Basis transformation properties of anomalous dimensions for hard exclusive processes

Sam Van Thurenhout

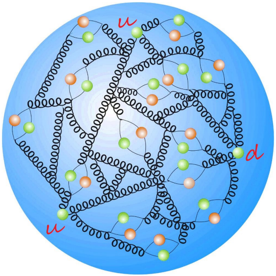
Theory seminar HUN-REN Wigner RCP

1 December 2023

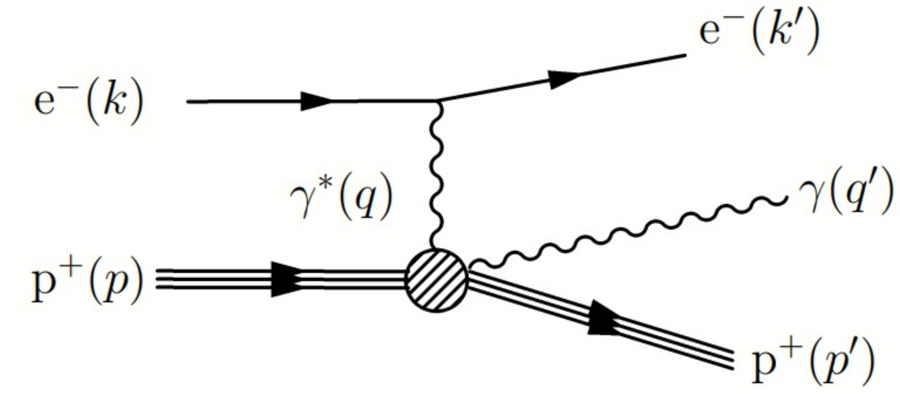
hep-ph/2309.16236 (under review @ Nucl. Phys. B)

How to gain insight into the structure of hadrons

- Important question: How do hadronic properties emerge from the properties of the constituent partons?
- Experimentally: Perform high-energy scattering experiments that can resolve the inner hadron structure (e.g. scatter electrons off a proton)
- Depending on the kinematics (inclusive vs. exclusive): Different properties of hadron structure
- In QCD: Factorization between short range and long range physics
- Long range functions provide information on partons within proton



Hadronic structure from exclusive processes: DVCS



Assumptions:

- Photon highly virtual, $Q^2 \equiv -q^2 \gg p^2$
- $s \gg m_p^2$

Hadronic structure from exclusive processes: DVCS

The DVCS amplitude is determined by the hadronic tensor $T_{\mu\nu}$ which corresponds to a time-ordered product of EM currents

$$T_{\mu\nu} = i \int \mathrm{d}^4 x \, e^{i(q+q')\cdot x} \left\langle p' \right| \mathcal{T} J_\mu(x) J_\nu(0) \left| p \right\rangle$$

with

$$J_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f .$$

Applying the operator product expansion (OPE) [Wilson, 1969, Zimmermann, 1973] the product of currents is related to the leading-twist spin-N operators (focus on flavor non-singlet operators in this talk)

$$\mathcal{O}^{\mathrm{NS}}_{\mu_1 \ldots \mu_N} = \mathcal{S} \bar{\psi}' \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} \psi .$$

Hadronic structure from exclusive processes: GPDs

The corresponding hadronic matrix elements of these operators define the generalized parton distributions (GPDs), which are the non-forward generalizations of the standard parton distribution functions (PDFs)

- PDFs give the probability to find a quark inside the proton with momentum xp $(0 \leq x \leq 1)$. They encode the longitudinal momentum/polarization carried by partons within fast-moving hadrons.
- GPDs describe (a) transverse distributions of partons and (b) contributions partonic orbital angular momentum to total hadronic spin
  $\Rightarrow$ Important quantities for describing proton/hadron structure (e.g. proton spin puzzle [Aidala et al., 2013, Leader and Lorcé, 2014, Deur et al., 2018, Ji et al., 2021])
  $\Rightarrow$ Will be measured with unprecedented precision at a future EIC

  [Boer et al., 2011], [Abdul Khalek et al., 2021]

Scale dependence of GPDs

As GPDs are defined in terms of hadronic matrix elements of QCD operators, they **cannot be computed in perturbation theory**.

$\rightarrow$ Direct extraction from experimental data (see e.g. [Brock et al., 1995]) or using lattice QCD (see e.g. [Alexandrou et al., 2020], [Ji et al., 2021], [Wang et al., 2021])

Phenomenologically the dependence of the distributions on the energy scale of the experiment is also important. This is determined by the scale dependence of the operators, characterized by their **anomalous dimension**

$$\frac{\mathrm{d}[\mathcal{O}]}{\mathrm{d}\ln\mu^2} = \gamma[\mathcal{O}], \quad \gamma \equiv a_s\gamma^{(0)} + a_s^2\gamma^{(1)} + \ldots$$

The operator anomalous dimensions can be calculated perturbatively in QCD!

Composite operators and distributions

During this talk: Focus on leading-twist flavor-non-singlet quark operators

$$\mathcal{O} = \mathcal{S}\overline{\psi}\lambda^{\alpha}\Gamma D_{\nu_1} \dots D_{\nu_N}\psi$$

with $D_\mu = \partial_\mu - ig_s A_\mu$ the covariant derivative.

Different Dirac structure Γ depending on the physical process

- Wilson operators:

$$\mathcal{O}_{\mu_1 \dots \mu_N} = \mathcal{S}\overline{\psi}\lambda^{\alpha}\gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N}\psi$$

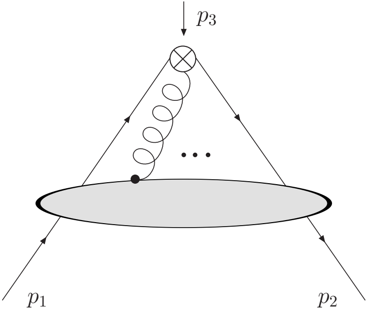
- Transversity operators:

$$\mathcal{O}^{T}_{\nu\mu_1 \dots \mu_N} = \mathcal{S}\overline{\psi}\lambda^{\alpha}\sigma_{\nu\mu_1} D_{\mu_2} \dots D_{\mu_N}\psi$$

Scale-dependence of distributions

In practice, the anomalous dimensions are extracted by renormalizing partonic matrix elements of the operators

$$\langle \psi(p_1) | \mathcal{O}(p_3) | \overline{\psi}(p_2) \rangle$$



In forward kinematics ($p_3 = 0$): Anomalous dimensions in the $\overline{\text{MS}}$-scheme related to $1/\varepsilon$-pole of bare OME in $D = 4 - 2\varepsilon$ dimreg in a simple way

Operator renormalization in non-forward kinematics

$$\langle\psi(p_1)|\mathcal{O}(p_3)|\overline{\psi}(p_2)\rangle$$

In non-forward kinematics ($p_3 \neq 0$), there is mixing with total derivative operators

$$\begin{pmatrix} \mathcal{O}_{N+1} \\ \partial\mathcal{O}_N \\ \vdots \\ \partial^N\mathcal{O}_1 \end{pmatrix} = \begin{pmatrix} Z_{N,N} & Z_{N,N-1} & \dots & Z_{N,0} \\ 0 & Z_{N-1,N-1} & \dots & Z_{N-1,0} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & Z_{0,0} \end{pmatrix} \begin{pmatrix} [\mathcal{O}_{N+1}] \\ [\partial\mathcal{O}_N] \\ \vdots \\ [\partial^N\mathcal{O}_1] \end{pmatrix}$$

Hence we now also have an anomalous dimension matrix (ADM)

$$\hat{\gamma} = -\frac{\mathrm{d}\ln\hat{Z}}{\mathrm{d}\ln\mu^2} = \begin{pmatrix} \gamma_{N,N} & \gamma_{N,N-1} & \cdots & \gamma_{N,0} \\ 0 & \gamma_{N-1,N-1} & \cdots & \gamma_{N-1,0} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \gamma_{0,0} \end{pmatrix}.$$

Hence now the $1/\varepsilon$-pole of the OME involves some combination of the elements of the ADM

Operator renormalization: Non-forward kinematics

The elements of the ADM determine the scale dependence of non-forward (exclusive) distributions through the ERBL equation

[Efremov and Radyushkin, 1980a],[Efremov and Radyushkin, 1980b],[Lepage and Brodsky, 1979],

[Lepage and Brodsky, 1980]

$$\frac{\mathrm{d}H(x,\chi,t,\mu^2)}{\mathrm{d}\ln\mu^2} = \frac{1}{|\chi|}\int_{-1}^{1}\mathrm{d}y\, V(x,y)H(y,\chi,t,\mu^2).$$

$$\sum_{k=0}^{N}\gamma_{N,k}y^k = -\int_0^1 \mathrm{d}x\, x^N V(x,y).$$

We now have to choose a basis for the total-derivative operators. In this talk we will focus on two convenient options: the total derivative basis and the Gegenbauer basis.

- Diagonal elements = forward anomalous dimensions
- Triangular

The total derivative basis

In this basis the operators are defined as

$$\mathcal{O}^{\mathcal{D}}_{k,N-k} = (\Delta \cdot \partial)^k \{\overline{\psi'}(\Delta \cdot \Gamma)(\Delta \cdot D)^{N-k}\psi\}$$

with $\Delta^2 = 0$. E.g. for $N = 0$ we simply have $\{\overline{\psi'}(\Delta \cdot \Gamma)\psi\}$ (conserved current if $\Gamma = \gamma_\mu$!) and for $N = 1$

$$\left\{\overline{\psi'}(\Delta \cdot \Gamma)(\Delta \cdot D)\psi, (\Delta \cdot \partial)\overline{\psi'}(\Delta \cdot \Gamma)\psi\right\}.$$

- This choice of operator basis is used for hadronic studies on the lattice, see e.g. [Göckeler et al., 2005] and [Gracey, 2009]
- In this basis, the anomalous dimensions for low-N operators were computed directly up to $O(a_s^3)$ [Gracey, 2009, Kniehl and Veretin, 2020]
- These low-N results have been extended in [Moch and Van Thurenhout, 2021] and [Van Thurenhout, 2022] to produce some all-N results based on a consistency relation following from the renormalization structure of the operators
  - Large n_f: 5-loop Wilson, 4-loop transversity anomalous dimensions
  - Large n_c: 2-loop Wilson anomalous dimensions

The total derivative basis

The consistency relation can be written in the following form

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,\, j+k}$$

$$+ \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,\, j}$$

$$\begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \cdots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \cdots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \gamma_{0,0} \end{pmatrix}$$

e.g.

$$\gamma^{\mathcal{D}}_{N,N-1} = \frac{N}{2}(\gamma_{N-1,N-1} - \gamma_{N,N})$$

NOTE: Diagonal elements independent of the operator basis (i.e. $\gamma^{\mathcal{D}}_{N,N} = \gamma^{\mathcal{B}}_{N,N} = \gamma_{N,N}$)

The total derivative basis

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,\, j+k}$$
$$+ \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,\, j}$$

To use this relation constructively, we need two ingredients:

- The forward anomalous dimensions $\gamma_{N,N}$, which are (partially) available up to five loops [Gross and Wilczek, 1973, Floratos et al., 1977, Moch et al., 2004, Blümlein et al., 2021, Gracey, 1994, Davies et al., 2017, Velizhanin, 2012, Velizhanin, 2020, Ruijl et al., 2016, Moch et al., 2017, Herzog et al., 2019, Blümlein, 2023, Gehrmann et al., 2023]
- The last column $\gamma^{\mathcal{D}}_{N,0}$ which serves as a boundary condition. The latter can be related to the operator matrix elements, the computation of which can be fully automated using computer algebra methods (e.g. *Forcer* [Ruijl et al., 2020] in *Form* [Vermaseren, 2000, Kuipers et al., 2013])

The Gegenbauer basis

In this basis the operators are written in terms of Gegenbauer polynomials

$$\mathcal{O}^{\mathcal{G}}_{N,k} = (\Delta \cdot \partial)^k \overline{\psi'} (\Delta \cdot \Gamma) C_N^{3/2} \left(\frac{\overleftarrow{D} \cdot \Delta - \Delta \cdot \overrightarrow{D}}{\overleftarrow{\partial} \cdot \Delta + \Delta \cdot \overrightarrow{\partial}} \right) \psi$$

with [Olver et al., 2010]

$$C_N^{\nu}(z) = \frac{\Gamma(\nu + 1/2)}{\Gamma(2\nu)} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{(N+j+2)!}{(j+1)!} \left(\frac{1}{2} - \frac{z}{2} \right)^j .$$

- This choice of operator basis is natural within conformal schemes
  [Efremov and Radyushkin, 1980a], [Belitsky and Müller, 1999], [Braun et al., 2017]
- The anomalous dimensions of the Wilson operators in this basis are known up to $O(a_s^3)$ [Braun et al., 2017].

The Gegenbauer basis

- The reconstruction of the anomalous dimensions in this conformal approach relies on consistency relations coming from the conformal algebra
- At one loop, exact conformal symmetry at leading order dictates that the ADM in the Gegenbauer basis is diagonal
- Beyond leading order, off-diagonal elements are generated by the so-called conformal anomaly which is currently known to two-loop accuracy [Braun et al., 2017, Müller, 1991, Braun et al., 2016] (although a closed-form expression at two loops is currently not available!)

$$\gamma^{\mathcal{G},(1)}_{N,k} = -\frac{\gamma^{(0)}_{N,N} - \gamma^{(0)}_{k,k}}{a(N,k)} \Bigg\{ -2(2k+3)\Big(\beta_0 + \gamma^{(0)}_{k,k}\Big)\vartheta_{N,k} + 2\Delta^{\mathcal{G},(0)}_{N,k} \Bigg\}$$

- As generically the L-loop anomalous dimensions depend only on the $(L-1)$-loop conformal anomaly [Müller, 1991], they could be calculated up to three loops using this approach

Goal of this work

In the present work we want to construct an explicit similarity transformation between both bases

$$\hat{\gamma}_N^{\mathcal{D}} = \hat{R}_N^{-1} \hat{\gamma}_N^{\mathcal{G}} \hat{R}_N$$

- Use the properties of both bases to ones advantage in the computation of the operator anomalous dimensions
- Use to validate computations

Setting up the similarity transformation

Start from relating local operators to non-local ones [Braun et al., 2003]

$$\mathcal{O}_N(0) = \mathcal{P}_N(\partial_1, \partial_2)\phi(z_1)\phi(z_2)\big|_{z_{1,2}\to 0}$$

Here $\mathcal{P}_N$ is the *characteristic polynomial* (of degree N). E.g. for the derivative operators in the Gegenbauer basis we have [Braun et al., 2017]

$$\mathcal{O}^{\mathcal{G}}_{N,k} = (\partial_1 + \partial_2)^k C_N^{3/2}\left(\frac{\partial_1 - \partial_2}{\partial_1 + \partial_2}\right)\mathcal{O}(z_1, z_2)\big|_{z_{1,2}\to 0}$$

with

$$\mathcal{O}(z_1, z_2) = \bar{\psi}(z_1 n)\not{n}\psi(z_2 n).$$

The corresponding characteristic polynomial can be seen to be

$$\mathcal{P}^{\mathcal{G}}_{N,k}(z_1, z_2) = (z_1 + z_2)^k C_N^{3/2}\left(\frac{z_1 - z_2}{z_1 + z_2}\right)$$

Setting up the similarity transformation

To set up the basis transformation, we now want the corresponding relation for the operators in the derivative basis, and then compare the derivative structure in both cases. We have

$$\mathcal{P}^{\mathcal{D}}_{N,k}(z_1, z_2) = z_2^k (z_1 + z_2)^N \Rightarrow \mathcal{O}^{\mathcal{D}}_{N,k} = \partial_2^k (\partial_1 + \partial_2)^N \mathcal{O}(z_1, z_2)|_{z_{1,2}\to 0}$$

Comparing with the result in the Gegenbauer basis (we focus on the conformal operators $k = N$ in what follows)

$$\mathcal{P}^{\mathcal{G}}_{N,N}(z_1, z_2) = (z_1 + z_2)^N C_N^{3/2}\left(\frac{z_1 - z_2}{z_1 + z_2}\right) = (z_1 + z_2)^N C_N^{3/2}\left(1 - \frac{2z_2}{z_1 + z_2}\right)$$

we see that the two bases can be related by expanding the Gegenbauer polynomials as

$$C_N^{3/2}(1 - 2z) = \sum_{k=0}^{N} c_{N,k} z^k.$$

The elements of the rotation matrix, $R_{N,k}$, will then simply correspond to the series coefficients $c_{N,k}$.

Setting up the similarity transformation

From the definition of the Gegenbauer polynomials it immediately follows that

$$C_N^{3/2}(1-2z) = \frac{1}{2N!}\sum_{k=0}^{N}(-1)^k\binom{N}{k}\frac{(N+k+2)!}{(k+1)!}z^k$$

and hence

$$R_{N,k} = c_{N,k} = \frac{1}{2N!}(-1)^k\binom{N}{k}\frac{(N+k+2)!}{(k+1)!}.$$

To compute the elements of the inverse transformation matrix we start from

$$C_N^{3/2}(1-2z) = \sum_{k=0}^{N} c_{N,k} z^k$$

to write

$$z^N = \sum_{j=0}^{N} \tilde{R}_{N,j}\; C_j^{3/2}(1-2z)$$

with $R^{-1}_{N,k} \equiv \tilde{R}_{N,k}$.

Setting up the similarity transformation

The elements of the inverse transformation matrix can then be obtained using the properties of the Gegenbauer polynomials [Gradshteyn and Ryzhik, 2007]

- Orthogonality:

$$\int_0^1 \mathrm{d}z\, C_N^{3/2}(1-2z)C_k^{3/2}(1-2z)[4z(1-z)] = \frac{(N+1)(N+2)}{3+2N}\delta_{N,k}$$

$\Rightarrow$

$$\tilde{R}_{N,k} = \frac{4(3+2k)}{(k+1)(k+2)}\int_0^1 \mathrm{d}z\, z^{N+1}(1-z)C_k^{3/2}(1-2z)$$

- DE representation:

$$C_N^{3/2}(t) = \frac{(-1)^N(N+2)}{2^{N+1}N!}\frac{1}{1-t^2}\frac{\mathrm{d}^N}{\mathrm{d}t^N}(1-t^2)^{N+1}$$

$\Rightarrow$

$$\tilde{R}_{N,k} = \frac{3+2k}{2^{2k+1}(k+1)k!}\int_0^1 \mathrm{d}z\, z^N\frac{\mathrm{d}^k}{\mathrm{d}z^k}[4z(1-z)]^{k+1}$$

Setting up the similarity transformation

The remaining integral can be evaluated using IBP and we find

$$\tilde{R}_{N,k} = \frac{2(-1)^k(3+2k)k!\,(N+1)!}{(N+k+3)!}\binom{N}{k}.$$

Hence the similarity transformation, in component form, becomes

$$\gamma^{\mathcal{D}}_{N,k} = \frac{(-1)^k(N+1)!}{(k+1)!}\sum_{l=k}^{N}(-1)^l\binom{N}{l}\frac{l!\,(3+2l)}{(N+l+3)!}\sum_{j=k}^{l}\binom{j}{k}\frac{(j+k+2)!}{j!}\gamma^{\mathcal{G}}_{l,j}$$

$$\gamma^{\mathcal{G}}_{N,k} = (-1)^k\frac{k!}{N!}(3+2k)\sum_{l=k}^{N}(-1)^l\binom{N}{l}\frac{(N+l+2)!}{(l+1)!}\sum_{j=k}^{l}\binom{j}{k}\frac{(j+1)!}{(j+k+3)!}\gamma^{\mathcal{D}}_{l,j}$$

Neat consequence: 1-loop off-diagonal elements in the $\mathcal{D}$-basis ADM admit a representation in terms of (sums of) the diagonal ones

Illustration: 1-loop quark anomalous dimensions

$$\gamma_{N,k}^{\mathcal{D},(0)} = \frac{(-1)^k N!\,(N+1)!}{(k+1)!} \sum_{l=k}^{N} (-1)^l \binom{l}{k} \frac{(3+2l)(l+k+2)!}{l!\,(N-l)!\,(N+l+3)!} \gamma_{l,l}^{(0)}$$
$$= 2\,C_F \left(\frac{1}{N+2} - \frac{1}{N-k} \right)$$

with

$$\gamma_{N,N}^{(0)} = C_F \left(4S_1(N) + \frac{2}{N+1} + \frac{2}{N+2} - 3 \right)$$

for Wilson operators ($C_F = (n_c^2 - 1)/(2n_c)$) and

$$\gamma_{N,k}^{T,\mathcal{D},(0)} = \frac{(-1)^k N!\,(N+1)!}{(k+1)!} \sum_{l=k}^{N} (-1)^l \binom{l}{k} \frac{(3+2l)(l+k+2)!}{l!\,(N-l)!\,(N+l+3)!} \gamma_{l,l}^{T,(0)}$$
$$= 2\,C_F \left(\frac{1}{N+1} - \frac{1}{N-k} \right)$$

with

$$\gamma_{N,N}^{T,(0)} = C_F \left(4S_1(N) + \frac{4}{N+1} - 3 \right)$$

for transversity ones.

Illustration: 1-loop quark anomalous dimensions

The anomalous dimensions can generically be written in terms of denominators in N and k and harmonic sums. The latter are recursively defined by [Vermaseren, 1999, Blümlein and Kurth, 1999]

$$S_{\pm m}(N) = \sum_{i=1}^{N} (\pm 1)^i \, i^{-m} \,,$$

$$S_{\pm m_1, m_2, \ldots, m_d}(N) = \sum_{i=1}^{N} (\pm 1)^i \, i^{-m_1} \, S_{m_2, \ldots, m_d}(i) \,.$$

The expressions for $\gamma_{N,k}^{\mathcal{D},(0)}$ and $\gamma_{N,k}^{T,\mathcal{D},(0)}$ above agree with previous calculations [Artru and Mekhfi, 1990, Shifman and Vysotsky, 1981, Baldracchini et al., 1981, Blümlein, 2001, Moch and Van Thurenhout, 2021, Van Thurenhout, 2022]. The sums were evaluated using the *MATHEMATICA* package *SIGMA* [Schneider, 2007, Schneider, 2013].

Consistency relation from CP-symmetry

Consider the inverse basis transformation,

$$\gamma^{\mathcal{G}}_{N,k} = R_{N,l}\gamma^{\mathcal{D}}_{l,j}\tilde{R}_{j,k}.$$

The operators in the Gegenbauer basis are CP-even. At the level of the anomalous dimensions, this means that $\gamma^{\mathcal{G}}_{N,k} = 0$ whenever $N - k$ is odd. Substituting into the equation above we then find a tower of relations for the anomalous dimensions in the derivative basis

$$\sum_{l=k}^{N}(-1)^l\binom{N}{l}\frac{(N+l+2)!}{(l+1)!}\sum_{j=k}^{l}\binom{j}{k}\frac{(j+1)!}{(j+k+3)!}\gamma^{\mathcal{D}}_{l,j} = 0 \quad (N-k\ \text{odd}).$$

Note that these relations are valid to all orders of perturbation theory. Setting $k = N - 1$ we reproduce

$$\gamma^{\mathcal{D}}_{N,N-1} = \frac{N}{2}(\gamma_{N-1,N-1} - \gamma_{N,N})$$

Consistency relation from CP-symmetry

This exactly matches the relation derived from our consistency relation,

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,\, j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,\, j}$$

and the same holds for other $(N-\alpha)$-values with α odd. Hence, we can conclude that the physical origin of this consistency condition is directly related to CP symmetry.

Similarity transformation in the singlet sector

The above analysis can be repeated for the flavor singlet operators, for which mixing between quark and gluon operators has to be taken into account. The gluon operators in the derivative basis are written as

$$\mathcal{O}^{g,\mathcal{D}}_{k,N-1} = (\Delta \cdot \partial)^k F_{\mu\Delta} (\Delta \cdot D)^{N-1} F_{\Delta}{}^{\mu}$$

and in the Gegenbauer basis [Braun et al., 2022]

$$\mathcal{O}^{g,\mathcal{G}}_{N,k} = 6(\Delta \cdot \partial)^{k-1} F_{\mu\Delta} C^{5/2}_{N-1}\left(\frac{\overleftarrow{D} \cdot \Delta - \Delta \cdot \vec{D}}{\overleftarrow{\partial} \cdot \Delta + \Delta \cdot \vec{\partial}}\right) F^{\mu\Delta}.$$

Here we defined

$$F_{\mu\Delta} = F_{\mu\nu}\Delta^{\nu}.$$

The similarity transformation is written as

$$\hat{\gamma}^{g,\mathcal{D}} = \hat{G}^{-1} \hat{\gamma}^{g,\mathcal{G}} \hat{G}.$$

Similarity transformation in the singlet sector

Following similar steps as before

- Orthogonality:

$$\int_0^1 \mathrm{d}z\, C_N^{5/2}(1-2z)C_k^{5/2}(1-2z)[4z(1-z)]^2 = \frac{(N+1)(N+2)(N+3)(N+4)}{9(5+2N)}\delta_{N,k}$$

- DE representation:

$$C_N^{5/2}(t) = \frac{(-1)^N(N+3)(N+4)}{12\cdot 2^N N!}\frac{1}{(1-t^2)^2}\frac{\mathrm{d}^N}{\mathrm{d}t^N}(1-t^2)^{N+2}$$

we find

$$G_{N,k} = -\frac{1}{2(N-1)!}(-1)^k\binom{N-1}{k-1}\frac{(N+k+2)!}{(k+1)!}$$

$$\tilde{G}_{N,k} = -\frac{2(-1)^k(3+2k)k!\,(N+1)!}{N(N+k+3)!}\binom{N}{k}$$

Two-loop anomalous dimensions beyond leading color

Let us decompose the two-loop anomalous dimensions as follows

$$\gamma^{(1)}_{N,k} = \gamma^{(1)}_{N,k}\Big|_{n_f} + \gamma^{(1)}_{N,k}\Big|_{\mathrm{LC}} + \gamma^{(1)}_{N,k}\Big|_{\mathrm{slc}}.$$

We will focus on the Wilson operators in what follows. The first term represents the large-n_f limit while the remaining terms represent the leading- and subleading-color terms respectively. In terms of the color and flavor factors we have

$$\gamma^{(1)}_{N,k}\Big|_{n_f} \sim n_f\, C_F\,,$$

$$\gamma^{(1)}_{N,k}\Big|_{\mathrm{LC}} \sim C_F^2,$$

$$\gamma^{(1)}_{N,k}\Big|_{\mathrm{slc}} \sim C_F\left(C_F - \frac{C_A}{2}\right)$$

with $C_F = \frac{n_c^2-1}{2n_c}$, $C_A = n_c$.

Two-loop anomalous dimensions beyond leading color

- The expressions for $\gamma_{N,k}^{\mathcal{D},(1)}\Big|_{n_f}$ and $\gamma_{N,k}^{\mathcal{D},(1)}\Big|_{\mathrm{LC}}$ in the derivative basis were computed in [Moch and Van Thurenhout, 2021]
- Fully analytic expression for $\gamma_{N,k}^{\mathcal{D},(1)}\Big|_{\mathrm{slc}}$ currently not known
  - $\gamma_{N,N-1}^{\mathcal{D},(1)}\Big|_{\mathrm{slc}} = \frac{N}{2}(\gamma_{N-1,N-1} - \gamma_{N,N})\Big|_{\mathrm{slc}}$; $\gamma_{N,0}^{\mathcal{D},(1)}\Big|_{\mathrm{slc}}$ from OMEs

In the Gegenbauer basis we can write [Braun et al., 2017]

$$\gamma_{N,k}^{\mathcal{G},(1)}\Big|_{\mathrm{slc}} = \frac{2(\gamma_{N,N}^{(0)} - \gamma_{k,k}^{(0)})}{a(N,k)}[-2(2k+3)\beta_0|_{C_A}\vartheta_{N,k}]$$

with

$$\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f \quad \text{and } \beta_0|_{C_A} = \frac{11}{3}C_A.$$

Hence the analytic structure of $\gamma_{N,k}^{\mathcal{G},(1)}\Big|_{\mathrm{slc}}$ is the same as that of $\gamma_{N,k}^{\mathcal{G},(1)}\Big|_{n_f}$!

Two-loop anomalous dimensions beyond leading color

The expansion of $\gamma^{\mathcal{G},(1)}_{N,k}\Big|_{n_f}$ in terms of harmonic sums is [Moch et al., 2017]

$$\gamma^{\mathcal{G},(1)}_{N,k}\Big|_{n_f} = \frac{8}{3}\frac{n_f C_F}{a(N,k)}\vartheta_{N,k}\Big\{-2\Big(S_1(N)-S_1(k)\Big)(2k+3)-(2k+3)\Big(\frac{1}{N+1}+\frac{1}{N+2}\Big) + 4 + \frac{1}{k+1} - \frac{1}{k+2}\Big\}$$

and leads to

$$\gamma^{\mathcal{G},(1)}_{N,k}\Big|_{\text{slc}} = \frac{88}{3}C_F\Big(C_F-\frac{C_A}{2}\Big)\frac{\vartheta_{N,k}}{a(N,k)}\Big\{-2\Big(S_1(N)-S_1(k)\Big)(2k+3)-(2k+3)\Big(\frac{1}{N+1} + \frac{1}{N+2}\Big) + 4 + \frac{1}{k+1} - \frac{1}{k+2}\Big\}$$

Using our similarity transformation one can then generate the fixed-N anomalous dimensions as

$$\gamma^{\mathcal{D},(1)}_{N,k}\Big|_{\text{slc}} = \frac{(-1)^k (N+1)!}{(k+1)!}\sum_{l=k}^{N}(-1)^l\binom{N}{l}\frac{l!\,(3+2l)}{(N+l+3)!}\sum_{j=k}^{l}\binom{j}{k}\frac{(j+k+2)!}{j!}\gamma^{\mathcal{G},(1)}_{l,j}\Big|_{\text{slc}}$$

Two-loop anomalous dimensions beyond leading color

- Agreement with low-N computations in [Gracey, 2009]
- Agreement with $\gamma^{\mathcal{D},(1)}_{N,N-1}\Big|_{\text{slc}}$ and $\gamma^{\mathcal{D},(1)}_{N,0}\Big|_{\text{slc}}$ [Van Thurenhout, 2023]
- Agreement with values obtained from x-space expression in e.g. [Mikhailov and Radyushkin, 1985]
- Fixed moments up to $N = 10$ explicitly computed, see [Van Thurenhout, 2023]

Illustrative(?) plot time!

Two-loop anomalous dimensions beyond leading color

$$\sum_{k=0}^{N} \gamma_{N,k} y^k = -\int_0^1 \mathrm{d}x\, x^N V(x,y).$$

$$\frac{(1-y)^2}{\gamma_{N,N}^{(1)}} \sum_{k=0}^{N} \gamma_{N,k}^{\mathcal{D},(1)} y^k,$$

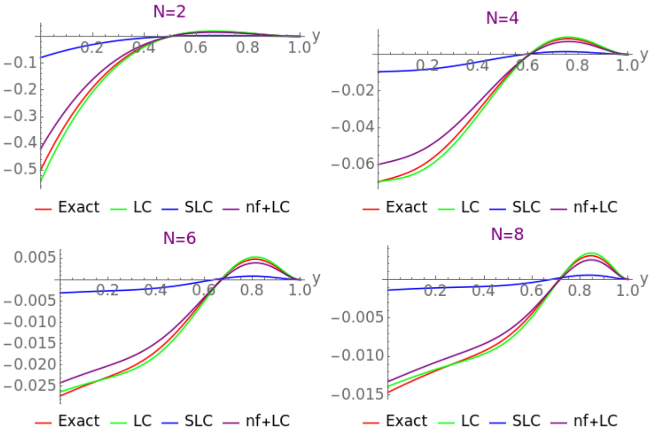
$$\frac{(1-y)^2}{\gamma_{N,N}^{(1)}} \sum_{k=0}^{N} \gamma_{N,k}^{\mathcal{D},(1)}\Big|_{\mathrm{LC}} y^k,$$

$$\frac{(1-y)^2}{\gamma_{N,N}^{(1)}} \sum_{k=0}^{N} \left(\gamma_{N,k}^{\mathcal{D},(1)}\Big|_{\mathrm{LC}} + \gamma_{N,k}^{\mathcal{D},(1)}\Big|_{n_f} \right) y^k,$$

$$\frac{(1-y)^2}{\gamma_{N,N}^{(1)}} \sum_{k=0}^{N} \gamma_{N,k}^{\mathcal{D},(1)}\Big|_{\mathrm{slc}} y^k$$

We plot the results in QCD ($C_A = 3$ and $C_F = 4/3$) and set $n_f = 3$

Two-loop anomalous dimensions beyond leading color



Two-loop anomalous dimensions beyond leading color

The analysis presented above can be repeated to compute the two-loop subleading-color part of the transversity anomalous dimensions. The corresponding leading-n_f expression in the Gegenbauer basis was computed in [Van Thurenhout, 2022] to be

$$\gamma_{N,k}^{T,\mathcal{G},(1)}\Big|_{n_f} = -\frac{16}{3}\frac{n_f C_F}{a(N,k)}\vartheta_{N,k}\left\{(3+2k)\left(S_1(N)-S_1(k)+\frac{1}{N+1}\right)-\frac{1}{k+1}-2\right\}.$$

Hence the expression for the subleading-color part becomes

$$\gamma_{N,k}^{T,\mathcal{G},(1)}\Big|_{\text{slc}} = -\frac{176}{3}\frac{\vartheta_{N,k}}{a(N,k)}C_F\left(C_F-\frac{C_A}{2}\right)\left\{(3+2k)\left(S_1(N)-S_1(k)+\frac{1}{N+1}\right)\right.$$
$$\left.-\frac{1}{k+1}-2\right\}$$

such that in the derivative basis we have

$$\gamma_{N,k}^{T,\mathcal{D},(1)}\Big|_{\text{slc}} = \frac{(-1)^k(N+1)!}{(k+1)!}\sum_{l=k}^{N}(-1)^l\binom{N}{l}\frac{l!\,(3+2l)}{(N+l+3)!}\sum_{j=k}^{l}\binom{j}{k}\frac{(j+k+2)!}{j!}\gamma_{l,j}^{T,\mathcal{G},(1)}\Big|_{\text{slc}}.$$

Two-loop anomalous dimensions beyond leading color

$$\sum_{k=0}^{N} \gamma_{N,k}^{T} y^k = -\int_0^1 \mathrm{d}x\, x^N V^T(x,y).$$

$$\frac{(1-y)^2}{\gamma_{N,N}^{T,(1)}} \sum_{k=0}^{N} \gamma_{N,k}^{T,\mathcal{D},(1)} y^k,$$

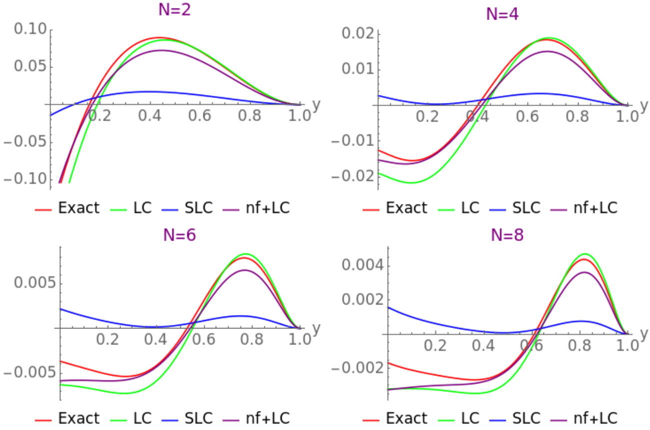
$$\frac{(1-y)^2}{\gamma_{N,N}^{T,(1)}} \sum_{k=0}^{N} \gamma_{N,k}^{T,\mathcal{D},(1)}\Big|_{\mathrm{LC}} y^k,$$

$$\frac{(1-y)^2}{\gamma_{N,N}^{(1)}} \sum_{k=0}^{N} \left(\gamma_{N,k}^{T,\mathcal{D},(1)}\Big|_{\mathrm{LC}} + \gamma_{N,k}^{T,\mathcal{D},(1)}\Big|_{n_f} \right) y^k,$$

$$\frac{(1-y)^2}{\gamma_{N,N}^{T,(1)}} \sum_{k=0}^{N} \gamma_{N,k}^{T,\mathcal{D},(1)}\Big|_{\mathrm{SLC}} y^k$$

We plot the results in QCD ($C_A = 3$ and $C_F = 4/3$) and set $n_f = 3$

Two-loop anomalous dimensions beyond leading color



Validation of conformal anomaly computations

In conformal schemes, the conformal anomaly constitutes vital input for the computation of the off-diagonal elements of the ADM. At the 1-loop level it is given by [Braun et al., 2017, Braun et al., 2016, Braun and Manashov, 2014]

$$\Delta_{N,k}^{\mathcal{G},(0)} = 2C_F(2k+3)a(N,k)\left(\frac{A_{N,k} - S_1(N+1)}{(k+1)(k+2)} + \frac{2A_{N,k}}{a(N,k)}\right)\vartheta_{N,k}$$

with

$$A_{N,k} = S_1\left(\frac{N+k+2}{2}\right) - S_1\left(\frac{N-k-2}{2}\right) + 2S_1(N-k-1) - S_1(N+1).$$

The two-loop conformal anomaly is also known, although no closed-form expression exists thus far [Braun et al., 2017].

Validation of conformal anomaly computations

Using our new similarity transformation, the conformal anomaly (in the $\mathcal{G}$-basis) can be written in terms of the leading-color anomalous dimensions in the $\mathcal{D}$-basis at 1 order in a_s higher. For example the 1-loop conformal anomaly can be written as

$$\Delta_{N,k}^{\mathcal{G},(0)} = (2k+3)\left(\gamma_{k,k}^{(0)} + \frac{22}{3}C_F\right) - \frac{(N-k)(N+k+3)}{2(\gamma_{N,N}^{(0)} - \gamma_{k,k}^{(0)})} R_{N,l}\tilde{R}_{j,k}\gamma_{l,j}^{\mathcal{D},(1)}\Bigg|_{\text{LC}}$$

Using the expression for $\gamma_{N,k}^{\mathcal{D},(1)}$ computed in [Moch and Van Thurenhout, 2021] we find exact agreement for $\Delta_{N,k}^{\mathcal{G},(0)}$.

- Similar expressions at higher orders
- Leading-color anomalous dimensions in $\mathcal{D}$-basis in principle straightforward to compute
- Representation of the anomaly in terms of sums, which can be efficiently evaluated for fixed (N, k) using e.g. the *SUMMER* package [Vermaseren, 1999] in *FORM* [Vermaseren, 2000, Kuipers et al., 2013]

1-loop gluon ADM

In the flavor-singlet sector:

$$\begin{pmatrix} \mathcal{O}^{q,\mathcal{D}} \\ \mathcal{O}^{g,\mathcal{D}} \end{pmatrix} = \begin{pmatrix} Z^{qq,\mathcal{D}} & Z^{qg,\mathcal{D}} \\ Z^{gq,\mathcal{D}} & Z^{gg,\mathcal{D}} \end{pmatrix} \begin{pmatrix} [\mathcal{O}^{q,\mathcal{D}}] \\ [\mathcal{O}^{g,\mathcal{D}}] \end{pmatrix}$$

In the case of non-forward kinematics these operators additionally mix with total-derivative ones and the $Z^{ij,\mathcal{D}}$ (and the corresponding anomalous dimensions) become matrices. Focussing on $O(a_s)$ and the purely gluonic case for illustration we have

$$\hat{Z}^{gg,\mathcal{D}} = \mathbb{1} + \frac{a_s}{\varepsilon}\hat{\gamma}^{gg,\mathcal{D},(0)} + O(a_s^2)$$

- Diagonal elements (forward anomalous dimension) [Balitsky and Braun, 1989, Braunschweig et al., 1987]:

$$\gamma_{N,N}^{gg,(0)} = C_A\left(4S_1(N+1) + \frac{4}{N+3} - \frac{4}{N+2} + \frac{4}{N+1} - \frac{4}{N} - \frac{11}{3}\right) + \frac{2}{3}n_f$$

1-loop gluon ADM

- Off-diagonal elements (use diagonality in Gegenbauer basis)

$$\gamma_{N,k}^{gg,(0),\mathcal{D}} = \frac{(-1)^k (N+1)!}{N(k+1)!} \sum_{l=k}^{N} (-1)^l \binom{N}{l} \binom{l-1}{k-1} \frac{(3+2l)l!\,(l+k+2)!}{(N+l+3)!\,(l-1)!} \gamma_{l,l}^{gg,(0)}$$

$$= C_A \left(\frac{-4(2+k)}{N+3} + \frac{4(3+2k)}{N+2} - \frac{6k}{N+1} + \frac{2(1-k)}{N} - \frac{2}{N-k} \right)$$

e.g.

$$\hat{\gamma}_{N=5}^{gg,\mathcal{D},(0)} = C_A \begin{pmatrix} \frac{181}{35} & -\frac{51}{35} & -\frac{109}{210} & -\frac{26}{105} \\ 0 & \frac{21}{5} & -\frac{7}{5} & -\frac{1}{2} \\ 0 & 0 & \frac{14}{5} & -\frac{7}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{2}{3} n_f \mathbb{1}_{4\times 4}$$

Validation of gluon ADM

The result we presented for $\gamma_{N,k}^{gg,(0),\mathcal{D}}$ can be compared against a computation in yet another operator basis due to B. Geyer and friends [Geyer, 1982, Blümlein et al., 1999] in which the gluon operators are written as

$$\mathcal{O}_{N,k}^{\text{Geyer}} = F_{\mu\Delta}(\overleftarrow{D}\cdot\Delta + \Delta\cdot\vec{D})^{N-k}(\overleftarrow{D}\cdot\Delta - \Delta\cdot\vec{D})^{k-1}F^{\mu\Delta}.$$

By transforming to the $\mathcal{D}$ basis (see e.g. [Moch and Van Thurenhout, 2021] for more details) one can write an implicit relation between the anomalous dimensions in the Geyer and derivative bases (for even N)

$$\gamma_{N+1,N+1}^{gg,(0)} + \sum_{k=0}^{N}\frac{1-(-1)^k}{2}\gamma_{N+1,k}^{(0),\text{Geyer}} = (-1)^N\sum_{l=0}^{N}(-1)^l\,2^l\binom{N}{l}\gamma_{l+1,1}^{gg,\mathcal{D},(0)}.$$

It was checked that our result obeys this consistency relation.

Conclusions and outlook

- Scale dependence of non-perturbative parton distributions is set by the anomalous dimensions of composite operators
- For exclusive processes, there is mixing with total-derivative operators which complicates the extraction of the anomalous dimensions
- Choosing a convenient basis for the operator can simplify ones life

(a) Gegenbauer basis:
  - ADM diagonal at $O(a_s)$ due to exact conformal symmetry at leading order
  - Off-diagonal elements generated at higher orders due to conformal symmetry breaking $\rightarrow$ conformal anomaly

(b) Derivative basis:
  - ADM non-diagonal already at $O(a_s)$
  - Using consistency relation, off-diagonal elements can be determined from the diagonal elements + matrix elements, where the latter can be computed using computer algebra methods

Conclusions and outlook

In this talk: Explicit similarity transformation between these 2 bases for leading-twist operators (quarks+gluons)

- Generation of fixed moments for the 2-loop $\mathcal{D}$-basis anomalous dimensions beyond leading color
  $\rightarrow$ full analytic structure would be valuable for higher order computation! (in progress)
- Independent validation of the conformal anomaly using leading-color anomalous dimensions in the $\mathcal{D}$-basis
  - Leads to representation of the anomaly in terms of sums which can be efficiently evaluated using modern computer algebra packages
  - 1-loop validation done
  - For 2-loop validation: compute 3-loop leading-color ADM in $\mathcal{D}$-basis (TODO)
- As application of the gluonic transformation: 1-loop off-diagonal elements for $\hat{\gamma}^{gg,\mathcal{D},(0)}$
  $\rightarrow$ Generalize to off-diagonal elements (γ^{qg}, γ^{gq}) + higher orders (TODO)

End[1]

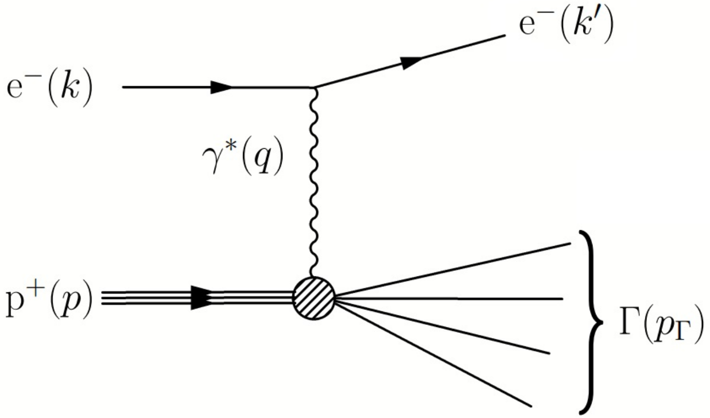
Thank you for your attention!

[1] This work has been supported by grant K143451 of the National Research, Development and Innovation Fund in Hungary.

Appendices and references

1. DIS
2. Sample processes transversity
3. Some comments on *FORCER*
4. GPDs
5. Characteristic polynomial in the $\mathcal{D}$-basis
6. Subleading color expressions
7. References

DIS



Assumptions:

- Photon highly virtual, $Q^2 \equiv -q^2 \gg p^2$
- $s \gg m_p^2$

The DIS cross section

The physical cross section of DIS is proportional to

$$\frac{1}{q^4} L_{\mu\nu} W^{\mu\nu}$$

Here, $L_{\mu\nu}$ represents the leptonic tensor and $W_{\mu\nu}$ the hadronic one.

- $L_{\mu\nu}$ encodes the polarization information of the electrons and the off-shell photon. Applying standard techniques it is easy to find that

$$L_{\mu\nu} = \frac{1}{2} Tr[\not{k}' \gamma_\mu \not{k} \gamma_\nu].$$

- $W^{\mu\nu}$ encodes the information of the $\gamma^* p^+ \to \Gamma$ process, the amplitude of which is

$$\mathcal{M}(\gamma^* p^+ \to \Gamma) \sim \langle \Gamma | \, J_\mu \, | p^+(p) \rangle$$

  with

$$J_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f \text{ the electromagnetic current.}$$

The DIS hadronic tensor

The hadronic tensor appearing in the DIS cross section can then be written as

$$W_{\mu\nu} = \int \mathrm{d}^4x \, e^{iq\cdot x} \left\langle p^+(p) \right| J_\mu(x) J_\nu(0) \left| p^+(p) \right\rangle .$$

Note that this is independent of the final states Γ.

Hence, the calculation of the hadronic tensor of DIS boils down to calculating the product of two current operators.

The standard formalism to deal with this type of problem is the operator product expansion (OPE).

The OPE

The OPE was first introduced by Wilson [Wilson, 1969] and later proven in perturbation theory by Zimmerman [Zimmermann, 1973].

The main idea is that the time-ordered product of two local operators $J(x)$ and $J'(y)$ can be expanded in a series of regular operators, multiplied by functions (called Wilson coefficients) encoding the singularity of the operator product as $x = y$

$$\mathcal{T} J(x) J'(y) = \sum_{n=0}^{\infty} C_n(x-y) \mathcal{O}_n\Big(\frac{x-y}{2}\Big).$$

To apply the OPE to the DIS hadronic tensor, we use the optical theorem to relate the rate of $\gamma^* p^+ \to \Gamma$ to the imaginary part of the forward scattering rate $\gamma^* p^+ \to \gamma^* p^+$:

$$W_{\mu\nu} = 2\,\mathrm{Im}\, T_{\mu\nu},$$

$$T_{\mu\nu} = i \int \mathrm{d}^4 x\, e^{iq\cdot x} \left\langle p^+(p) \right| \mathcal{T} J_\mu(x) J_\nu(0) \left| p^+(p) \right\rangle.$$

Application of the OPE to DIS

$T_{\mu\nu}$ can be explicitly calculated as the forward matrix element for Compton scattering, $\gamma^* q \to \gamma^* q$ (photon off-shell and no polarizations included). This gives

$$T_{\mu\nu} \sim -\bar{u}(p)\frac{\gamma_\mu(\not{p}+\not{q})\gamma_\nu}{(p+q)^2}u(p).$$

As we are interested in the regime of large Q^2, we expand the denominator for $Q^2 \gg p^2$

$$\frac{1}{(p+q)^2} = -\frac{1}{Q^2}\sum_n \left(\frac{2p\cdot q}{Q^2}\right)^n$$

such that

$$T_{\mu\nu} \sim \frac{1}{Q^2}\bar{u}(p)\gamma_\mu(\not{p}+\not{q})\gamma_\nu u(p)\sum_n \left(\frac{2p\cdot q}{Q^2}\right)^n.$$

Application of the OPE to DIS

The ingredients of the OPE, i.e. the Wilson coefficients and the operators, can be read of from the momentum expansion in a relatively straightforward manner:

- Factors of p_μ should come from factors of $i\partial_\mu$ from the operators, acting on the external states
- The dependence on the short-distance scale should be incorporated into the Wilson coefficients

This implies that the Wilson coefficients for DIS will be of the following form

$$C^{\mu_1 \cdots \mu_n} \sim \frac{2^n}{Q^{2n+1}} q^{\mu_1 \cdots \mu_n}.$$

Application of the OPE to DIS

For the extraction of the operators, it is customary to use a basis of gauge-invariant operators, meaning that ordinary derivatives are replaced by covariant ones

$$\partial_\mu \to D_\mu = \partial_\mu - ig_s A_\mu .$$

Furthermore, the OPE is dominated by leading-twist operators, where *twist = dimension - spin*. These operators are symmetric in the Lorentz indices and traceless.

Hence, the operators appearing in the OPE for DIS are gauge-invariant leading-twist spin-N operators (focus on flavor non-singlet operators in this talk)

$$\mathcal{O}^{\text{NS}}_{\mu_1\ldots\mu_N} = \mathcal{S}\bar{\psi}'\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi .$$

PDFs and DIS

Finally, one has to consider the forward matrix element of these operators

$$\left\langle p^+(p) \right| \mathcal{O}_{\mu_1 \ldots \mu_N} \left| p^+(p) \right\rangle \sim \mathcal{M}_N(Q)\, p_{\mu_1} \ldots p_{\mu_N}.$$

The functions $\mathcal{M}_N$ are directly related to the parton distribution functions (PDFs)

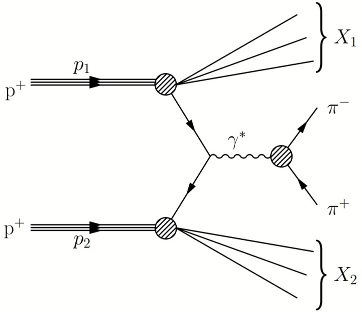
$$f_q(x) \sim \sum_n \frac{\text{Im}\, \mathcal{M}_n}{x^n},$$

which can be interpreted to give the probability to find a quark inside the proton with momentum xp $(0 \leq x \leq 1)$. They encode the longitudinal momentum/polarization carried by partons within fast-moving hadrons.

Since the PDFs are defined in terms of hadronic states, they are non-perturbative
$\Rightarrow$ Direct extraction from experimental data (see e.g. [Brock et al., 1995]) or using lattice QCD (see e.g. [Alexandrou et al., 2020], [Ji et al., 2021], [Wang et al., 2021])

Example of an inclusive process

- Inclusive polarized Drell-Yan



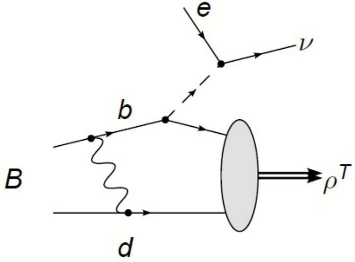
Distributions: Transversity distributions (TDFs) $h^T(x, \mu_f^2)$

[Ralston and Soper, 1979], [Artru and Mekhfi, 1990], [Jaffe and Ji, 1991], [Jaffe and Ji, 1992], [Cortes et al., 1992]

- ◇ Difference in probabilities of finding a parton in a transversely polarized nucleon polarized parallel to the nucleon spin and an oppositely polarized one
- ◇ Studied e.g. by the STAR experiment at RHIC [?]

Example of an exclusive process

- Exclusive production of transversely polarized ρ-meson



Distributions: Transverse distribution amplitudes (DAs) $\phi(x, \mu_F^2)$

[Lepage and Brodsky, 1980]

- ◇ Measure parton distributions within mesons
- ◇ Important input for e.g. LHCb [?]

Some comments on *FORCER*

- *FORM* [Vermaseren, 2000], [Kuipers et al., 2013] program for the reduction of four-loop massless propagator-type integrals to master integrals
- Parametric IBP reductions
- Often possible to avoid explicit IBP reductions by reducing topologies to simpler ones (1-loop integrals, triangle rule, ...) $\rightarrow$ Automatized!
- Less diagrams for which actual IBP reductions are necessary, special rules for these

More details can be found in the original paper [Ruijl et al., 2020].

GPDs

$$F^q \equiv \int \frac{dz^-}{2\pi} e^{ix\chi^+ z^-} \langle p' | \overline{\psi}(-z/2)\gamma^+\psi(z/2) | p\rangle \sim H(x,\chi,t)\overline{\psi}(p')\gamma^+\psi(p)$$

$$+ E(x,\chi,t)\overline{\psi}(p')\frac{i\sigma^{+\nu}\tilde{\Delta}_\nu}{2m_p}\psi(p)$$

$$+ \text{higher twist}$$

$$\int dx\, x^N F^q \sim \overline{\psi}(0)\gamma^+ D^N \psi(0)$$

see e.g. [Diehl, 2003]. Here χ is the skewedness

$$\chi = \frac{p^+ - p'^+}{p^+ + p'^+}.$$

For some four-vector $v \equiv (v^0, v^1, v^2, v^3)$ light-cone coordinates are defined as

$$v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^3), \quad \vec{v} = (v^1, v^2).$$

Characteristic polynomial in the $\mathcal{D}$-basis

Here we derive the characteristic polynomial for the operators in the derivative basis. For this, we start from the fact that the non-local operators act as generating functions for local ones [Braun et al., 2017]

$$\mathcal{O}(z_1, z_2) = \sum_{k,m} \frac{z_1^k z_2^m}{k!\, m!} \mathcal{O}^{\mathcal{D}}_{0,k,m}.$$

Note that the local operators appearing in the right-hand side are already written in the derivative basis. Using now the identity

[Van Thurenhout and Moch, 2022]

$$\mathcal{O}^{\mathcal{D}}_{0,N-k,k} = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathcal{O}^{\mathcal{D}}_{j,N-j,0}$$

this can be rewritten in terms of operators with covariant derivatives acting only on the $\bar{\psi}$ field

$$\mathcal{O}(z_1, z_2) = \sum_{k=0}^{N} \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} \frac{z_1^{N-k} z_2^k}{k!\,(N-k)!} \mathcal{O}^{\mathcal{D}}_{j,N-j,0}.$$

Characteristic polynomial in the $\mathcal{D}$-basis

From the latter relation, the following two identities immediately follow

$$\partial_1^N \mathcal{O}(z_1, z_2)|_{z_{1,2}\to 0} = \mathcal{O}^{\mathcal{D}}_{0,N,0},$$
$$\partial_2^N \mathcal{O}(z_1, z_2)|_{z_{1,2}\to 0} = \sum_{i=0}^{N} (-1)^i \binom{N}{i} \mathcal{O}^{\mathcal{D}}_{N-i,i,0}.$$

To determine the characteristic polynomial, we first consider the spin-two case, for which we only have two local operators $\{\mathcal{O}^{\mathcal{D}}_{1,0,0}, \mathcal{O}^{\mathcal{D}}_{0,1,0}\}$. Using the identities just presented we find

$$\partial_1 \mathcal{O}(z_1, z_2)|_{z_{1,2}\to 0} = \mathcal{O}^{\mathcal{D}}_{0,1,0},$$
$$\partial_2 \mathcal{O}(z_1, z_2)|_{z_{1,2}\to 0} = \mathcal{O}^{\mathcal{D}}_{1,0,0} - \mathcal{O}^{\mathcal{D}}_{0,1,0}$$

and hence, upon inversion,

$$\mathcal{O}^{\mathcal{D}}_{1,0,0} = (\partial_1 + \partial_2)\mathcal{O}(z_1, z_2)|_{z_{1,2}\to 0},$$
$$\mathcal{O}^{\mathcal{D}}_{0,1,0} = \partial_1 \mathcal{O}(z_1, z_2)|_{z_{1,2}\to 0}.$$

Characteristic polynomial in the $\mathcal{D}$-basis

This implies that

$$\mathcal{P}^{\mathcal{D}}_{1,0}(z_1, z_2) = z_1 + z_2,$$
$$\mathcal{P}^{\mathcal{D}}_{0,1}(z_1, z_2) = z_1.$$

These polynomials are homogeneous with

$$(z_1\partial_1 + z_2\partial_2 - N - k)\mathcal{P}^{\mathcal{D}}_{N,k}(z_1, z_2) = 0.$$

Here $N, k \in \{0, 1\}$ and $N + k = 1$ represents the total number of derivatives. The above example can easily be generalized to higher-spin operators and we find

$$\mathcal{P}^{\mathcal{D}}_{N,k}(z_1, z_2) = z_1^k(z_1 + z_2)^N.$$

This is a homogeneous polynomial of degree $N + k$, which is the total number of derivatives,

$$(z_1\partial_1 + z_2\partial_2 - N - k)\mathcal{P}^{\mathcal{D}}_{N,k}(z_1, z_2) = 0.$$

Characteristic polynomial in the $\mathcal{D}$-basis

Hence we can conclude that the local operators in the derivative basis can be derived from the non-local one as

$$\mathcal{O}^{\mathcal{D}}_{N,k,0} = \partial_1^k (\partial_1 + \partial_2)^N \mathcal{O}(z_1, z_2)|_{z_{1,2} \to 0}.$$

The equivalent relation for the operators with covariant derivatives acting on the ψ field reads

$$\mathcal{O}^{\mathcal{D}}_{N,0,k} = \partial_2^k (\partial_1 + \partial_2)^N \mathcal{O}(z_1, z_2)|_{z_{1,2} \to 0}.$$

Subleading color expressions

Diagonal:

$$\gamma_{N,N} = C_F\left(C_F - \frac{C_A}{2}\right)\left(\frac{17}{3} - \frac{8}{(N+1)^3} + \frac{8(-1)^N}{(N+1)^3} + \frac{20}{3}\frac{1}{(N+1)^2} + \frac{8(-1)^N}{(N+1)^2} + \frac{212}{9}\frac{1}{N+1} - \frac{8}{(N+2)^3}\right.$$
$$- \frac{8(-1)^N}{(N+2)^3} + \frac{20}{3}\frac{1}{(N+2)^2} - \frac{8(-1)^N}{(N+2)^2} - \frac{748}{9}\frac{1}{N+2} + 16S_{-3}(N) - \frac{16S_{-2}(N)}{N+1} - \frac{16S_{-2}(N)}{N+2}$$
$$\left. - \frac{536}{9}S_1(N) + \frac{88}{3}S_2(N) - 16S_3(N) - 32S_{1,-2}(N)\right)$$

Next-to-diagonal:

$$\gamma^{\mathcal{D},(1)}_{N,N-1}\Big|_{\mathbf{SLC}} = C_F\left(C_F - \frac{C_A}{2}\right)\left(\frac{268}{9} - \frac{34}{3}\frac{1}{N} + \frac{4(-1)^N}{N} + \frac{4}{N^2} - \frac{4(-1)^N}{N^2} + \frac{160}{3}\frac{1}{N+1} - \frac{8(-1)^N}{N+1} - \frac{778}{9}\frac{1}{N+2}\right.$$
$$\left. + \frac{4(-1)^N}{N+2} + \frac{32}{3}\frac{1}{(N+2)^2} - \frac{4(-1)^N}{(N+2)^2} - \frac{8}{(N+2)^3} - \frac{8(-1)^N}{(N+2)^3} + 16S_{-2}(N) - \frac{16S_{-2}(N)}{N+2}\right)$$

Last column:

$$\gamma^{\mathcal{D},(1)}_{N,0}\Big|_{\mathbf{SLC}} = C_F\left(C_F - \frac{C_A}{2}\right)\left\{\frac{1}{N+1}\left(\frac{136}{3} + 16S_1(N)\right) + \frac{1}{(N+2)^2}\left(\frac{20}{3} + 8S_1(N)\right)\right.$$
$$\left. + \frac{1}{N+2}\left(-\frac{676}{9} + \frac{20}{3}S_1(N) - 8S_2(N)\right) - \frac{8}{N^2}S_1(N) + \frac{1}{N}\left(\frac{268}{9} - \frac{68}{3}S_1(N) + 8S_2(N)\right)\right\}$$

References I

[Abdul Khalek et al., 2021] Abdul Khalek, R. et al. (2021).
Science Requirements and Detector Concepts for the Electron-Ion Collider: EIC Yellow Report.

[Aidala et al., 2013] Aidala, C. A., Bass, S. D., Hasch, D., and Mallot, G. K. (2013).
The Spin Structure of the Nucleon.
Rev. Mod. Phys., 85:655–691.

[Alexandrou et al., 2020] Alexandrou, C., Bacchio, S., Constantinou, M., Finkenrath, J., Hadjiyiannakou, K., Jansen, K., Koutsou, G., Panagopoulos, H., and Spanoudes, G. (2020).
Complete flavor decomposition of the spin and momentum fraction of the proton using lattice QCD simulations at physical pion mass.
Phys. Rev. D, 101(9):094513.

[Artru and Mekhfi, 1990] Artru, X. and Mekhfi, M. (1990).
Transversely Polarized Parton Densities, their Evolution and their Measurement.
Z. Phys. C, 45:669.

[Baldracchini et al., 1981] Baldracchini, F., Craigie, N. S., Roberto, V., and Socolovsky, M. (1981).
A Survey of Polarization Asymmetries Predicted by QCD.
Fortschritte der Physik, 29(11-12):505–550.

[Balitsky and Braun, 1989] Balitsky, I. I. and Braun, V. M. (1989).
Evolution Equations for QCD String Operators.
Nucl. Phys. B, 311:541–584.

References III

[Belitsky and Müller, 1999] Belitsky, A. V. and Müller, D. (1999).
Broken conformal invariance and spectrum of anomalous dimensions in QCD.
Nucl. Phys. B, 537:397–442.

[Blümlein, 2001] Blümlein, J. (2001).
On the anomalous dimension of the transversity distribution $h_1(x, Q^2)$.
Eur. Phys. J. C, 20:683–687.

[Blümlein, 2023] Blümlein, J. (2023).
Deep-Inelastic Scattering: What do we know ?

[Blümlein et al., 1999] Blümlein, J., Geyer, B., and Robaschik, D. (1999).
The Virtual Compton amplitude in the generalized Bjorken region: twist-2 contributions.
Nucl. Phys. B, 560:283–344.

References IV

[Blümlein and Kurth, 1999] Blümlein, J. and Kurth, S. (1999).
Harmonic sums and Mellin transforms up to two loop order.
Phys. Rev. D, 60:014018.

[Blümlein et al., 2021] Blümlein, J., Marquard, P., Schneider, C., and Schönwald, K. (2021).
The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements.
Nucl. Phys. B, 971:115542.

[Boer et al., 2011] Boer, D. et al. (2011).
Gluons and the quark sea at high energies: Distributions, polarization, tomography.

References V

[Braun et al., 2022] Braun, V. M., Chetyrkin, K. G., and Manashov, A. N. (2022).
NNLO anomalous dimension matrix for twist-two flavor-singlet operators.
Phys. Lett. B, 834:137409.

[Braun et al., 2003] Braun, V. M., Korchemsky, G. P., and Müller, D. (2003).
The Uses of conformal symmetry in QCD.
Prog. Part. Nucl. Phys., 51:311–398.

[Braun and Manashov, 2014] Braun, V. M. and Manashov, A. N. (2014).
Two-loop evolution equations for light-ray operators.
Phys. Lett. B, 734:137–143.

[Braun et al., 2016] Braun, V. M., Manashov, A. N., Moch, S., and Strohmaier, M. (2016).
Two-loop conformal generators for leading-twist operators in QCD.
JHEP, 03:142.

[Braun et al., 2017] Braun, V. M., Manashov, A. N., Moch, S., and Strohmaier, M. (2017).
Three-loop evolution equation for flavor-nonsinglet operators in off-forward kinematics.
JHEP, 06:037.

[Braunschweig et al., 1987] Braunschweig, T., Geyer, B., and Robaschik, D. (1987).
Anomalous Dimensions of Flavor Singlet Light Cone Operators.
Annalen Phys., 44:403–411.

References VII

[Brock et al., 1995] Brock, R. et al. (1995).
Handbook of perturbative QCD: Version 1.0.
Rev. Mod. Phys., 67:157–248.

[Cortes et al., 1992] Cortes, J. L., Pire, B., and Ralston, J. P. (1992).
Measuring the transverse polarization of quarks in the proton.
Z. Phys. C, 55:409–416.

[Davies et al., 2017] Davies, J., Vogt, A., Ruijl, B., Ueda, T., and Vermaseren, J. A. M. (2017).
Large-n_f contributions to the four-loop splitting functions in QCD.
Nucl. Phys. B, 915:335–362.

[Deur et al., 2018] Deur, A., Brodsky, S. J., and De Téramond, G. F. (2018).
The Spin Structure of the Nucleon.

References VIII

[Diehl, 2003] Diehl, M. (2003).
Generalized parton distributions.
Phys. Rept., 388:41–277.

[Efremov and Radyushkin, 1980a] Efremov, A. V. and Radyushkin, A. V. (1980a).
Asymptotical Behavior of Pion Electromagnetic Form-Factor in QCD.
Theor. Math. Phys., 42:97–110.

[Efremov and Radyushkin, 1980b] Efremov, A. V. and Radyushkin, A. V. (1980b).
Factorization and Asymptotical Behavior of Pion Form-Factor in QCD.
Phys. Lett. B, 94:245–250.

References IX

[Floratos et al., 1977] Floratos, E. G., Ross, D. A., and Sachrajda, C. T. (1977).
Higher Order Effects in Asymptotically Free Gauge Theories: The Anomalous Dimensions of Wilson Operators.
Nucl. Phys. B, 129:66–88.
[Erratum: Nucl.Phys.B 139, 545–546 (1978)].

[Gehrmann et al., 2023] Gehrmann, T., von Manteuffel, A., Sotnikov, V., and Yang, T.-Z. (2023).
The $N_f\ C_F^3$ contribution to the non-singlet splitting function at four-loop order.

[Geyer, 1982] Geyer, B. (1982).
Anomalous dimensions in local and non-local light cone expansion. (Talk).
Czech. J. Phys. B, 32:645–652.

References X

[Göckeler et al., 2005] Göckeler, M., Horsley, R., Pleiter, D., Rakow, P. E. L., and Schierholz, G. (2005).
A Lattice determination of moments of unpolarised nucleon structure functions using improved Wilson fermions.
Phys. Rev. D, 71:114511.

[Gracey, 1994] Gracey, J. A. (1994).
Anomalous dimension of nonsinglet Wilson operators at $O(1/n_f)$ in deep inelastic scattering.
Phys. Lett. B, 322:141–146.

[Gracey, 2009] Gracey, J. A. (2009).
Three loop anti-MS operator correlation functions for deep inelastic scattering in the chiral limit.
JHEP, 04:127.

References XI

[Gradshteyn and Ryzhik, 2007] Gradshteyn, I. S. and Ryzhik, I. M. (2007).
Table of integrals, series, and products.
Elsevier/Academic Press, Amsterdam, seventh edition.

[Gross and Wilczek, 1973] Gross, D. J. and Wilczek, F. (1973).
Asymptotically Free Gauge Theories - I.
Phys. Rev. D, 8:3633–3652.

[Herzog et al., 2019] Herzog, F., Moch, S., Ruijl, B., Ueda, T., Vermaseren, J. A. M., and Vogt, A. (2019).
Five-loop contributions to low-N non-singlet anomalous dimensions in QCD.
Phys. Lett. B, 790:436–443.

[Jaffe and Ji, 1991] Jaffe, R. L. and Ji, X.-D. (1991).
Chiral odd parton distributions and polarized Drell-Yan.
Phys. Rev. Lett., 67:552–555.

[Jaffe and Ji, 1992] Jaffe, R. L. and Ji, X.-D. (1992).
Chiral odd parton distributions and Drell-Yan processes.
Nucl. Phys. B, 375:527–560.

[Ji et al., 2021] Ji, X., Yuan, F., and Zhao, Y. (2021).
What we know and what we don't know about the proton spin after 30 years.
Nature Rev. Phys., 3(1):27–38.

[Kniehl and Veretin, 2020] Kniehl, B. A. and Veretin, O. L. (2020).
Moments $n = 2$ and $n = 3$ of the Wilson twist-two operators at three loops in the RI$'$/SMOM scheme.
Nucl. Phys. B, 961:115229.

References XIII

[Kuipers et al., 2013] Kuipers, J., Ueda, T., Vermaseren, J. A. M., and Vollinga, J. (2013).
FORM version 4.0.
Comput. Phys. Commun., 184:1453–1467.

[Leader and Lorcé, 2014] Leader, E. and Lorcé, C. (2014).
The angular momentum controversy: What's it all about and does it matter?
Phys. Rept., 541(3):163–248.

[Lepage and Brodsky, 1979] Lepage, G. P. and Brodsky, S. J. (1979).
Exclusive Processes in Quantum Chromodynamics: Evolution Equations for Hadronic Wave Functions and the Form-Factors of Mesons.
Phys. Lett. B, 87:359–365.

References XIV

[Lepage and Brodsky, 1980] Lepage, G. P. and Brodsky, S. J. (1980).
Exclusive Processes in Perturbative Quantum Chromodynamics.
Phys. Rev. D, 22:2157.

[Mikhailov and Radyushkin, 1985] Mikhailov, S. V. and Radyushkin, A. V. (1985).
Evolution Kernels in QCD: Two Loop Calculation in Feynman Gauge.
Nucl. Phys. B, 254:89–126.

[Moch et al., 2017] Moch, S., Ruijl, B., Ueda, T., Vermaseren, J. A. M., and Vogt, A. (2017).
Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond.
JHEP, 10:041.

References XV

[Moch and Van Thurenhout, 2021] Moch, S. and Van Thurenhout, S. (2021).
Renormalization of non-singlet quark operator matrix elements for off-forward hard scattering.
Nucl. Phys. B, 971:115536.

[Moch et al., 2004] Moch, S., Vermaseren, J. A. M., and Vogt, A. (2004).
The Three loop splitting functions in QCD: The Nonsinglet case.
Nucl. Phys. B, 688:101–134.

[Müller, 1991] Müller, D. (1991).
Constraints for anomalous dimensions of local light cone operators in ϕ^3 in six-dimensions theory.
Z. Phys. C, 49:293–300.

References XVI

[Olver et al., 2010] Olver, F. W. J., Lozier, D. W., Boisvert, R. F., and Clark, C. W. (2010).
The NIST Handbook of Mathematical Functions.
Cambridge Univ. Press.

[Ralston and Soper, 1979] Ralston, J. P. and Soper, D. E. (1979).
Production of Dimuons from High-Energy Polarized Proton Proton Collisions.
Nucl. Phys. B, 152:109.

[Ruijl et al., 2020] Ruijl, B., Ueda, T., and Vermaseren, J. A. M. (2020).
Forcer, a FORM program for the parametric reduction of four-loop massless propagator diagrams.
Comput. Phys. Commun., 253:107198.

References XVII

[Ruijl et al., 2016] Ruijl, B., Ueda, T., Vermaseren, J. A. M., Davies, J., and Vogt, A. (2016).
First Forcer results on deep-inelastic scattering and related quantities.
PoS, LL2016:071.

[Schneider, 2007] Schneider, C. (2007).
Symbolic summation assists combinatorics.
Seminaire Lotharingien de Combinatoire, 56:1–36.

[Schneider, 2013] Schneider, C. (2013).
Simplifying Multiple Sums in Difference Fields.
In *LHCPhenoNet School: Integration, Summation and Special Functions in Quantum Field Theory*, pages 325–360.

[Shifman and Vysotsky, 1981] Shifman, M. A. and Vysotsky, M. I. (1981).
Form-Factors of Heavy Mesons in QCD.
Nucl. Phys. B, 186:475–518.

[Van Thurenhout, 2022] Van Thurenhout, S. (2022).
Off-forward anomalous dimensions of non-singlet transversity operators.
Nucl. Phys. B, 980:115835.

[Van Thurenhout, 2023] Van Thurenhout, S. (2023).
Basis transformation properties of anomalous dimensions for hard exclusive processes.

[Van Thurenhout and Moch, 2022] Van Thurenhout, S. and Moch, S.-O. (2022).
Off-forward anomalous dimensions in the leading-n_f limit.
PoS, LL2022:076.

[Velizhanin, 2012] Velizhanin, V. N. (2012).
Four loop anomalous dimension of the second moment of the non-singlet twist-2 operator in QCD.
Nucl. Phys. B, 860:288–294.

[Velizhanin, 2020] Velizhanin, V. N. (2020).
Four-loop anomalous dimension of the third and fourth moments of the nonsinglet twist-2 operator in QCD.
Int. J. Mod. Phys. A, 35(32):2050199.

[Vermaseren, 1999] Vermaseren, J. A. M. (1999).
Harmonic sums, Mellin transforms and integrals.
Int. J. Mod. Phys. A, 14:2037–2076.

[Vermaseren, 2000] Vermaseren, J. A. M. (2000).
New features of FORM.

[Wang et al., 2021] Wang, G., Yang, Y.-B., Liang, J., Draper, T., and Liu, K.-F. (2021).
Proton momentum and angular momentum decompositions with overlap fermions.

References XX

[Wilson, 1969] Wilson, K. G. (1969).
Nonlagrangian models of current algebra.
Phys. Rev., 179:1499–1512.

[Zimmermann, 1973] Zimmermann, W. (1973).
Normal products and the short distance expansion in the perturbation theory of renormalizable interactions.
Annals Phys., 77:570–601.