Automatic computation of Soft anomalous dimension matrices for the Heavy Quark Hadroproduction

Based on hep-ph/2206.10977 + work in progress

August 18, 2023

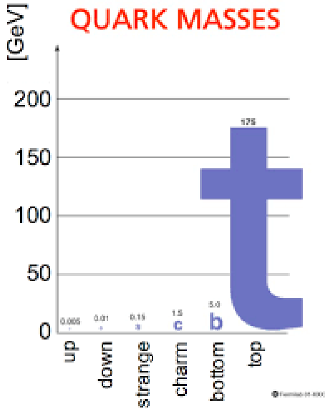
Bakar Chargeishvili, Maria Vittoria Garzelli, Sven-Olaf Moch

II. Institut für Theoretische Physik
Universität Hamburg

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Top quarks are special

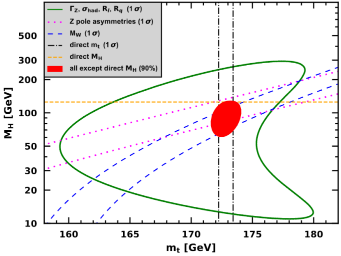
- Pointlike particle with mass of gold atom, 35x heavier than b-quark $\rightarrow$ Why?
- Current mass estimate (pole mass, from cross-section measurement):
  - $m_{pole} = 173.1 \pm 0.9$ GeV
- Participates in all type of interactions and has a strongest coupling with the Higgs boson.
- Higgs vev: $v/\sqrt{2} \simeq 175$ GeV $\rightarrow$ A role in EW symmetry breaking or coincidence?
- Top = the only "free" quark $\rightarrow$ no bound states.
  - Spin/polarization passed on decay products without dilution $\rightarrow$ direct access to quark properties
- Events containing top quarks are backgrounds to new physics searches



Having precise measurement of the top quark properties is important!

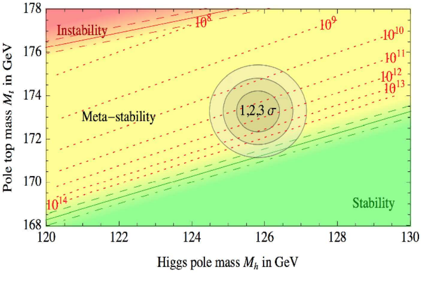
Motivation

- The mass of the top is an important input to the global electroweak fits, which assess the self-consistancy within SM.



[PDG 2022]

- Top-quark mass is related together with the Higgs-boson mass to the vacuum stability of the SM.

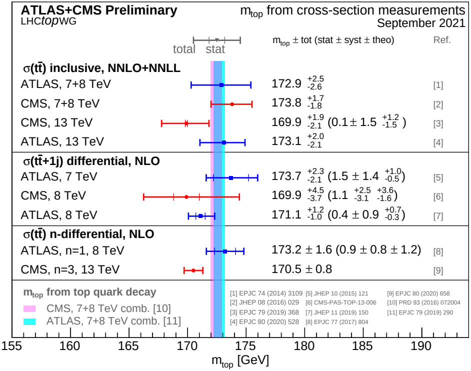


[Alekhin, Djouadi, Moch, B716 (2012) 214]

Precise value of the top quark mass might determine the fate of the universe!

Top quark mass measurement

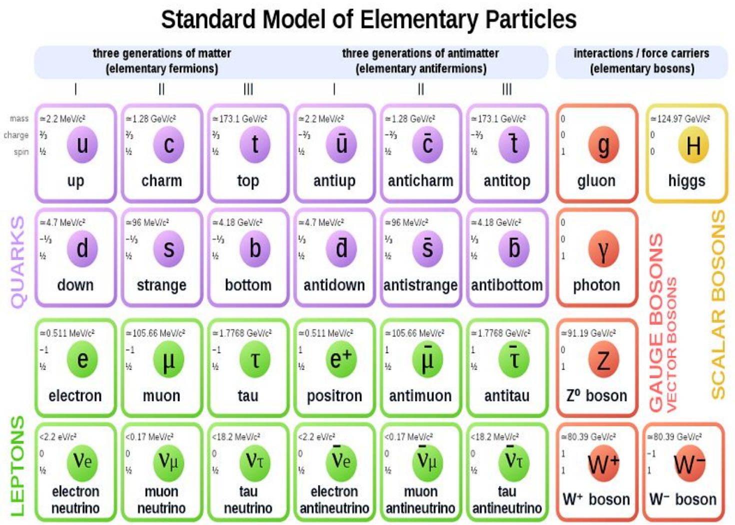
The indirect top-quark mass measurement are becoming a clear competitor of the direct methods.



[ATL-PHYS-PUB-2022-016]

- It has been shown that in case of tt+jet, the observable $\rho_s = 2m_t/\sqrt{s_{t\bar{t}j}}$ shows very good sensitivity on the top quark mass [hep-ph/1303.6415]
- Latest studies using tt+jet cross-section use NLO (NLO+PS) predictions for the mass determination
- Elevating the accuracy of the theory predictions beyond NLO would improve the current m_t estimates

Standard Model of Elementary Particles



- Lagrangian of QCD:

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f}^{N_f} \overline{\psi}_f \left(i\gamma_\mu D^\mu - m_f\right) \psi_f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$
$$- \frac{1}{2\xi} \left(\partial^\mu A^a_\mu\right)\left(\partial^\nu A^a_\nu\right) + g f^{abc} \bar{\chi}^a \partial^\mu \left(A^c_\mu \chi^b\right) - \bar{\chi}^a \partial^\mu \partial_\mu \chi^a,$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

and

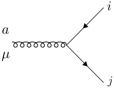
$$D_\mu = \partial_\mu - i g A^a_\mu t^a$$

- Generators t^a form $SU(3)$ algebra:

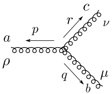
$$t^a \equiv \frac{\lambda^a}{2}, \qquad \mathrm{Tr}\left[t^a \cdot t^b\right] = T_F \delta^{ab}, \qquad [t^a, t^b] = i f^{abc} t^c.$$

Feynman rules of QCD

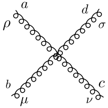
Vertex factors:



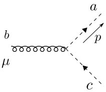
$$ig_s\gamma_\mu t^a_{ji}$$



$$-g_s f^{abc}\left[(p-q)_\nu g_{\rho\mu} + (q-r)_\rho g_{\mu\nu} + (r-p)_\mu g_{\nu\rho}\right]$$

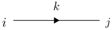


$$\begin{aligned}&-ig_s^2 f^{abe} f^{cde}\left(g_{\rho\nu}g_{\mu\sigma} - g_{\rho\sigma}g_{\mu\nu}\right)\\&-ig_s^2 f^{ace} f^{bde}\left(g_{\rho\mu}g_{\nu\sigma} - g_{\rho\sigma}g_{\mu\nu}\right)\\&-ig_s^2 f^{ade} f^{cbe}\left(g_{\rho\nu}g_{\mu\sigma} - g_{\rho\mu}g_{\sigma\nu}\right)\end{aligned}$$

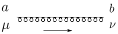


$$g_s f^{abc} p_\mu$$

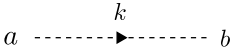
Propagators:



$$i\delta_{ij}\frac{(\not{k}+m)}{k^2-m^2+i\epsilon}$$



$$\frac{-i\delta_{ab}}{k^2+i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right] \qquad \xi = \begin{cases}1, & \text{Feynman gauge}\\ 0, & \text{Landau gauge}\end{cases}$$



$$\frac{-i\delta_{ab}}{k^2+i\epsilon}$$

Singularities in QCD

In QCD one encounters two type of singularities:

- One is coming from the **low energy** limit and are called **infrared (IR)** divergences.
- The other source is the **high energy** limit, designated as **ultraviolet (UV)** divergences.

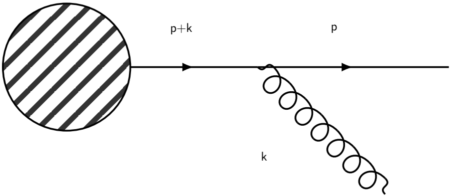
Solution for UV singularities $\rightarrow$ Regularization/Renormalization

- Dimensional regularization:

$$g \rightarrow \mu^{(4-D)/2} g, \qquad \text{where} \quad D = 4 - 2\epsilon$$

1. If $\epsilon < 0$ the expression diverges, then it has an UV divergence
2. If $\epsilon > 0$ the expression diverges, then it has an IR divergence

- Renormalization: All the UV divergences can be absorbed in the definitions of the coupling constant, field strength and mass terms.



Gluon emission

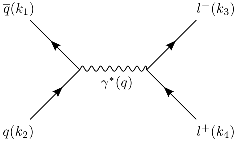
The corresponding propagator is given as:

$$\frac{1}{(p+k)^2} = \frac{1}{2pk} = \frac{1}{2E_q E_g (1-\cos\theta)}$$

- If gluon is soft: $E_g \to 0$ one gets soft singularity, manifested as $1/\epsilon$ pole
- $\cos\theta \to 1$ collinear singularity, manifested as another $1/\epsilon$ pole.
- Soft emission might be collinear as well, manifested as $1/\epsilon^2$ pole.

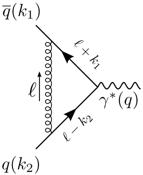
Case study: Drell-Yan production

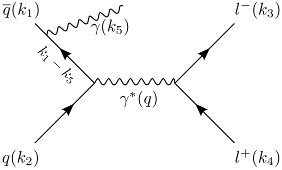
Tree level

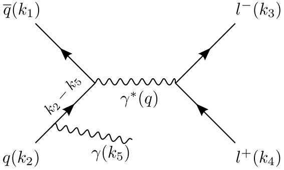


$$\sigma_{\mathrm{B}} = \frac{4\pi Q_f^2 \alpha_s^2}{3 N_{\mathrm{C}} s}$$

NLO







$$\frac{\mathrm{d}\sigma_V^{(1)}}{\mathrm{d}q^2} = C_\mathrm{F}\frac{\alpha_s}{2\pi}\frac{1}{s}\sigma^{(0)}\left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\times \left[-\frac{2}{\epsilon_\mathrm{IR}^2}-\frac{3}{\epsilon_\mathrm{IR}}-8+\frac{2\pi^2}{3}+\mathcal{O}(\epsilon)\right]\delta(1-z)$$

$$\frac{d\sigma_R^{(1)}}{dq^2} = C_F\frac{\alpha_s}{2\pi}\frac{1}{s}\sigma^{(0)}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon \times \left(\frac{2}{\epsilon_\mathrm{IR}^2}\delta(1-z)-\frac{2}{\epsilon_\mathrm{IR}}\frac{1+z^2}{(1-z)_+}+4\left(1+z^2\right)\left[\frac{\ln(1-z)}{1-z}\right]_+-2\frac{1+z^2}{1-z}\ln z+\mathcal{O}(\epsilon)\right)$$

$$\frac{\mathrm{d}\sigma_\mathrm{DY}^\mathrm{NLO}}{\mathrm{d}q^2} = \frac{\mathrm{d}\sigma_V^{(1)}}{\mathrm{d}q^2}+\frac{\mathrm{d}\sigma_R^{(1)}}{\mathrm{d}q^2} = \frac{\alpha_s}{2\pi}\frac{1}{s}\sigma^{(0)}\omega_{qq}(z)-\frac{2}{\epsilon}\frac{\alpha_s}{2\pi}\frac{1}{s}\sigma^{(0)}P_{qq}^{(1)}(z)\left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}+\mathcal{O}(\epsilon)$$

where $\omega_{qq}(z)$ is defined as:

$$\omega_{qq}(z) = C_\mathrm{F}\left\{\left[\frac{2}{3}\pi^2-8\right]\delta(1-z)-2\frac{1+z^2}{1-z}\ln z+4(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+\right\}$$

Challenges of the NNLO calculation

Fully accurate NNLO computation for $2 \to 3$ process with partons present in both initial and final states imposes a serious technical challenge:

- 2-loop master integrals for $2 \to 3$ kinematics with massive partons are mostly unknown.
- No easy way to deal with soft and collinear singularities at NNLO level. Multiple approaches are under development:
  - Antenna subtraction [Gehrmann et al.]
  - CoLoRFul subtraction [Del Duca et al.]
  - Sector-improved residue subtraction [Czakon et al.]
  - Local analytic sector subtraction [Magnea et al.]
  - qT-slicing [Catani et al.]
  - ...
- Automation of these procedures will take some time...

Meanwhile, it is possible to work in the threshold limit of the $t\bar{t}$+jet production and by summing the large logarithms to all orders of the perturbation theory, estimate the NNLO correction.

The problem of large logarithms

- The perturbative expansion of observable relies on the smallness of α_s and expansion coefficients c_k:

$$\hat{O}(\alpha_s) = \hat{O}_0 \left[1 + \frac{\alpha_s}{2\pi} c_1 + \left(\frac{\alpha_s}{2\pi}\right)^2 c_2 + \dots\right]$$

- At higher orders the cross-section is given as sum of virtual and real contributions
- In dimensional regularization $(D = 4 - \epsilon, \epsilon < 0)$ one finds cancellation of the form

$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{\left(Q^2\right)^{\epsilon} \int_0^{m_{\text{jet}}^2} \frac{dk^2}{\left(k^2\right)^{1+\epsilon}}}_{\text{real}} \implies \ln\left(m_{\text{jet}}^2 / Q^2\right)$$

- When $m_{\text{jet}}^2 \ll Q^2$ these logs can become quite large
- Parametrized using the threshold variable $z = \frac{M}{\hat{s}}$, at the n-th order of the perturbation theory there are terms proportional to:

$$-\frac{\alpha_s^n}{n!} \left[\frac{\ln^m (1 - z)^{-1}}{1 - z}\right]_+, \qquad m \leq 2n - 1$$

- Action of $-\frac{\alpha_s^n}{n!}\left[\frac{\ln^m(1-z)}{1-z}\right]_+$ on a smooth function $\mathcal{F}$:

$$-\frac{\alpha_s^n}{n!}\int_0^1 dz\,\frac{\mathcal{F}(z)-\mathcal{F}(1)}{1-z}\ln^{2n-1}\left((1-z)^{-1}\right) = \frac{\alpha_s^n}{n!}\int_0^1 dz\,\mathcal{F}'(1)\ln^{2n-1}\left((1-z)^{-1}\right)+\dots$$
$$\sim \frac{\alpha_s^n}{n!}(2n-1)!+\dots$$

- At n-th order these contributions grow faster than $n!$ and spoil the convergence of the perturbative series

- Solution $\Longrightarrow$ Resummation

- Sum up these logs order-by-order and rearrange them into exponentials:

$$\frac{\sigma^N}{\sigma_B^N} = 1 + \sum_{n=1}^{\infty} \alpha_s^n \sum_{m=0}^{2n} \tilde{c}_{n,m} L^m = \exp\left\{ \sum_{n=1}^{\infty} \alpha_s^n \sum_{m=1}^{n+1} c_{n,m} L^m \right\} \underbrace{C(\alpha_s)}_{\text{constants}}$$

$$= \exp\left[\mathrm{LL} + \mathrm{NLL} + \mathrm{NNLL} + \ldots\right] C(\alpha_s)$$

	LL	NLL	NNLL	...
LO	1			
NLO	$\alpha_s L^2$	$\alpha_s L$		
NNLO	$\alpha_s^2 L^4$	$\alpha_s^2 L^{\{3,2\}}$	$\alpha_s^2 L$	
$\vdots$	...	...	...	$\ddots$

- In the threshold limit the partonic cross-section can be factorized as [Collins, Soper, Sterman, 1983]:

$$\hat{\sigma} = \psi_i \otimes \psi_j \otimes H \otimes S \otimes J$$

- ψ_i - initial state jet functions, modeling the initial state collinear radiation
- S - soft gluon exchange
- H - hard amplitude matrix
- J - final state jet function

General strategy:

- Evaluate each of these integrals perturbatively at scales at which they are free of large logs
- By means of the RGEs evolve everything back to the common scale
- Expand the result to the desired fixed order
- The strategy has been successfully applied on $t\bar{t}$ production processes with associated bosons: $t\bar{t} + W/Z/H$ [Kulesza et al.], [Broggio et al.].

In case of $t\bar{t}$+jet complications come from the:

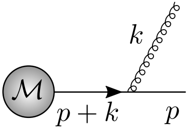
(i) Involved calculations because of richer color structure

(ii) Appearance of the final state jet, which has highly nontrivial soft singularity structure

Currently we focus on calculation of the soft function.

Factorization of soft gluons

- Soft emissions factorize:

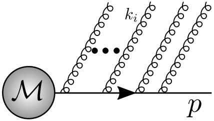


$$= \mathcal{M}\frac{\not{p}+\not{k}}{(p+k)^2}\gamma^\mu T^a u(p) = \mathcal{M}\frac{\not{p}+\not{k}}{2p\cdot k}\gamma^\mu T^a u(p) \overset{k\ll p}{\simeq} \mathcal{M}\frac{p^\mu}{p\cdot k}T^a u(p)$$

Emission of multiple gluons:

- Define a Wilson line:

$$\Phi_\beta^{(f)}(\lambda_2,\lambda_1;x) = \mathcal{P}\exp\left[-ig\mu^\epsilon\int_{\lambda_1}^{\lambda_2} d\eta\beta\cdot A^{(f)}(\eta\beta+x)\right]$$



$$\sim \mathcal{M}\langle 0|\Phi_\beta(\infty,0)|0\rangle u(p)$$

$$\Phi_\beta(\infty,0) = 1 + g\mu^\epsilon\int\frac{d^Dk}{(2\pi)^D}\frac{\beta^\mu}{k\cdot\beta}\tilde{A}_\mu(k) + \frac{g^2\mu^{2\epsilon}}{2}\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{\beta^\mu}{k_1\cdot\beta}\frac{\beta^\nu}{k_2\cdot v}\tilde{A}_\mu(k_1)\tilde{A}_\nu(k_2)+\dots$$

Soft function

- For the process $a(\beta_a)b(\beta_b) \to 1(\beta_1)2(\beta_2)3(\beta_3)$ define a eikonal nonlocal operator ω_I:

$$\omega_I^{(f)}(x)_{\{c_k\}} = \sum_{d_i} \Phi_{\beta_3}^{f_3}(\infty, 0; x)_{c_3,d_3} \Phi_{\beta_2}^{f_2}(\infty, 0; x)_{c_2,d_2} \Phi_{\beta_1}^{f_1}(\infty, 0; x)_{c_1,d_1} \left(c_I^{(f)}\right)_{d_3 d_2 d_1, d_b d_a}$$
$$\times \Phi_{\beta_a}^{f_a}(0, -\infty; x)_{d_a,c_a} \Phi_{\beta_b}^{f_b}(0, -\infty; x)_{d_b,c_b}$$

- Eikonal cross-section is given as:

$$\sigma_{LI}^{(f),\mathrm{eik}}(\alpha_s, \epsilon) = \sum_{\xi} \delta(w - w(\xi)) \times \left\langle 0 \left| \overline{T} \left[\left(\omega_L^{(f)}(0) \right)^{\dagger}_{\{b_i\}} \right] \right| \xi \right\rangle \left\langle \xi \left| T \left[\omega_I^{(f)}(0)_{\{b_i\}} \right] \right| 0 \right\rangle$$

- Define the soft function as the part of this cross-section which is free of collinear divergences

$$\sigma_{LI}^{(f),eik^N} = S_{JI}^N j_a^N j_b^N j_1^N j_2^N j_3^N$$

- Because the soft matrix is defined as a product of two operators it has to be renormalized multiplicatively:

$$S_{LI}^{(f)^{(B)}} = \left(Z_S^{(f)^\dagger}\right)_{LB} S_{BA}^{(f)} \left(Z_S^{(f)}\right)_{AI} \tag{1}$$

- Derive RGE:

$$\mu \frac{d}{d\mu} S_{LI}^{(f)} = \left(\mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s}\right) S_{LI}^{(f)} = -\left(\Gamma_S^{(f)}\right)^\dagger_{LB} S_{BI}^{(f)} - S_{LA}^{(f)} \left(\Gamma_S^{(f)}\right)_{AI}, \tag{2}$$

- where at 1-loop level:

$$\left(\Gamma_S^{(f)}\right)_{LI}(g) = -\alpha_s \frac{\partial}{\partial \alpha_s} \mathrm{Res}_{\epsilon \to 0} \left(Z_S^{(f)}\right)_{LI}(g, \epsilon) \tag{3}$$

- Z_S are UV-divergent parts of the eikonal amplitudes. Solution of eq. (2):

$$S(\mu) = \bar{\mathcal{P}} \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_S^\dagger(\alpha_s(\mu'^2))\right] S(\mu_0) \mathcal{P} \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_S(\alpha_s(\mu'^2))\right] \tag{4}$$

The strategy of choosing the color basis is explained in [Sjödahl 2008], here we quote the result:

gg-channel

$$\mathbf{c}^1_{abcde} = t^e_{\ cd}\delta_{ab}$$
$$\mathbf{c}^2_{abcde} = if_{abe}\delta_{cd}$$
$$\mathbf{c}^3_{abcde} = id_{abe}\delta_{cd}$$
$$\mathbf{c}^4_{abcde} = if_{abn}if_{men}t^m_{\ cd}$$
$$\mathbf{c}^5_{abcde} = d_{abn}if_{men}t^m_{\ cd}$$
$$\mathbf{c}^6_{abcde} = if_{abn}d_{men}t^m_{\ cd}$$
$$\mathbf{c}^7_{abcde} = d_{abn}d_{men}t^m_{\ cd}$$
$$\mathbf{c}^8_{abcde} = P^{10+\overline{10}}_{abme}t^m_{\ cd}$$
$$\mathbf{c}^9_{abcde} = P^{10-\overline{10}}_{abme}t^m_{\ cd}$$
$$\mathbf{c}^{10}_{abcde} = -P^{27}_{abme}t^m_{\ cd}$$
$$\mathbf{c}^{11}_{abcde} = P^{0}_{abme}t^m_{\ cd}$$

$q\bar{q}$-channel

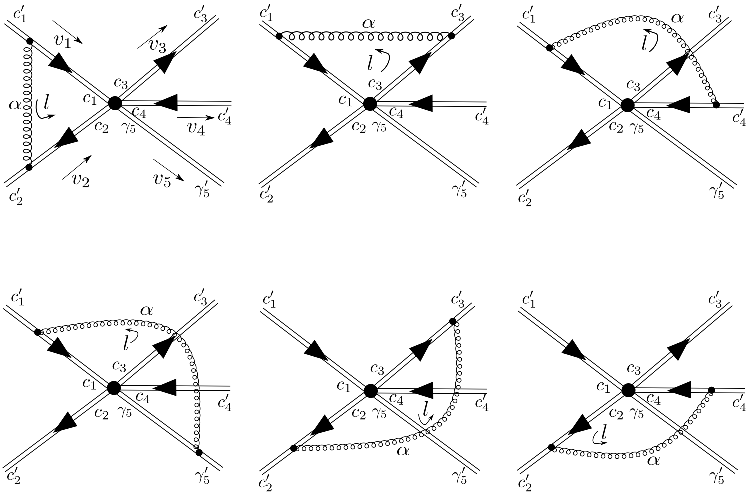
$$\mathbf{c}^1_{abcde} = t^e_{\ cd}\delta_{ab}$$
$$\mathbf{c}^2_{abcde} = t^e_{\ ab}\delta_{cd}$$
$$\mathbf{c}^3_{abcde} = t^m_{\ ba}t^n_{\ cd}if_{mne}$$
$$\mathbf{c}^4_{abcde} = t^m_{\ ba}t^n_{\ cd}d_{mne}$$

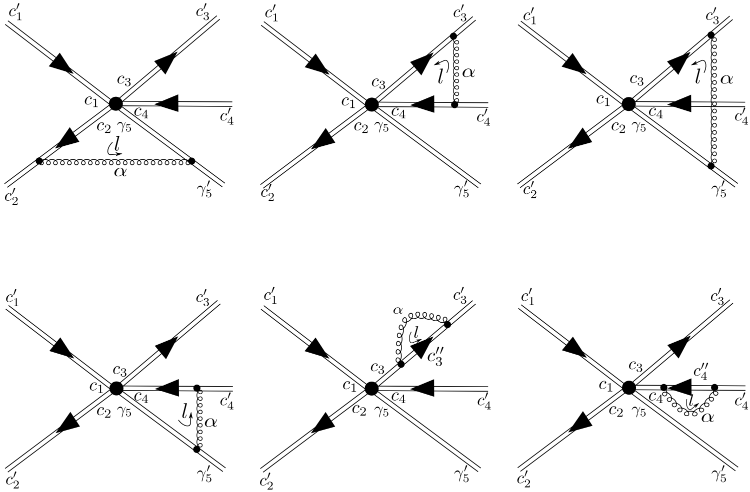
P^i are projectors: $P^i_{ABmn}P^j_{mnCD} = \delta_{ij}P^i_{ABCD}$

Break down of the color structure for $q\bar{q}$

Consider the process $q_a \bar{q}_b \to t_c \bar{t}_d g_e$:

- Initial state singlet $\Rightarrow$ final state is singlet $\Rightarrow$ gluon octet must matched by $t\bar{t}$ octet $\Rightarrow \delta_{ab} t^e_{cd}$
- Initial state octet $\Rightarrow$ final state is octet $\Rightarrow$
  - gluon matches the intial octet $\Rightarrow t\bar{t}$ is singlet $\Rightarrow t^e_{ab}\delta_{cd}$
  - if $t\bar{t}$ is octet $\Rightarrow 8 \otimes 8$ contains 2 octets to match the inital octet:
    - 8^a: $t^m_{\ ba} t^n_{\ cd} i f_{mne}$
    - 8^s: $t^m_{\ ba} t^n_{\ cd} d_{mne}$


c_1' c_3' v_1 v_3 α l c_1 c_3 c_2 c_4 γ_5 c_4' v_4 v_2 v_5 c_2' γ_5'
c_1' α c_3' l c_3 c_1 c_4 c_2 γ_5 c_4' c_2' γ_5'
c_1' α c_3' l c_3 c_1 c_4 c_2 γ_5 c_4' c_2' γ_5'
c_1' α c_3' l c_3 c_1 c_4 c_2 γ_5 c_4' c_2' γ_5'
c_1' c_3' c_3 c_1 c_4 c_2 γ_5 l c_4' α c_2' γ_5'
c_1' c_3' c_3 c_1 c_4 c_2 γ_5 c_4' l α c_2' γ_5'


c_1'
c_3'
c_3
c_1
c_4
c_4'
c_2 γ_5
l
α
c_2'
γ_5'
c_3''
c_4''

Eikonal amplitudes

Connection $(i-j)$	Kinematical part (κ_{ij}) before integration	Color part $(\mathscr{F}_{ij})$
$1-2$	$\frac{v_1^\mu}{-v_1\cdot l+i\epsilon}\,\frac{-v_2^\nu}{v_2\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_1c_1'}T^\alpha_{c_2'c_2}\delta_{c_3c_3'}\delta_{c_4c_4'}\delta_{\gamma_5\gamma_5'}$
$1-3$	$\frac{v_1^\mu}{v_1\cdot l+i\epsilon}\,\frac{v_3^\nu}{v_3\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_1c_1'}T^\alpha_{c_3'c_3}\delta_{c_2c_2'}\delta_{c_4c_4'}\delta_{\gamma_5\gamma_5'}$
$1-4$	$\frac{v_1^\mu}{v_1\cdot l+i\epsilon}\,\frac{-v_4^\nu}{v_4\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_1c_1'}T^\alpha_{c_4c_4'}\delta_{c_2c_2'}\delta_{c_3c_3'}\delta_{\gamma_5\gamma_5'}$
$1-5$	$\frac{v_1^\mu}{v_1\cdot l+i\epsilon}\,\frac{-v_5^\nu}{v_5\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_1c_1'}(-if^{\alpha\gamma_5\gamma_5'})\delta_{c_2c_2'}\delta_{c_3c_3'}\delta_{c_4c_4'}$
$2-3$	$\frac{-v_2^\mu}{-v_2\cdot l+i\epsilon}\,\frac{v_3^\nu}{-v_3\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_2'c_2}T^\alpha_{c_3'c_3}\delta_{c_1c_1'}\delta_{c_4c_4'}\delta_{\gamma_5\gamma_5'}$
$2-4$	$\frac{-v_2^\mu}{-v_2\cdot l+i\epsilon}\,\frac{-v_4^\nu}{-v_4\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_2'c_2}T^\alpha_{c_4c_4'}\delta_{c_1c_1'}\delta_{c_3c_3'}\delta_{\gamma_5\gamma_5'}$
$2-5$	$\frac{-v_2^\mu}{-v_2\cdot l+i\epsilon}\,\frac{v_5^\nu}{-v_5\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_2'c_2}(-if^{\alpha\gamma_5'\gamma_5})\delta_{c_1c_1'}\delta_{c_3c_3'}\delta_{c_4'c_4}$
$3-4$	$\frac{v_3^\mu}{-v_3\cdot l+i\epsilon}\,\frac{-v_4^\nu}{v_4\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_3'c_3}T^\alpha_{c_4c_4'}\delta_{c_1c_1'}\delta_{c_2c_2'}\delta_{\gamma_5\gamma_5'}$
$3-5$	$\frac{v_3^\mu}{-v_3\cdot l+i\epsilon}\,\frac{-v_5^\nu}{v_5\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_3'c_3}(-if^{\alpha\gamma_5\gamma_5'})\delta_{c_1c_1'}\delta_{c_2c_2'}\delta_{c_4c_4'}$
$4-5$	$\frac{v_4^\mu}{-v_4\cdot l+i\epsilon}\,\frac{-v_5^\nu}{v_5\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_4c_4'}(-if^{\alpha\gamma_5\gamma_5'})\delta_{\gamma_1\gamma_1'}\delta_{\gamma_2\gamma_2'}\delta_{c_3c_3'}$
$3-3$	$\frac{v_3^\mu}{v_3\cdot l+i\epsilon}\,\frac{v_3^\nu}{-v_3\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_3''c_3}T^\alpha_{c_3'c_3''}\delta_{c_1c_1'}\delta_{c_2c_2'}\delta_{c_4c_4'}\delta_{\gamma_5\gamma_5'}$
$4-4$	$\frac{-v_4^\mu}{v_4\cdot l+i\epsilon}\,\frac{-v_4^\nu}{-v_4\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_4''c_4}T^\alpha_{c_4'c_4''}\delta_{c_1c_1'}\delta_{c_2c_2'}\delta_{c_3c_3'}\delta_{\gamma_5\gamma_5'}$

The gluon propagator in a general axial gauge is given as:

$$N^{\mu\nu}(k) = g^{\mu\nu} - \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k} + n^2 \frac{k^\mu k^\nu}{(n \cdot k)^2} \tag{5}$$

To deal with the unphysical singularity introduced by the axial gauge we use the principal value prescription [Leibbrandt 1987]:

$$\frac{\mathcal{P}}{(l \cdot n)^\beta} = \frac{1}{2}\left(\frac{1}{(l \cdot n + i\epsilon)} + (-1)^\beta \frac{1}{(-l \cdot n + i\epsilon)}\right) \tag{6}$$

As a result, each integral over the kinematical part can be reduced to the following form:

$$\omega_{ij}(\delta_i v_i, \delta_j v_j, \Delta_i, \Delta_j) = \Delta_i \Delta_j \delta_i \delta_j \Bigg(I_1(\delta_i v_i, \delta_j v_j) - \frac{1}{2} I_2(\delta_i v_i, n) - \frac{1}{2} I_2(\delta_i v_i, -n) - \frac{1}{2} I_3(\delta_j v_j, n) - \frac{1}{2} I_3(\delta_j v_j, -n) + I_4(n^2)\Bigg) \tag{7}$$

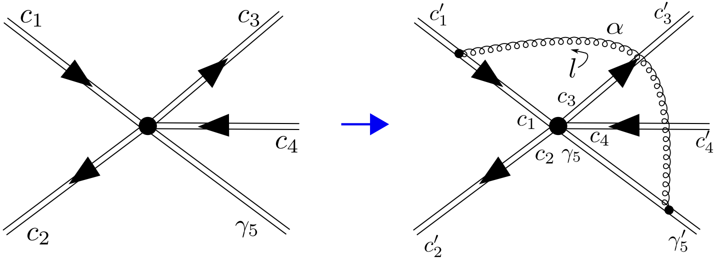
The integrals $I_1 - I_4$ are evaluated in [Kidonakis 1997].
For example, when both partons are massless:

$$I_1^{\text{UV pole}} = \frac{\alpha}{2\pi}\left\{\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left[\gamma + \ln\left(\delta_i\delta_j\frac{v_{ij}}{2}\right) - \ln(4\pi)\right]\right\}$$
$$I_2^{\text{UV pole}} = \frac{\alpha}{2\pi}\left\{\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left[\gamma + \ln(\nu_i) - \ln(4\pi)\right]\right\}$$
$$I_3^{\text{UV pole}} = \frac{\alpha}{2\pi}\left\{\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left[\gamma + \ln(\nu_j) - \ln(4\pi)\right]\right\}$$
$$I_4^{\text{UV pole}} = -\frac{\alpha}{\pi}\frac{1}{\epsilon}.$$

where

$$\nu_a = \frac{(\nu_a \cdot n)^2}{|n|^2} \tag{8}$$

and $v_{ij} = v_i \cdot v_j = \frac{2p_ip_j}{s}$.



Before the gluon exchange:

$$\mathbf{c}_1 = \delta_{c_1 c_2} T^{\gamma_5}_{c_3 c_4}$$
$$\mathbf{c}_2 = \delta_{c_3 c_4} T^{\gamma_5}_{c_2 c_1}$$
$$\mathbf{c}_3 = T^{\alpha}_{c_3 c_4} T^{\beta}_{c_3 c_4} i f^{\alpha\beta\gamma_5}$$
$$\mathbf{c}_4 = T^{\alpha}_{c_3 c_4} T^{\beta}_{c_3 c_4} d^{\alpha\beta\gamma_5}$$

After the gluon exchange:

$$\mathbf{c}'_1 = \delta_{c'_1 c'_2} T^{\gamma'_5}_{c'_3 c'_4}$$
$$\mathbf{c}'_2 = \delta_{c'_3 c'_4} T^{\gamma_5}_{c'_2 c'_1}$$
$$\mathbf{c}'_3 = T^{\alpha}_{c'_3 c'_4} T^{\beta}_{c'_3 c'_4} i f^{\alpha\beta\gamma'_5}$$
$$\mathbf{c}'_4 = T^{\alpha}_{c'_3 c'_4} T^{\beta}_{c'_3 c'_4} d^{\alpha\beta\gamma'_5}$$

The vertex correction $\mathscr{F}_{15} = T^{\alpha}_{c_1 c_1'} i f^{\alpha \gamma_5' \gamma_5} \delta_{c_2 c_2'} \delta_{c_3 c_3'} \delta_{c_4 c_4'}$, modifies the Born basis in the following way:

$$\mathbf{c}_1 \mathscr{F}_{15} = -\mathbf{c}_3' \,, \tag{9}$$

$$\mathbf{c}_2 \mathscr{F}_{15} = -\frac{N_c}{2} \mathbf{c}_2' \,, \tag{10}$$

$$\mathbf{c}_3 \mathscr{F}_{15} = -\frac{1}{2} \mathbf{c}_1' - \frac{N_c}{4} \mathbf{c}_3' - \frac{N_c}{4} \mathbf{c}_4' \,, \tag{11}$$

$$\mathbf{c}_4 \mathscr{F}_{15} = \left(\frac{1}{N_c} - \frac{N_c}{4} \right) \mathbf{c}_3' - \frac{N_c}{4} \mathbf{c}_4' \,. \tag{12}$$

The linear transformation, which describes the modification of the Born-color structure caused by the soft gluon exchange between two partons:

$$\mathcal{F}_{15} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{N_c}{2} & 0 & 0 \\ -1 & 0 & -\frac{N_c}{4} & \frac{1}{N_c} - \frac{N_c}{4} \\ 0 & 0 & -\frac{N_c}{4} & -\frac{N_c}{4} \, , \end{pmatrix} \tag{13}$$

Let us define the following tensor:

$$G_{AB} = \mathrm{tr}\left(\mathbf{c}_A \mathbf{c}'^{\dagger}_B \sum_{i,j;i\leq j} \omega_{ij}^{(\mathrm{UV})} \mathscr{F}_{ij}\right). \tag{14}$$

The soft anomalous dimension matrix is then given as:

$$\Gamma_{IJ} = \left(S^{(0)}\right)^{-1}_{IK} G_{KJ}\,, \tag{15}$$

where $S^{(0)}$ is a Born-level soft matrix, defined as:

$$S^{0}_{LI} = \left\langle \mathbf{c}_L \middle| \mathbf{c}_I \right\rangle\,. \tag{16}$$

This procedure is fully automatized:

- Wilson diagrams and corresponding amplitudes are generated using a `C`-package `WilsonWebs`.
- The color decomposition is done using a `FORM` routine `FORMSoft`.

Example results

- Example components from $q\bar{q} \to t\bar{t}g$-channel

$$\Gamma^{(1)}_{1,1} = \frac{1}{2N_c}\left[2L_\beta + N_c^2\left(-2\log(\nu_5) + 2\log(v_{35}) + 2\log(v_{45}) + 2\log\left(\frac{s}{m_t^2}\right) - \log(16) + 2\right)\right.$$
$$\left. + \left(N_c^2 - 1\right)\left(-\log(\nu_1) - \log(\nu_2) + 2i\pi\right) + \log(4)\right]$$

$$\Gamma^{(1)}_{1,2} = \frac{1}{N_c}\left[\log(v_{13}) - \log(v_{14}) - \log(v_{23}) + \log(v_{24})\right]$$

- Example components from $gg \to t\bar{t}g$-channel

$$\Gamma^{(1)}_{2,9} = 0$$

$$\Gamma^{(1)}_{2,10} = \frac{N_c + 3}{4N_c + 4}\left[\log(v_{13}) - \log(v_{14}) - \log(v_{23}) + \log(v_{24})\right]$$

$$\Gamma^{(1)}_{11,8} = \frac{N_c(N_c + 1) - 2}{2N_c}\left[\log(v_{13}) + \log(v_{14}) - 2\log(v_{15}) - \log(v_{23}) - \log(v_{24}) + 2\log(v_{25})\right]$$

$$\Gamma^{(1)}_{11,9} = \frac{N_c(N_c + 1) - 2}{2N_c}\left[-\log(v_{13}) + \log(v_{14}) + \log(v_{23}) - \log(v_{24})\right]$$

The following checks are done:

(i) Correctness of the color decomposition procedure:

- $\mathrm{Tr}\left(H^{(0)} S^{(0)}\right)$ should evaluate back to the squared Born matrix element

(ii) Correctness of the 1-loop pole structure:

- The analytic pole structure at NLO is known, e.g. by evaluating the Catani-Seymour subtraction terms.
- The same structure can be generated using $H^{(0)}$ and $\Gamma^{(1)}$ matrices.

$$V_{\mathrm{poles}} = 2\left\langle M_0 \middle| I^{(1)} \middle| M_0 \right\rangle,$$

where

$$\boldsymbol{I}^{(1)}\left(\epsilon, \mu^2; \{p\}\right) = \frac{1}{2}\frac{e^{-\epsilon\psi(1)}}{\Gamma(1-\epsilon)} \sum_i \frac{1}{\boldsymbol{T}_i^2} \mathcal{V}_i^{\mathrm{sing}}(\epsilon) \sum_{j \neq i} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \left(\frac{\mu^2 e^{-i\lambda_{ij}\pi}}{2p_i \cdot p_j}\right)^{\epsilon}$$

Both of these checks have been successfully passed.

It is possible to perform the same calculation at 2 loop level using the general result from [Becher, Neubert, 2009]:

$$
\begin{aligned}
\mathbf{\Gamma}(\{p\},\{m\},\mu) = & \sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \gamma_{\text{cusp}}\,(\alpha_s) \ln\left(\frac{\mu^2}{-s_{ij}}\right) + \sum_i \gamma^i\,(\alpha_s) \\
& - \sum_{(I,J)} \frac{\mathbf{T}_I \cdot \mathbf{T}_J}{2} \gamma_{\text{cusp}}\,(\beta_{IJ}, \alpha_s) + \sum_I \gamma^I\,(\alpha_s) \\
& + \sum_{(I,j)} \mathbf{T}_I \cdot \mathbf{T}_j \gamma_{\text{cusp}}\,(\alpha_s) \ln\left(\frac{m_I \mu}{-s_{Ij}}\right) \\
& + \sum_{(I,J,K)} \mathrm{i} f^{abc} \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_K^c F_1\,(\beta_{IJ}, \beta_{JK}, \beta_{KI}) \\
& + \sum_{(I,J)} \sum_k \mathrm{i} f^{abc} \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_k^c f_2 \left(\beta_{IJ}, \ln\left(\frac{-\sigma_{Jk} v_J \cdot p_k}{-\sigma_{Ik} v_I \cdot p_k}\right)\right).
\end{aligned}
$$

Currently we are examining the resulting IR structure at 2 loops level.

- Analytical results of the soft anomalous dimension matrices at 1-loop for all partonic channels of $t\bar{t}j$ production have been presented.
- This allows evaluation of 1-loop soft function, an essential tool in the threshold resummation at NLL accuracy.
- Working on the calculation at 2-loop order.
- Numerous consistency checks have been successfully passed.
- The software tool for performing this calculation are openly available and they can be applicable to other projects as well:
  - WilsonWebs - automatic generation of n-loop Wilson diagrams.
  - FORMSoft - Soft anomalous dimension matrices at 1-loop.
  - FORMHard - Hard matrix generator (with or without the color decomposition).
  - pyDipole - NLO pole structure generator using Catani-Seymour formalism and $\Gamma^{(1)}$ and $H^{(0)}$ matrices.

Thanks for your attention!

Universität Hamburg

DER FORSCHUNG | DER LEHRE | DER BILDUNG

Backup

- The leading-order soft function:

$$S_{IJ}^{(0)} = \mathrm{tr}\left(c_I^\dagger c_J\right) \tag{17}$$

- The lowest-order hard function:

$$H_{ij,IJ}^{0} = h_{ij,I}^{0} h_{ij,J}^{*0} \tag{18}$$

- $h_{ij,J}^{(0)}$ are color projected amplitudes (A):

$$h_I^{(0)} = \left(S^{(0)}\right)_{IK}^{-1} \mathrm{tr}\left(c_K^\dagger A\right), \qquad h_I^{*(0)} = \mathrm{tr}\left(A^\dagger c_K\right) \left(S^{(0)}\right)_{IK}^{-1} \tag{19}$$

- Consistency check for the color decomposition procedure:

$$\mathrm{tr}\left(H_{AB}^{(0)} S_{BC}^{(0)}\right) = AA^\dagger \tag{20}$$