satisfies the energy constraint at τ_0, a **unique solution** $a(\tau)$ of the semiclassical equation exists in $[\tau_0, \tau]$ for sufficiently small τ

Linear Stability

The issue of runaway solutions

- Semiclassical theories of gravity seem to include unstable, **exponentially growing** solutions in time
  - *G. T. Horowitz and R. M. Wald. "Dynamics of Einstein's equation modified by a higher-order derivative term", PRD* **17***, 414–416 (1978).*
  - *G. T. Horowitz. "Semiclassical relativity: The weak-field limit", PRD* **21***, 1445–1461 (1980).*
  - *W. M. Suen. "Minkowski spacetime is unstable in semiclassical gravity", PRL* **62***, 2217–2220 (1989).*
  - *E. E. Flanagan and R. M. Wald. "Does back reaction enforce the averaged null energy condition in semiclassical gravity?", PRD* **36***, 6233–6283 (1996).*
- Perturbative approach (**linearization**):

$$g_{ab} = \eta_{ab} + h_{ab}.$$

  The background solution cannot be assumed to be stable if the linear perturbation becomes dominant at large times $t > 0$.
- Runaway solutions might invalidate the research of **global solutions**, which should describe the evolution of the early Universe at large times.
- Investigate the issue of stability using a **semiclassical toy model**.

Semiclassical toy model

- **Quantum massive scalar field** ϕ + **classical scalar field** ψ in flat spacetime

$$\begin{cases} \Box\phi - m^2\phi = \lambda\psi\phi, & \lambda \in \mathbb{R} \\ g_2\Box\psi - g_1\psi = \lambda_1 \left\langle :\phi^2: \right\rangle_\omega - \lambda_2\Box \left\langle :\phi^2: \right\rangle_\omega, & \lambda_1, \lambda_2, g_1, g_2 \in \mathbb{R} \end{cases}$$

- **Linearization**: $\psi = \psi_0 + \psi_1$.
    1. Quantization of ϕ is performed "on the **background field**" ψ_0.
    2. Formulate an interacting theory for the **classical perturbation** ψ_1.
    3. To simplify the analysis, choose $\psi_0 \in \mathbb{R}$ and the *Minkowski vacuum state*.

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- **Linearization**: $\psi = \psi_0 + \psi_1$.
  1. Quantization of ϕ is performed "on the **background field**" ψ_0.
  2. Formulate an interacting theory for the **classical perturbation** ψ_1.
  3. To simplify the analysis, choose $\psi_0 \in \mathbb{R}$ and the *Minkowski vacuum state*.
- The dynamics of ψ_1 is governed by the linearized equation

$$(g_2\Box - g_1)\,\psi_1 = (\lambda_1 - \lambda_2\Box)\left\langle :\phi^2: \right\rangle_0^{(\text{lin})}, \qquad \left\langle :\phi^2: \right\rangle_0^{(\text{lin})} = \hbar\lambda\,\mathcal{K}_a(\psi_1),$$ Fourier

$$\mathcal{K}_a : \mathcal{D}(\mathcal{M}) \to C^\infty(\mathcal{M})$$

$$\mathcal{K}_a(x) = -i\left(\Delta_F^2(x) - \Delta_+^2(x)\right) = (\Box + a)\int_{4m^2}^{\infty} \mathrm{d}M^2 \varrho(M^2)\frac{1}{M^2 + a}\Delta_R(x, M^2),$$

$$\varrho(M^2) = \frac{1}{16\pi^2}\sqrt{1 - \frac{4m^2}{M^2}}, \qquad -4m^2 < a < 0, \qquad (\Box - M^2)\Delta_R(x, M^2) = \delta_x$$

- The constant a encodes the **renormalization freedom** of Δ_F^2 Details

Steps of the work

Study the following **fourth-order differential equation** in ψ_1

$$\hbar\lambda(\lambda_2\Box-\lambda_1)\mathcal{K}_a(\psi_1)+(g_2\Box-g_1)\psi_1 = f, \qquad f \in \mathcal{D}(\mathcal{M}), \qquad \mathcal{K}_a \approx (\Box+a)\Delta_R.$$

1. Show that **past compact solutions** ψ_1 respect causality:
$$\text{supp}(\psi_1) \subset J^+(\text{supp} f).$$
2. Construct the **retarded fundamental solution** $D_R : \mathcal{D}(\mathcal{M}) \to C^\infty(\mathcal{M})$, such that past compact solutions
$$\psi_1 = D_R(f)$$
decay at zero for large $t > 0$.
3. Prove that
$$(g_2\Box - g_1)\,\psi_1 = (\lambda_1 - \lambda_2\Box)\left\langle :\phi^2: \right\rangle_0^{(\text{lin})}$$
$$\updownarrow$$
$$\hbar\lambda(\lambda_2\Box - \lambda_1)\mathcal{K}_a(\psi_1) + (g_2\Box - g_1)\psi_1 = 0$$
has a well-posed **initial-value problem** with initial data $\psi_1^{(0,j)}(0,\mathbf{x})$, with $j \in \{0,1\}$ or $j \in \{0,1,2,3\}$, and for wide ranges of values of $(a, g_1, g_2, \lambda, \lambda_1, \lambda_2)$.

Main Theorem

Consider the semiclassical equation

$$\hbar\lambda(\lambda_2\Box - \lambda_1)\mathcal{K}_a(\psi_1) + (g_2\Box - g_1)\psi_1 = f, \qquad f \in \mathcal{D}(\mathcal{M})$$

and its (formal) Fourier transform

$$S\left(-(p_0 - i0^+)^2 + |\mathbf{p}|^2\right)\hat{\psi}_1(p_0, \mathbf{p}) = \hat{f}(p_0, \mathbf{p}).$$

Fix λ_2 and at least one of the two g_i as non-vanishing constants, assume that the inequality $g_2\lambda_1 - \lambda_2 g_1 \geq 0$ holds, and set $-4m^2 < a < 0$. If $\mathcal{S} = \{z | S(z) = 0\}$ contains only real negative elements s (suff. cond. $\lambda_2 g_2 > 0$), then the Fourier transform of the retarded fundamental solution D_R reads

$$\hat{D}_R(p_0, \mathbf{p}) = \frac{1}{S(-(p_0 - i0^+)^2 + |\mathbf{p}|^2)},$$

and hence

$$D_R(x) = -\sum_{s\in\mathcal{S}} \frac{1}{S'(s)}\Delta_R(x, s) - \frac{\lambda\hbar}{16\pi^2}\int_{4m^2}^{\infty}\sqrt{1 - \frac{4m^2}{M^2}}\frac{(\lambda_2 M^2 - \lambda_1)}{|S(-M)|^2}\Delta_R(x, M^2)\mathrm{d}M^2,$$

for $s \in (-4m^2, \infty) \cup \{-\lambda_1/\lambda_2\}$ in $\mathcal{S}$. Also, $D_R : \mathcal{D}(\mathcal{M}) \to C^\infty(\mathcal{M})$ is a linear operator such that a past compact solution

$$\psi_1 = D_R(f)$$

decays as $1/t^{3/2}$ for large t. Details

Homogeneous equation

Theorem

The spatial Fourier transform of a smooth solution $\psi_1(t, \mathbf{x})$ *of*

$$(g_2\Box - g_1)\psi_1 = (\lambda_1 - \lambda_2\Box)\left\langle :\phi^2: \right\rangle_0^{(\mathrm{lin})}$$

reads

$$\hat{\psi}_1(t, \mathbf{p}) = \sum_{s\in\mathcal{S}} \left(C_+^s(\mathbf{p})\mathrm{e}^{+it\sqrt{|\mathbf{p}|^2-s}} + C_-^s(\mathbf{p})\mathrm{e}^{-it\sqrt{|\mathbf{p}|^2-s}} \right),$$

where $\mathcal{S} \subset (-4m^2, \infty) \cup \{-\lambda_1/\lambda_2\} \subset \mathbb{R}$*. If* $\mathcal{S}$ *contains only negative elements* $s < 0$*, then each solution* ψ_1 *is uniquely fixed by the initial values*

$$\psi_1^{(j)}(0, \mathbf{x}) = \varphi^j(\mathbf{x}), \qquad j \in \{0, \ldots, 2|\mathcal{S}|\},$$

where $|\mathcal{S}|$ *is the cardinality of* $\mathcal{S}$*, and* $\varphi^j \in C_0^\infty(\mathbb{R}^3)$*. In this case,* $\psi_1(t, \mathbf{x})$ *decays at least as* $1/t^{3/2}$ *at large times* t*.*

- There are **sufficient conditions** on the parameters $(\lambda, \lambda_1, \lambda_2, g_1, g_2)$ such that this Theorem holds ($s < 0$), with $g_2\lambda_1 - \lambda_2 g_1 \geq 0$;
- **Number of initial data** (two/four) depends on the number of negative distinct solutions (one/two).

The analogy with the cosmological problem

From the toy model to the linearized Semiclassical Einstein Equations

- **Formal correspondence**

$$-R + 4\Lambda = 8\pi G \left\langle :T: \right\rangle_\omega \quad \leftrightarrow \quad (g_2\Box - g_1)\psi_1 = (\lambda_1 - \lambda_2\Box)\left\langle :\phi^2: \right\rangle_0^{(\mathrm{lin})}.$$

viewing R as the perturbative external field ψ_1 around a spacetime with vanishing curvature $\psi_0 = 0$ (Minkowski spacetime)

$$\left\langle :T: \right\rangle_\omega = \left(3\left(\xi - \frac{1}{6}\right)\Box - m^2\right)\left\langle :\phi^2: \right\rangle_\omega + \left\langle :T: \right\rangle_\omega^{(\mathrm{an})} + c_1 m^4 + c_2 m^2 R + \gamma\Box R.$$

- Fixing $\Lambda = 0$, neglecting $\left\langle :T: \right\rangle_\omega^{(\mathrm{an})}$ as quadratic contribution in R, and setting

$$g_1 = -\frac{1}{8\pi G} = -\frac{m_P^2}{8\pi}, \qquad g_2 = \alpha_3, \qquad \lambda = \xi, \qquad \lambda_1 = m^2, \qquad \lambda_2 = 3\left(\xi - \frac{1}{6}\right).$$

$$g_2\lambda_1 - \lambda_2 g_1 \geq 0 \quad \leftrightarrow \quad \alpha_3\frac{m^2}{m_P^2} \geq -\frac{3}{8\pi}\left(\xi - \frac{1}{6}\right)$$

- **Guessing stability**: results are the same as in the toy model for

$$\xi > 1/6, \qquad \alpha_3 > 0, \qquad a > -4m^2, \qquad \text{sufficiently large } m^2$$

- The linearized model with source corresponds to include a classical source incorporating fluctuations (**Einstein-Langevin equations**)

Current work

Linear Stability of Minkowski spacetime

- **Backreaction** of a **massive** quantum scalar field ϕ, with $m^2 > 0$, $0 \leq \xi \leq 1/6$, over Minkowski spacetime $(\mathcal{M}, \eta)$
- **Steps of the work:**
  1. Show that $(\mathcal{M}, \eta)$ is solution of the **zeroth-order Semiclassical Einstein Equations** using the Minkowski vacuum state ω_0
  $$G^{(0)}_{ab}[\eta] = 8\pi G \, \langle :T_{ab}[\phi, \eta]: \rangle^{(0)}_{\omega} \tag{0}$$
  2. Study **linear stability** of Minkowski spacetime against linear perturbations h_{ab} using the **linearized Semiclassical Einstein Equations**
  $$G^{(1)}_{ab}[\eta, h] = 8\pi G \, \langle :T_{ab}[\phi, \eta, h]: \rangle^{(1)}_{\omega} \,, \tag{1}$$
  where $\langle :T_{ab}[\phi, \eta, h]: \rangle^{(1)}_{\omega}$ is obtained by perturbation theory.
  3. Show that **classical gravitational waves** in the radiative gauge ($\eta^{ab} h_{ab} = 0$) are the unique solutions of Eq. (1)
  4. There exist several choices of the **renormalization constants** of the model such that runaway solutions are ruled out.
- **The paper** containing the proof is now in preparation and should appear within 1-2 months... **Stay tuned!**

Summary

- Local existence of semiclassical solutions is established by using Banach fixed point theorem.
- An inversion procedure is crucial to prove existence and uniqueness.
- Linear stability holds for several choices of the renormalization constants.

Work in Progress and Future Outlooks

- Linear Stability for different choices of reference state (e.g., thermal states).
- Studying stability in other class of spacetimes (De Sitter, FLRW, etc.).
- Formulation of the theory of cosmological perturbations in Semiclassical Gravity.
- ...

Thanks a lot for the attention!

QFT in curved spacetimes (1/2)

Quantization of a field theory in flat spacetime is based on the choice of a **Fock space** built over a **vacuum** as unique Lorentz invariant state.

How to quantize on curved spacetimes?

- No preferred **vacuum state**
- No **symmetries**
- Infinite **inequivalent representations**

$\Rightarrow$

Algebra of Observables

- $\mathcal{A}(\mathcal{M})$: unital $*$-algebra of **observables**.
- **Generators**: smeared quantum fields $\phi(f), f \in \mathcal{D}(\mathcal{M})$,
$$\phi^*(f) = \phi(\bar{f}).$$
- **States** $\langle a \rangle_\omega : \mathcal{A} \to \mathbb{C}$ are linear positive normalized functionals.

Quantization procedure

- Assign to each spacetime a *-algebra $\mathcal{M} \mapsto \mathcal{A}(\mathcal{M})$.
- Identify quantum fields as **abstract observables** which can be multiplied with each other, without being represented as operators on a Hilbert space.
- Find a **physical state** ω to get measurements.

QFT in curved spacetimes (2/2)

Free Quantum Klein-Gordon field

- **Linear Klein-Gordon field**

$$\phi(Pf) = 0, \qquad P = -\Box + m^2 + \xi R, \qquad \Box = g_{ab}\nabla^a\nabla^b, \qquad \xi \in \mathbb{R}$$

- In **globally hyperbolic spacetimes**, such as Minkowski, FLRW, etc., there are unique *advanced* Δ_A and *retarded* Δ_R fundamental solutions $P\Delta_{A/R} = \delta$.
- *Local* and *covariant* quantum fields satisfy the **canonical commutation relations** (*CCR algebra*)

$$[\phi(f_1), \phi(f_2)] = i\Delta(f_1, f_2)\mathbb{1}, \qquad \Delta = \Delta_R - \Delta_A \text{ causal propagator}$$

- **Two-point functions** of quasi-free states

$$\omega_2(f_1, f_2) = \langle \phi(f_1)\phi(f_2)\rangle_\omega = \mu(f_1, f_2) + \frac{i}{2}\Delta(f_1, f_2) \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}).$$

- **GNS construction** to represent $\phi(f) \in \mathcal{A}(\mathcal{M})$ as operator over some Hilbert space, and to recover the Fock representation built over ω.
- Quantum scalar fields should enter the **Semiclassical Einstein Equations**

$$G_{ab}[g] = 8\pi G \langle :T_{ab}: \rangle_\omega [\phi, g].$$

- Extending $\mathcal{A}(\mathcal{M})$ to include **quadratic observables** such as $:T_{ab}:$

Expectation values of $:\phi^2:$ and $:T_{00}:$

Point-splitting regularization mode-wise

$$\langle :\phi^2: \rangle_\omega = \frac{1}{(2\pi)^3 a^2} \int_{\mathbb{R}^3} \left(|\zeta_k|^2 - C^{\mathcal{H}}_{\phi^2}(\tau, k) \right) \mathrm{d}\mathbf{k} + \frac{w(\tau)^2}{8\pi^2 a^2} \log\left(\frac{w(\tau_0)}{a(\tau)} \right) - \frac{w(\tau_0)^2}{16\pi^2 a^2} + \alpha_1 m^2 + \alpha_2 R(\tau)$$

$$\langle :T_{00}: \rangle_\omega = \frac{1}{(2\pi)^3 a^4} \int_{\mathbb{R}^3} \left(\frac{|\zeta_k'|^2}{2} + \left(k^2 + a^2 m^2 - (6\xi - 1)\, a^2 H^2 \right) \frac{|\zeta_k|^2}{2} + aH\,(6\xi - 1)\, 2\mathrm{Re}(\overline{\zeta}_k \zeta_k') - C^{\mathcal{H}}_{\varrho}(\tau, k) \right) \mathrm{d}\mathbf{k}$$
$$- \frac{H^4}{960\pi^2} + \left(\xi - \frac{1}{6} \right)^2 \frac{3H^2 R}{8\pi^2} + k_1 m^4 + k_2 m^2 G_{00} + k_3 I_{00}$$

Point-splitting functions

$$C^{\mathcal{H}}_{\phi^2}(\tau, k) \doteq \frac{1}{2k_0} - \frac{V(\tau)}{4k_0^3},$$

$$C^{\mathcal{H}}_{\varrho}(\tau, k) \doteq \frac{k}{2} + \frac{a^2 m^2 - a^2 H^2 (6\xi - 1)}{4k} - \frac{a^4 m^4 + 12\left(\xi - \frac{1}{6}\right) m^2 a^4 H^2 - a^4 \left(\xi - \frac{1}{6}\right)^2 2 I_{00}(\tau)}{16k\left(k^2 + \frac{a^2}{\lambda^2}\right)}$$

References

J. Schlemmer (*PhD Thesis*), A. Degner (*PhD Thesis*), T.P. Hack (*arXiv:1306.3074s*), D. Siemssen (*arXiv:1503.01826*)

Semiclassical toy-model: perturbative approach

- **Perturbation theory**

$$V=\int_{\mathcal{M}}\mathcal{L}_I(x)g(x)\mathrm{d}^4x=-\frac{\lambda}{2}\int_{\mathcal{M}}\phi^2(x)\psi_1(x)g(x)\mathrm{d}^4x,\qquad g\in\mathcal{D}(\mathcal{M}),$$

$$R_V(\phi^2)=S(V)^{-1}T(S(V)\phi^2),\qquad S(V)=T\left(\exp\left(\frac{i}{\hbar}V\right)\right).$$

- The **Bogoliubov map** R_V allows to obtain a perturbative expansion of the interacting ϕ^2 as formal power series in λ

$$\left\langle :\phi^2: \right\rangle_\omega=\omega(R_V(\phi^2))=\left\langle :\phi^2: \right\rangle_\omega^{(\mathrm{bac})}+\left\langle :\phi^2: \right\rangle_\omega^{(\mathrm{lin})}+\dots,$$

$$\left\langle :\phi^2: \right\rangle_\omega^{(\mathrm{bac})}=\omega(\phi^2)\overset{|0\rangle}{=}0,\qquad \left\langle :\phi^2: \right\rangle_\omega^{(\mathrm{lin})}=\frac{i}{\hbar}\left(\omega(T\left(V\phi^2\right))-\omega(V\phi^2)\right).$$

- The state for the interacting theory is constructed as $\omega\circ R_V$ by means of the **free state**, and it is fixed once and forever.
- **Linearized expectation value** of the Wick square in the adiabatic limit ($g=1$)

$$\left\langle :\phi^2: \right\rangle_\omega^{(\mathrm{lin})}(x)=-i\hbar\lambda\int_{\mathcal{M}}\left(\Delta_F^2(y-x)-\Delta_+^2(y-x)\right)\psi_1(y)\mathrm{d}y,$$

where $\Delta_F(y,x)=\hbar^{-1}\left\langle T\left(\phi(y)\phi(x)\right)\right\rangle_0$ and $\Delta_+(y,x)=\hbar^{-1}\left\langle\phi(y)\phi(x)\right\rangle_0$.

- **Hörmander's criterion** for multiplying distributions: given $u, v \in \mathcal{D}'(\mathcal{M}, \mathbb{C})$, if

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) = \{(x, k+p) : (x,k) \in \mathrm{WF}(u), (x,p) \in \mathrm{WF}(v)\}$$

  does not intersect the zero section, then $u \cdot v$ is well-defined in $\mathcal{D}'(\mathcal{M}, \mathbb{C})$

- **Wave Front Sets** of propagators: given $(x_1, k_1) \sim_\gamma (x_2, k_2)$,
  a. $\mathrm{WF}(\Delta_+) = \{(x_1, k_1, x_2, -k_2) \in \left(T^*(\mathcal{M})^2 \setminus \{\mathbf{0}\}\right) : k_1 \rhd 0\}$
  b. $\mathrm{WF}(\Delta_F) = \mathrm{WF}(\delta) \cup \{(x_1, k_1, x_2, -k_2) \in \left(T^*(\mathcal{M})^2 \setminus \{\mathbf{0}\}\right) : k_1 \rhd 0 \text{ if } x_1 \notin J^-(x_2), \text{ and } k_1 \lhd 0 \text{ if } x_1 \in J^-(x_2)\}$.

- **Epstein-Glaser renormalization**: extending time-ordered products to the diagonal

- **Steinmann's scaling degree**: for $u \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$,
  $\mathrm{sd}(u) \doteq \inf\{\sigma \in \mathbb{R} : \lim_{\lambda \to 0^+} \lambda^\sigma u(f_\lambda) = 0\}$.
  a. If $\mathrm{sd}(u) < d$, then the extension $u_e \in \mathcal{D}'(\mathbb{R}^d)$ is unique
  b. If $d \leq \mathrm{sd}(u) < \infty$, then

$$\tilde{u}_e = u_e + \sum_{|\alpha| \leq \mathrm{sd}(u) - d} c_\alpha \partial^\alpha \delta_x, \qquad u_e, \tilde{u}_e \in \mathcal{D}'(\mathbb{R}^d).$$

  c. If $\mathrm{sd}(u) = \infty$, then u is not extensible.

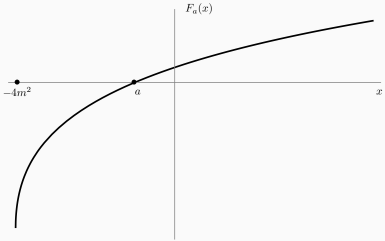
- $\mathrm{sd}(\Delta_F) = 2$, then $\mathrm{sd}(\Delta_F^2) = 4$, and hence $\tilde{\Delta}_F^2 = \Delta_F^2 + c\delta_x$

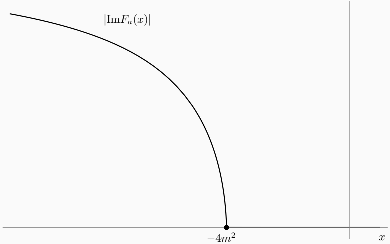
Fourier transform of the Wick square

$$\mathcal{F}\{\langle :\phi^2: \rangle_0^{(\text{lin})}\}(p_0, \mathbf{p}) = \lim_{\epsilon \to 0^+} \frac{\lambda \hbar}{16\pi^2} F_a(-(p_0 - i\epsilon)^2 + |\mathbf{p}|^2)\hat{\psi}_1(p_0, \mathbf{p}),$$

$$F_a(z) = \int_{4m^2}^{\infty} \sqrt{1 - \frac{4m^2}{M^2}} \left(\frac{1}{M^2 + a} - \frac{1}{M^2 + z} \right) \mathrm{d}M^2 \qquad z = -(p_0 - i\epsilon)^2 + |\mathbf{p}|^2.$$

$F_a(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, -4m^2]$, and has a branch cut on $z \in (-\infty, -4m^2)$.





In the massless case [G. T. Horowitz 1980]

$$F_a(-p_0^2 + |\mathbf{p}|^2) = \log\left(\frac{-p_0^2 + |\mathbf{p}|^2}{a} \right), \qquad -p_0^2 + |\mathbf{p}|^2 > 0, a > 0.$$

Nature of past compact solutions

- **Decomposition** of a past compact solution $\psi_1 = D_R(f)$, $f \in \mathcal{D}(\mathcal{M})$.

$$D_R(x) = -\sum_{s\in\mathcal{S}} \frac{1}{S'(s)}\Delta_R(x,s) - \frac{\lambda\hbar}{16\pi^2}\int_{4m^2}^{\infty}\sqrt{1-\frac{4m^2}{M^2}}\frac{(\lambda_2 M^2-\lambda_1)}{|S(-M)|^2}\Delta_R(x,M^2)\mathrm{d}M^2,$$

$$\updownarrow$$

$$\psi_1(x) = \psi_1^O(x) + \psi_1^C(x).$$

- Unlike $\psi_1^O(x)$, $\psi_1^C(x)$ **cannot** be determined by a finite number of initial conditions, because the integration in M^2 is over uncountably many points.
- However, the kernel of the operator

$$T(z) = \frac{S(z)}{\prod_{s\in\mathcal{S}}(z-s)}, \qquad z = -(p_0 - i\epsilon)^2 + |\mathbf{p}|^2$$

  does not contain non-vanishing elements, then $T(z)$ can be **inverted**, and hence it disappears from the homogeneous equation $S(z)\hat{\psi}_1 = 0$.
- $T(z)$ should be related to the **unbounded operator** $\mathcal{T}[f]$ seen in the local case!
- Unlike **branch cuts**, only the contributions due to the **poles** can give origin to non trivial solutions of $S(z)\hat{\psi}_1 = 0$.