What about quarkonia in the (non)commutative space?

Bukor Benedek

Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava

Seminar of the Theory Department of the HUN-REN Wigner RCP
26^{th} January 2024, Budapest

coauthor, supervisor: *Mgr. Juraj Tekel, PhD.*

arXiv:2209.09028 [hep-ph]
Eur. Phys. J. Plus (2023) 138: 499

Outline

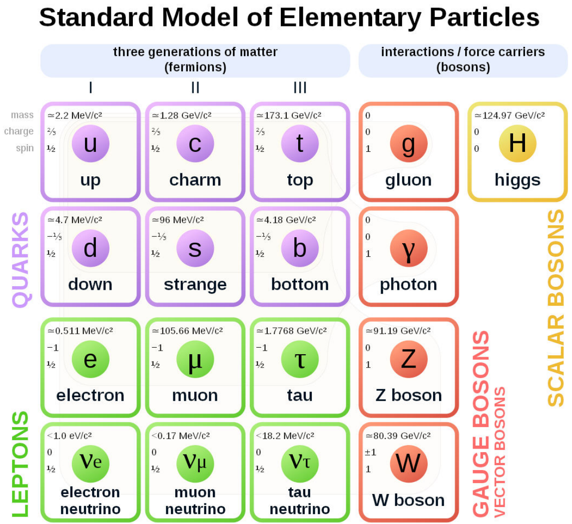
1. Cornell potential & quarkonium masses
   in standard/commutative quantum mechanics (QM)
   - approximation methods: WKB & Pekeris-type

2. introduction to the 3D non-commutative (NC) QM
   - Coulomb potential/hydrogen atom in NC QM

3. Cornell potential & quarkonium masses
   in non-commutative quantum mechanics (QM)

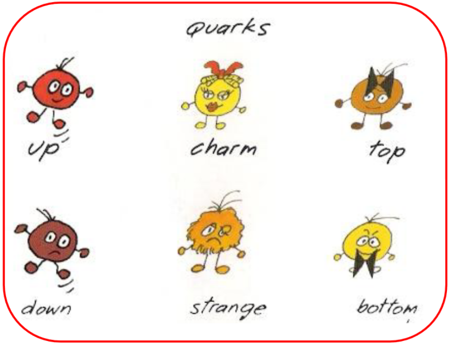
What about quarkonia in the (non)commutative space?

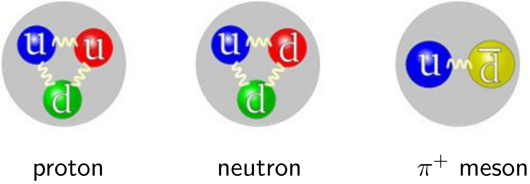
Elementary particles

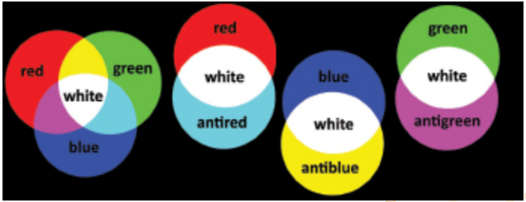


Colour of quarks

- hadron is a particle made of 2 or more quarks
  - meson is a particle made of 2 quarks
  - baryon is a particle made of 3 quarks

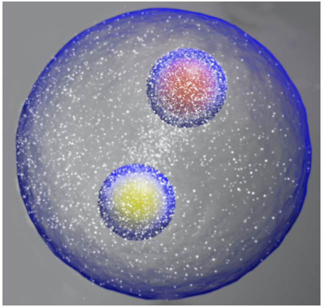


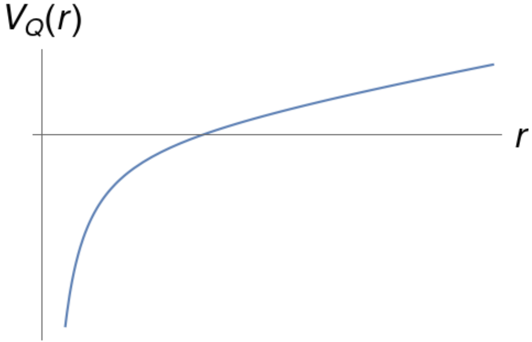




- **Cornell potential**

$$V_Q(r) = -\frac{C}{r} + B\,r$$

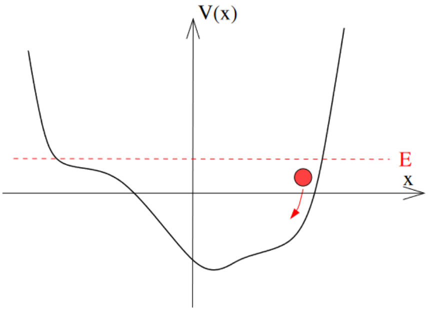




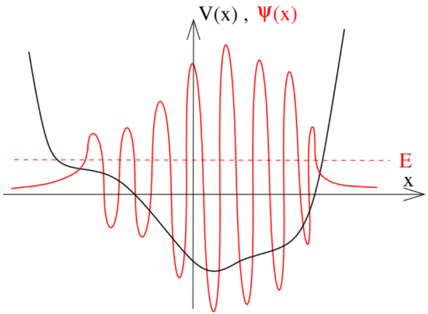
- the Schrödinger equation in standard QM for $V_Q(r)$ cannot be solved analytically

$$\left[-\frac{\hbar^2}{2\mu}\Delta + V_Q(r)\right]\Psi(\mathbf{r}) = E\,\Psi(\mathbf{r})$$

WKB approximation



ball in the well



particle in the potential well

https://www.damtp.cam.ac.uk/user/tong/aqm/topics2.pdf

WKB approximation in 1D

- Schrödinger equation in 1D for $\Psi(x)$ [Griffiths '04 textbook]

$$-\frac{\hbar^2}{2m}\Psi''(x) + V(x)\Psi(x) = E\Psi(x)$$

- ansatz for $\Psi(x)$

$$\Psi(x) = e^{\frac{i}{\hbar}W(x)}, \text{ where } W(x) = W_0(x) + \hbar W_1(x) + \hbar^2 W_2(x) + \ldots$$

- plugging into the Schrödinger equation

$$\left[-\left(W_0'\right)^2 + p^2\right] + \hbar\left[iW_0'' - 2W_0'W_1'\right] + \cdots = 0$$

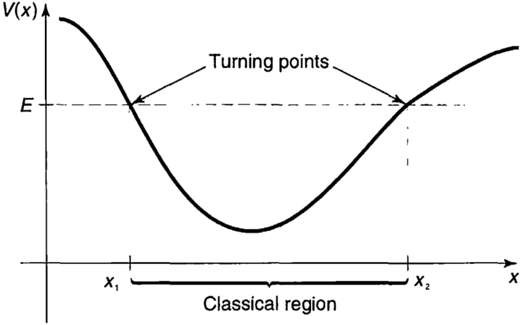
where $p(x) = \sqrt{2m[E - V(x)]}$ is the semiclassical momentum of the particle

WKB approximation in 1D

- the wave function in this potential [Griffiths '04 textbook]

$$\psi(x) = \begin{cases} \frac{C}{\sqrt{|p(x)|}} e^{\pm \frac{i}{\hbar} \int^x p(s) \mathrm{d}s} , & \text{for classical region} \\ \frac{C}{\sqrt{|p(x)|}} e^{\pm \frac{1}{\hbar} \int^x p(s) \mathrm{d}s} , & \text{for nonclassical region} \end{cases}$$

- probability of finding the particle in region $(x,\ x + \mathrm{d}x)$ is $|\psi(x)|^2 \mathrm{d}x = \frac{|C|^2}{p(x)} \mathrm{d}x$



WKB approximation in 1D

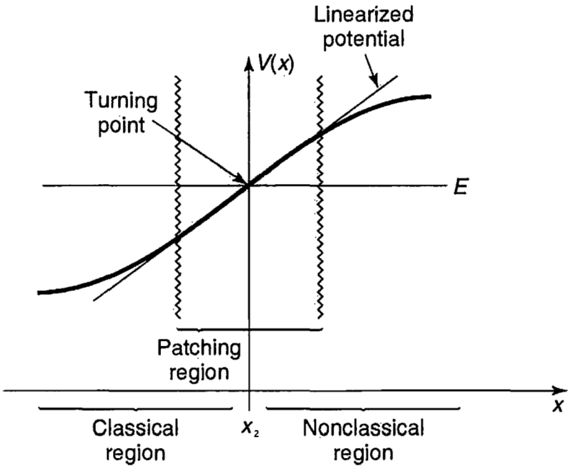
- the potential is linearized in the vicinity of the turning points x_T where $E = V(x_T)$

$$V(x) = E + V'(x_T)(x - x_T) + \dots$$

- the Schrödinger equation for this potential is

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\psi_p(u) - u\,\psi_p(u) = 0$$

where $u = \sqrt[3]{\frac{2mV'(x_T)}{\hbar^2}}(x - x_T)$

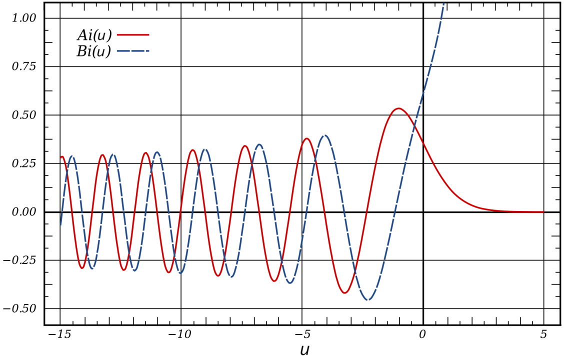


David J. Griffiths: Introduction to quantum mechanics

WKB approximation in 1D

- Airy equation and Airy functions [Griffiths '04 textbook]

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\psi_p(u) - u\,\psi_p(u) = 0 \qquad \Rightarrow \qquad \psi_p(u) \in \{Ai(u), Bi(u)\}$$

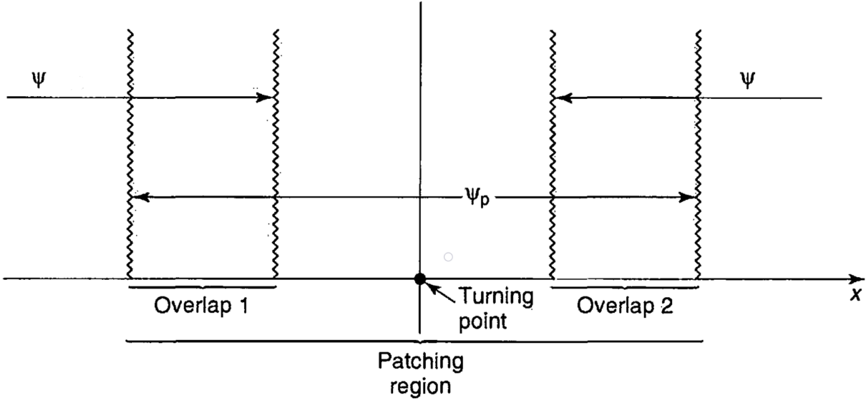


- because of the normalizability $\psi_p(u) = Ai(u) = \frac{1}{\pi}\int_0^\infty \mathrm{d}v \cos\left(\frac{v^3}{3} + uv\right)$

WKB approximation in 1D

- asymptotic behavior of the Airy function [Griffiths '04 textbook]

$$\psi_p(u) = Ai(u) \approx \begin{cases} \frac{1}{2\sqrt{\pi}u^{1/4}} e^{-\frac{2}{3}u^{3/2}} , & \text{for } u \gg 0 \iff x \gg x_T \\ \frac{1}{\sqrt{\pi}(-u)^{1/4}} \sin\left[\frac{2}{3}(-u)^{3/2} + \frac{\pi}{4}\right] , & \text{for } u \ll 0 \iff x \ll x_T \end{cases}$$



WKB approximation in 1D

- **WKB condition** in 1D [Griffiths '04 textbook]

$$\boxed{\frac{1}{\hbar}\int_{x_1}^{x_2} \mathrm{d}x\, p(x) = \left(n + \frac{1}{2}\right)\pi} \;,\; n \in \mathbb{Z}_0^+ \;,\; p(x_1) = p(x_2) = 0$$

- we have tested for the potential:
  - 1D linear harmonic oscillator $(E_n)_{\text{WKB}} = \left(n + \frac{1}{2}\right)\hbar\omega = E_n \;,\; n \in \mathbb{N}$
- Schrödinger equation in 1D for $\Psi(x)$

$$-\frac{\hbar^2}{2m}\Psi''(x) + V(x)\Psi(x) = E\Psi(x) \implies \Psi''(x) + \frac{1}{\hbar^2}\underbrace{2m\left[E - V(x)\right]}_{p^2(x)}\Psi(x) = 0$$

WKB approximation in 3D

[Band, Avishai '12 textbook]

- motivation for the WKB approximation in 3D for the radial wave function $R(r)$

$$0 = R''(r) + \frac{2}{r}R'(r) + \frac{2m}{\hbar^2}\left(E - V(r) - \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right)R(r)$$

  - in 1D we had $x \in \mathbb{R}$, in 3D we have $r \in \mathbb{R}_0^+$
    solution: mapping $r = e^x$
  - in 1D term $\Psi'(x)$ was not contained
    solution: we are searching for $R(r) \to R(x)$ in the form $R(x) = U(x)e^{f(x)}$, where $f(x)$ is chosen in the way that term $U'(x)$ vanishes

WKB approximation in 3D

- the Schrödinger equation obtains the form [Band, Avishai '12 textbook]

$$U''(x) + \frac{e^{2x}}{\hbar^2} \underbrace{2m\left(E - V(e^x) - \frac{(l+\frac{1}{2})^2\hbar^2}{2me^{2x}}\right)}_{p^2(e^x)} U(x) = 0$$

- **WKB condition** in 3D

$$\boxed{\frac{1}{\hbar}\int_{r_1}^{r_2} \mathrm{d}r\, p(r) = \left(n + \frac{1}{2}\right)\pi} \;, \quad n \in \mathbb{Z}_0^+ \;, \quad p(r_1) = p(r_2) = 0$$

- we have tested for the potential:
  - Coulomb potential $(E_N)_{\mathrm{WKB}} = -\frac{1}{2} m_e c^2 \alpha^2 \frac{1}{N^2} = E_N \;, \quad N \equiv (n + l + 1) \in \mathbb{N}$

Pekeris-type approximation

- WKB condition for the Cornell potential [BB, Tekel '23]

$$\frac{1}{\hbar}\int_{r_1}^{r_2} \mathrm{d}r \sqrt{2\mu\left(E-\left(-\frac{C}{r}+Br\right)-\frac{(l+\frac{1}{2})^2\hbar^2}{2\mu r^2}\right)}=\left(n+\frac{1}{2}\right)\pi\ ,\ n\in\mathbb{Z}_0^+$$

where $\mu=\frac{m_1 m_2}{m_1+m_2}$ is the reduced mass for the quarks

- **Pekeris-type approximation** – expand the integrand around the characteristic distance of the problem r_Q

$$r_Q=\sqrt{\frac{C}{B}}$$

Mass spectrum of the Cornell potential

- the energy levels of the Cornell potential [BB, Tekel '23]

$$E_{nl} = -\frac{2\mu}{\hbar^2}\left[\frac{2C}{n+\frac{1}{2}+\sqrt{\frac{2\mu}{\hbar^2}C\sqrt{\frac{C}{B}}+\left(l+\frac{1}{2}\right)^2}}\right]^2 + 3\sqrt{BC}$$

- the mass of the bound states is $M_{nl} = m_1 + m_2 + E_{nl}$
- the **mass spectrum of the mesons** [BB, Tekel '23]

$$\boxed{M_{nl} = (m_1+m_2) - \frac{2\mu}{\hbar^2}\left[\frac{2C}{n+\frac{1}{2}+\sqrt{\frac{2\mu}{\hbar^2}C\sqrt{\frac{C}{B}}+\left(l+\frac{1}{2}\right)^2}}\right]^2 + 3\sqrt{BC}}$$

Mass of quarks

https://fineartamerica.com/art/bottom+quark

$c\bar{c}$ meson

[1] [BB, Tekel '23]

measurement [Particle Data Group]

$c\bar{c}$ meson	$m_q = 1.27$ GeV	$B = 0.322$ GeV^2	$C = 0.891$

n	l	state	particle	**[1]** M_{nl} [GeV]	**[2]** [GeV]	**[3]** [GeV]	**meas.** [GeV]
0	0	1S	$J/\psi(1S)$	used for B, C	–	–	3.097
1	0	2S	$\psi(2S)$	used for B, C	–	–	3.686
2	0	3S	$\psi(4040)$	3.889	–	–	4.039
3	0	4S	$\psi(4230)$	3.982	4.266	4.262	4.223
0	1	1P	$\chi_{C1}(1P)$	3.518	3.262	3.258	3.511
1	1	2P	$\chi_{C2}(3930)$	3.823	3.784	3.779	3.923
0	2	1D	$\psi(3770)$	3.787	3.515	3.510	3.774

[2] [Omugbe, Osafile, Onyeaju '20]

[3] [Shady, Karim, Alarab '19]

$b\bar{b}$ meson

[1] [BB, Tekel '23]

measurement [Particle Data Group]

$b\bar{b}$ meson	$m_q = 4.18$ GeV	$B = 1.266$ GeV^2	$C = 0.344$

n	l	state	particle	**[1]** M_{nl} [GeV]	**[2]** [GeV]	**[3]** [GeV]	**meas.** [GeV]
0	0	1S	$\Upsilon(1S)$	used for B, C	–	–	9.460
1	0	2S	$\Upsilon(2S)$	used for B, C	–	–	10.023
2	0	3S	$\Upsilon(3S)$	10.178	10.365	10.360	10.355
3	0	4S	$\Upsilon(4S)$	10.242	10.588	10.580	10.579
0	1	1P	$h_b(1P)$	9.942	9.608	9.609	9.899
1	1	2P	$h_b(2P)$	10.150	10.110	10.109	10.260
0	2	1D	$\Upsilon_2(1D)$	10.140	9.841	9.846	10.164

[2] [Omugbe, Osafile, Onyeaju '20]

[3] [Shady, Karim, Alarab '19]

[1] [BB, Tekel '23] **measurement** [Particle Data Group]

$c\bar{b}$ meson	$\mu = 0.97$ GeV	$B = 604$ GeV^2	$C = 0.603$

n	l	state	particle	**[1]** M_{nl} [GeV]	**[2]** [GeV]	**[3]** [GeV]	**meas.** [GeV]
0	0	1S	B_c^+	used for B, C	–	–	6.274
1	0	2S	$B_c^\pm(2S)$	used for B, C	6.845	7.383	6.871
2	0	3S	–	7.054	7.125	7.206	none
3	0	4S	–	7.132	7.283	–	none
0	1	1P	–	6.749	6.519	7.042	none
1	1	2P	–	7.009	6.959	6.663	none
0	2	1D	–	6.989	6.813	–	none

[2] [Omugbe, Osafile, Onyeaju '20] **[3]** [Shady, Karim, Alarab '19]

Length scale of the mesons

[BB, Tekel '23]

- the **typical distance of the mesons** is the characteristic distance of the potential

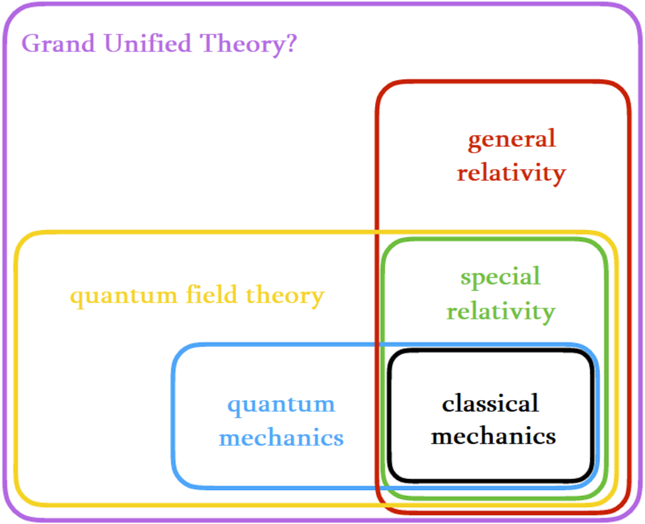
$$\boxed{r_Q = \sqrt{\frac{C}{B}}}$$

meson	μ [GeV c^{-2}]	B [GeV fm^{-1}]	C [GeV fm]	r_Q [10^{-16} m]
$c\bar{c}$	0.64	1.633	0.175	3.28
$b\bar{b}$	2.09	6.425	0.068	1.03
$c\bar{b}$	0.97	3.067	0.119	1.97

- for the typical distance of the mesons r_Q we got 10^{-16} m, which is in correspondence with the measurement [Satz '06]

What about quarkonia in the (non)commutative space?

Why are we interested in non-commutativity?

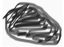


What does "non-commutative" mean?

- commutative/standard quantum mechanics

$$[\hat{x}_i, \hat{x}_j] = 0 \;\Rightarrow\; \Delta x_i \, \Delta x_j \geq 0$$

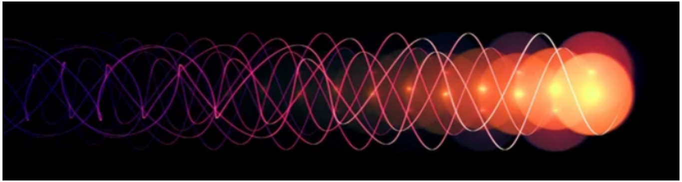
- non-commutative (NC) quantum mechanics

$$[\hat{x}_i, \hat{x}_j] \neq 0 \;\Rightarrow\; \Delta x_i \, \Delta x_j \geq$$ 

Planck length – the scale of non-commutativity

$$\text{reduced Planck constant } \hbar = 1.055 \times 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}$$
$$\text{gravitational constant } \kappa = 6.674 \times 10^{-11}\,\mathrm{kg^{-1}\,m^3\,s^{-2}}$$
$$\text{the speed of light in vacuum } c = 2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$$

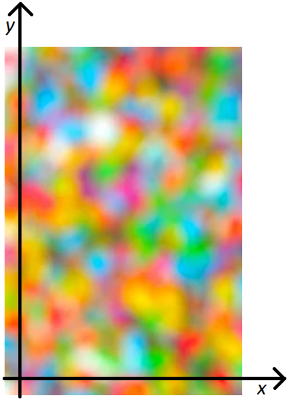
Planck length $\lambda = \sqrt{\frac{\hbar\,\kappa}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$



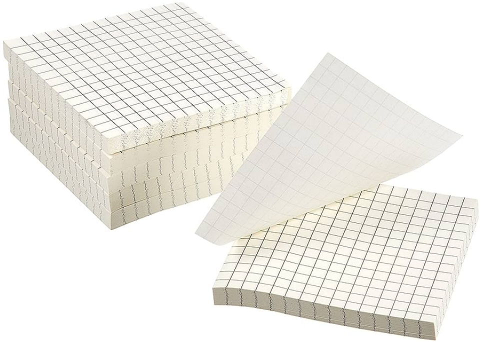
How we can(not) model the 3D non-commutative space...

- an obvious choice is to take the commutator

$$[\hat{x}, \hat{y}] = i\theta^2 \;\Rightarrow\; \Delta x\, \Delta y \geq \tfrac{1}{2}\theta^2$$



- and layer these non-commutative planes

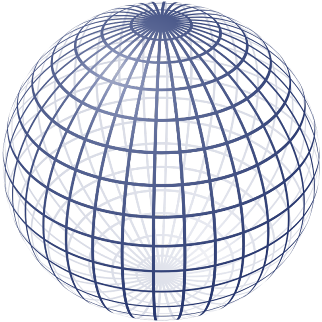


- this setup breaks the rotational invariance of the space!

fuzzy sphere

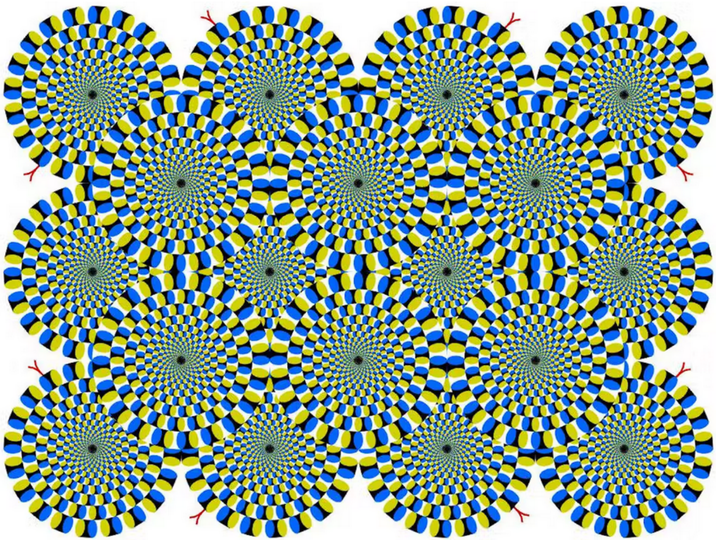
- **fuzzy sphere** – rotationally invariant non-commutative geometric model: $[x_i, x_j] = 2i\lambda\varepsilon_{ijk}x_k$, where $\lambda \approx 10^{-35}$ m is the Planck length [Hoppe '87] [Madore '91]

usual sphere
$[x_i, x_j] = 0$



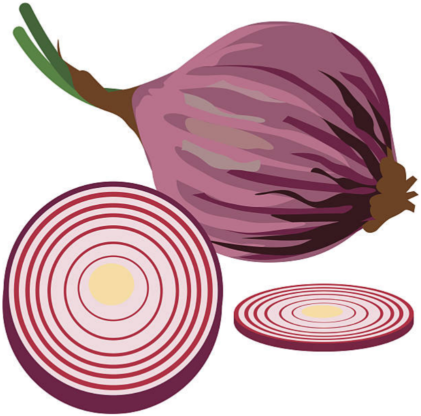
fuzzy sphere
$[x_i, x_j] = 2i\lambda\varepsilon_{ijk}x_k$

fuzzy sphere



...and how we can model the 3D non-commutative space

- the NC space can be modelled as a set of concentric fuzzy spheres with increasing radius by λ [Hammou, Lagraa, Sheikh-Jabbari '02]



The auxiliary Fock space

- non-commuting x_js have lost their meaning of being coordinates and these !!! non-commuting x_js become operators !!! [Balachandran, Kürkcüoğlu, Vaidya '06]
- we introduce the auxiliary operators
  - creation $a^\dagger$
  - annihilation a

$$\left[a_\alpha, a_\beta^\dagger\right] = \delta_{\alpha\beta} \ , \ \left[a_\alpha, a_\beta\right] = \left[a_\alpha^\dagger, a_\beta^\dagger\right] = 0 \ ; \quad \alpha, \beta \in \{1, 2\}$$

- the bosonic operators create vector states spanning the auxiliary Fock space

$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{\sqrt{n_1!\, n_2!}} |0, 0\rangle \ , \quad \text{where} \quad a_1 |0, 0\rangle = a_2 |0, 0\rangle = 0$$

- !!! these bosonic states have nothing common with the probed real system !!!

The position operator

[Balachandran, Kürkcüoğlu, Vaidya '06]

- the position operator $x_j(a, a^\dagger)$ in the form

$$x_j = \lambda \sigma^j_{\alpha\beta} a^\dagger_\alpha a_\beta \ , \ j \in \{1, 2, 3\}$$

is obeying $[x_i, x_j] = 2\mathrm{i}\lambda \varepsilon_{ijk} x_k$, where σ^j is the j$^{\text{th}}$ Pauli matrix

- operator of the length of the radius vector $r = \lambda \left(a^\dagger_\alpha a_\alpha + 1 \right)$, where $x_i x_i + \lambda^2 = r^2$
- the wave functions Ψ become

$$\Psi(x_i) \Longleftrightarrow \Psi(a, a^\dagger) = \sum C_{m_1 m_2 n_1 n_2} (a^\dagger_1)^{m_1} (a^\dagger_2)^{m_2} (a_1)^{n_1} (a_2)^{n_2}$$

The states Ψ in the Hilbert space

[Grosse, Klimčík, Prešnajder '96]

- integration over the space becomes Hilbert-Schmidt norm with the weight $w(r)$

$$\int \mathrm{d}^3 x \cdot \longrightarrow \mathrm{Tr}\,[\,\mathrm{w(r)} \cdot\,]$$

- the weight $w(r) = 4\pi\lambda^2 r$ is determined by the known volume of the ball

$$V_{\mathrm{ball}} = \mathrm{Tr}\,[\,4\pi\lambda^2 r\,]_{n_1+n_2 \le n} = 4\pi\lambda^2 \sum_{n_1,\,n_2}^{n_1+n_2=n} \langle n_1, n_2|\, r\,|n_1, n_2\rangle = \frac{4\pi}{3} r^3 + o\left(\frac{\lambda}{r}\right)$$

- the probability density of the wave function Ψ is

$$||\Psi||^2 = 4\pi\lambda^2 \mathrm{Tr}\,[\,r\,\Psi^\dagger \Psi\,]$$

Separation of variables

[Grosse, Klimčík, Prešnajder '96]

- we probe only spherically symmetric potentials, i.e. $V(\mathbf{r}) = V(r)$
- separation of variables, where $\tilde{r} = \lambda a_\alpha^\dagger a_\alpha$

$$\Psi_{lm} = \lambda^l \sum_{l,m} \frac{(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}}{m_1!m_2!} : \mathcal{K}_l(\tilde{r}) : \frac{a_1^{n_1}(-a_2)^{n_2}}{n_1!n_2!} \iff \Psi(\mathbf{r}) = R_l(r) Y_{lm}(\vartheta, \varphi)$$

$$: \mathcal{R}_l(\tilde{r}) : \; = \; : \mathcal{K}_l(\tilde{r}) \tilde{r}^l : \iff R_l(r) = K_l(r) r^l$$

- $: \;\; :$ denotes the normal ordering, i.e. $: a_\alpha a_\alpha^\dagger : \; = a_\alpha^\dagger a_\alpha$

The essence of normal ordering

[Grosse, Klimčík, Prešnajder '96]

- we need the analogy of derivation
  - linear
  - obeying Leibniz rule
  - resembles $\partial_x x^k = k\, x^{k-1}$
- normal ordering, where N is the particle number operator $N = a^\dagger_\alpha a_\alpha$

$$N^k = (a^\dagger_{\beta_1} a_{\beta_1}) \dots (a^\dagger_{\beta_k} a_{\beta_k}) \qquad :N^k: \; = (a^\dagger_{\beta_1} \dots a^\dagger_{\beta_k})(a_{\beta_1} \dots a_{\beta_k})$$

- the analogy of the derivative

$$\left[a_\alpha, :N^k:\right] = k :N^{k-1}: a_\alpha \qquad \left[a^\dagger_\alpha, :N^k:\right] = -k\, a^\dagger_\alpha :N^{k-1}:$$

Construction of the Laplace operator

[Gáliková, Prešnajder, Kováčik '15]

- *1. step*: it is differential operator $\Longrightarrow$ we need $\left[a_\alpha, \cdot\right]$ and $\left[a^\dagger_\alpha, \cdot\right]$
- *2. step*: it is operator of the 2nd order $\Longrightarrow$ we have $\hat{\Delta}_\lambda \Psi \sim [\cdot, [\cdot, \Psi]]$

  putting them together $\hat{\Delta}_\lambda \Psi \sim \left[\hat{a}^\dagger_\alpha, \left[\hat{a}_\alpha, \Psi\right]\right] = \left[\hat{a}_\alpha, \left[\hat{a}^\dagger_\alpha, \Psi\right]\right]$
- *3. step*: insure the right dimension $\Longrightarrow$ we have $\hat{\Delta}_\lambda \Psi \sim \frac{1}{\lambda r}\left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \Psi]\right]$
- *4. step*: test on elementary functions $\Longrightarrow$ we acquire $\hat{\Delta}_\lambda \Psi = -\frac{1}{\lambda r}\left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \Psi]\right]$

Action of the operators

[Gáliková, Prešnajder, Kováčik '15]

- operator of the length of the radius vector $r = \lambda\left(a_\alpha^\dagger a_\alpha + 1\right)$

$$\hat{r}\Psi_{lm} = \lambda^l \sum_{l,m} \frac{(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}}{m_1!m_2!} : \left[\tilde{r}\mathcal{K} + \lambda(l+1)\mathcal{K} + \lambda\tilde{r}\mathcal{K}'\right] : \frac{a_1^{n_1}\left(-a_2\right)^{n_2}}{n_1!n_2!}$$

- Laplace operator $\hat{\Delta}_\lambda\Psi = -\frac{1}{\lambda r}\left[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \Psi]\right]$

$$\hat{\Delta}_\lambda\Psi_{lm} = -\frac{\lambda^l}{\lambda r}\sum_{l,m} \frac{(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}}{m_1!m_2!} : \left[-\lambda\tilde{r}\mathcal{K}'' - 2(l+1)\lambda\mathcal{K}'\right] : \frac{a_1^{n_1}\left(-a_2\right)^{n_2}}{n_1!n_2!}$$

- Hamilton operator $\hat{H}_\lambda\Psi = \left[-\frac{\hbar^2}{2\mu}\hat{\Delta}_\lambda + V(\hat{r})\right]\Psi$

Hydrogen atom in the noncommutative space

Exact solution for the NC Coulomb potential

- the Schrödinger equation for the NC Coulomb potential [Gáliková, Prešnajder, Kováčik '15]

$$\frac{\hbar^2}{2m_e\lambda r}\left[\hat{a}^\dagger_\alpha,[\hat{a}_\alpha,\Psi]\right]-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\Psi=E\Psi$$

- switching to dimensionless quantities

$$\epsilon=\frac{E}{-\frac{1}{2}m_ec^2\alpha^2}\ ,\quad \rho=\frac{r}{a_B}\ ,\quad \sigma=\frac{\lambda}{a_B}\ ,\quad a_B=\frac{\hbar}{m_ec\alpha}\ ,\quad \alpha=\frac{e^2}{4\pi\varepsilon_0\hbar c}$$

- the radial Schrödinger equation for the operators in the NC space...

$$:\tilde{\rho}\mathcal{K}''+[-\epsilon\sigma\tilde{\rho}+2(l+1)]\mathcal{K}'+[-\epsilon\tilde{\rho}-\epsilon\sigma(l+1)+2]\mathcal{K}:\ =\ 0$$

Exact solution for the NC Coulomb potential

- ...to which we assign the equivalent ordinary differential equation

$$\rho K'' + [-\epsilon\sigma\rho + 2(l+1)]K' + [-\epsilon\rho - \epsilon\sigma(l+1) + 2]K = 0$$

- the radial Schrödinger equation of the NC Coulomb potential for the wave function $R(r) = K(r)r^l$

$$R'' + \left(\frac{2}{\rho} - \sigma\epsilon\right)R' + \left(-\epsilon + \frac{2}{\rho} - \frac{l(l+1)}{\rho^2} - \sigma\frac{\epsilon}{\rho}\right)R = 0$$

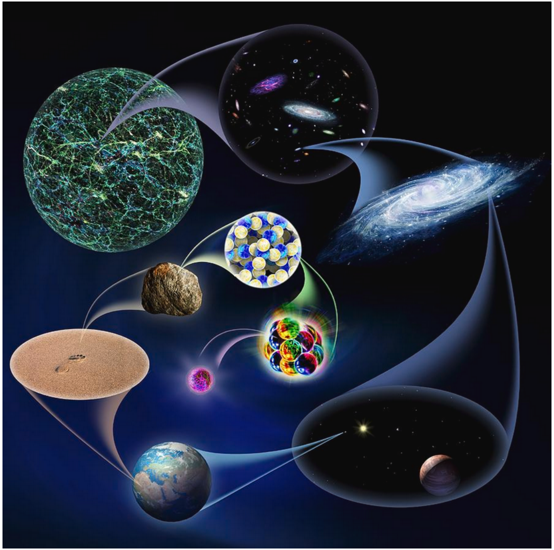
- the **modified energy spectrum**

[Gáliková, Prešnajder, Kováčik '15] [BB, Tekel '23]

$$E_N^\lambda = \frac{\hbar^2}{m_e\lambda^2}\left(1 - \sqrt{1 + \frac{m_e^2c^2\alpha^2}{\hbar^2}\frac{\lambda^2}{N^2}}\right) = -\frac{1}{2}m_ec^2\alpha^2\frac{1}{N^2} + \frac{\lambda^2}{a_B^2}\frac{1}{2}m_ec^2\alpha^2\frac{1}{4N^4} + \ldots = (E_N^\lambda)_{\mathrm{WKB}}$$

What about quarkonia in the noncommutative space?

Length scale of the mesons



- typical distance of the mesons [BB, Tekel '23]

meson	r_Q $[10^{-16}\,\mathrm{m}]$
$c\bar{c}$	3.28
$b\bar{b}$	1.03
$c\bar{b}$	1.97

Approximate solution for the NC Cornell potential

- the Schrödinger equation for the NC Cornell potential [BB, Tekel '23]

$$\frac{\hbar^2}{2\mu\lambda r}\left[\hat{a}^\dagger_\alpha,[\hat{a}_\alpha,\Psi]\right]+\left(-\frac{C}{r}+B\hat{r}\right)\Psi=E\Psi$$

- switching to dimensionless quantities

$$\zeta=\frac{r}{r_Q}\,,\quad r_Q=\sqrt{\frac{C}{B}}\,,\quad \epsilon=\frac{2\mu E r_Q^2}{\hbar^2}\,,\quad c=\frac{2\mu C r_Q}{\hbar^2}\,,\quad b=\frac{2\mu B r_Q^3}{\hbar^2}\,,\quad \sigma=\frac{\lambda}{r_Q}$$

- the radial Schrödinger equation for the NC Cornell potential

$$0=R''+\frac{2}{\zeta}R'-\frac{l(l+1)}{\zeta^2}R+\left(\frac{c}{\zeta}-b\zeta\right)R+\epsilon R+\sigma\left(\epsilon R'+\frac{\epsilon}{\zeta}R-2b\zeta R'-3bR\right)+$$
$$+\sigma^2\left(-b\zeta R''-3bR'-\frac{b}{\zeta}R\right)$$

Approximate solution for the NC Cornell potential

- the **modified mass spectrum** of the mesons is $M_{nl}^{\sigma} = M_{nl} + \sigma^2 M_{nl}^{(2)} + \ldots$

$$M_{nl}^{\sigma} = \left((m_1 + m_2) - \frac{2\mu}{\hbar^2} \left[\frac{2C}{n + \frac{1}{2} + \sqrt{\frac{2\mu}{\hbar^2} C \sqrt{\frac{C}{B}} + \left(l + \frac{1}{2}\right)^2}} \right]^2 + 3\sqrt{BC} \right) +$$

$$+ \sigma^2 \frac{\hbar^2}{2\mu} \frac{B}{C} \left(\frac{b\left(105b^2 + 62bc + 9c^2\right)}{8\left[n + \frac{1}{2} + \sqrt{b + (l + \frac{1}{2})^2}\right]^2} + \frac{b(c+3b)l(l+1)}{2\left[n + \frac{1}{2} + \sqrt{b + (l + \frac{1}{2})^2}\right]^2} + \right.$$

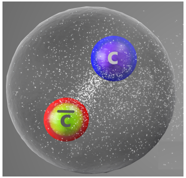
$$+ \frac{b(c+3b)^4}{8\sqrt{b + (l + \frac{1}{2})^2}\left[n + \frac{1}{2} + \sqrt{b + (l + \frac{1}{2})^2}\right]^5} - \frac{b}{4}(15b + 4c) -$$

$$\left. - \frac{(45b - c)(c+3b)^3}{64\left[n + \frac{1}{2} + \sqrt{b + (l + \frac{1}{2})^2}\right]^4} \right) + \ldots$$

NC $c\bar{c}$ meson

[BB, Tekel '23]

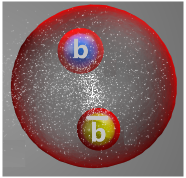
NC $c\bar{c}$ meson	$b = c = 1.883\ ,\quad \sigma^2 \approx 0.93 \times 10^{-39}$
state	correction to the mass $\sigma^2 M_{nl}^{(2)}$ [GeV]
1S	$0.522\,\sigma^2$
2S	$-1.422\,\sigma^2$
3S	$-2.613\,\sigma^2$
4S	$-3.301\,\sigma^2$
1P	$-0.456\,\sigma^2$
2P	$-1.936\,\sigma^2$
1D	$-1.062\,\sigma^2$



NC $b\bar{b}$ meson

[BB, Tekel '23]

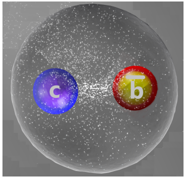
NC $b\bar{b}$ meson	$b = c = 0.750\ ,\quad \sigma^2 \approx 9.43 \times 10^{-39}$
state	correction to the mass $\sigma^2 M_{nl}^{(2)}$ [GeV]
1S	$-0.261\,\sigma^2$
2S	$-1.289\,\sigma^2$
3S	$-1.753\,\sigma^2$
4S	$-1.973\,\sigma^2$
1P	$-0.738\,\sigma^2$
2P	$-1.480\,\sigma^2$
1D	$-1.042\,\sigma^2$



NC c$\bar{b}$ meson

[BB, Tekel '23]

NC c$\bar{b}$ meson	$b = c = 1.174\ ,\quad \sigma^2 \approx 2.58 \times 10^{-39}$
state	correction to the mass $\sigma^2 M_{nl}^{(2)}$ [GeV]
1S	$-0.001\,\sigma^2$
2S	$-1.442\,\sigma^2$
3S	$-2.210\,\sigma^2$
4S	$-2.609\,\sigma^2$
1P	$-0.672\,\sigma^2$
2P	$-1.777\,\sigma^2$
1D	$-1.147\,\sigma^2$



Upper limit for the length scale λ

- the most precisely measured particle is the $J/\psi(1S)$, a $c\bar{c}$ meson [Particle Data Group]

$$M_{00} = 3096.900 \pm 0.006\,\text{MeV}$$

- with the help of this we can estimate the upper limit for the fundamental length scale of the non-commutativity [BB, Tekel '23]

$$\lambda \leq 1.11 \times 10^{-18}\,\text{m}$$

Future prospect

- NC hydrogen atom

$$E_N^\lambda = -\frac{1}{2} m_e c^2 \alpha^2 \frac{1}{N^2} + \boxed{\frac{\lambda^2}{a_B^2}} \frac{1}{2} m_e c^2 \alpha^2 \frac{1}{4N^4} + \dots$$

- NC quarkonium

$$M_{nl}^\sigma = \left((m_1 + m_2) - \frac{2\mu}{\hbar^2} \left[\frac{2C}{n + \frac{1}{2} + \sqrt{\frac{2\mu}{\hbar^2} C \sqrt{\frac{C}{B}} + \left(l + \frac{1}{2}\right)^2}} \right]^2 + 3\sqrt{BC} \right) +$$

$$+ \boxed{\sigma^2} \frac{\hbar^2}{2\mu} \frac{B}{C} \left(\frac{b\left(105b^2 + 62bc + 9c^2\right)}{8\left[n + \frac{1}{2} + \sqrt{b + (l + \frac{1}{2})^2}\right]^2} + \frac{b(c+3b)l(l+1)}{2\left[n + \frac{1}{2} + \sqrt{b + (l + \frac{1}{2})^2}\right]^2} + \right.$$

$$\left. + \frac{b(c+3b)^4}{8\sqrt{b + (l + \frac{1}{2})^2}\left[n + \frac{1}{2} + \sqrt{b + (l + \frac{1}{2})^2}\right]^5} - \frac{b}{4}(15b + 4c) - \frac{(45b - c)(c+3b)^3}{64\left[n + \frac{1}{2} + \sqrt{b + (l + \frac{1}{2})^2}\right]^4} \right) + \dots$$

- find a system or an effect for which the NC correction is the first order in σ

Source of the images

- https://www.nikhef.nl/www/wp-content/uploads/2021/07/TccLoosely.png
- https://www.splashlearn.com/math-vocabulary/geometry/sphere
- https://images.theconversation.com/files/305381/original/file-20191205-38984-btnq03.jpg?ixlib=rb-1.1.0q=30auto=formatw=600h=600fit=cropdpr=2
- https://hdclipartall.com/onion-clipart-red-onion-vector-art-illustration-612-225778.html
- https://www.canstockphoto.com/abstract-blur-color-background-12165881.html
- https://www.amazon.com/Pieces-Stickiness-Self-Stick-Teachers-Students/dp/B093758FZM
- https://www.icr.org/article/subatomic-particles-part-3-mesons
- https://pixels.com/featured/1-universe-to-quark-harald-ritschscience-photo-library.html
- https://www.quora.com/What-are-quarks-made-of
- https://cds.cern.ch/images/CERN-GRAPHICS-2022-027-6/file?size=large
- https://en.wikipedia.org/wiki/Sphere
- http://www.rmki.kfki.hu/ jancso/RDM_Mappak/3_RF_Dioban/1_Reszecskek.pdf

Köszönöm a figyelmet!

Thank you for your attention!