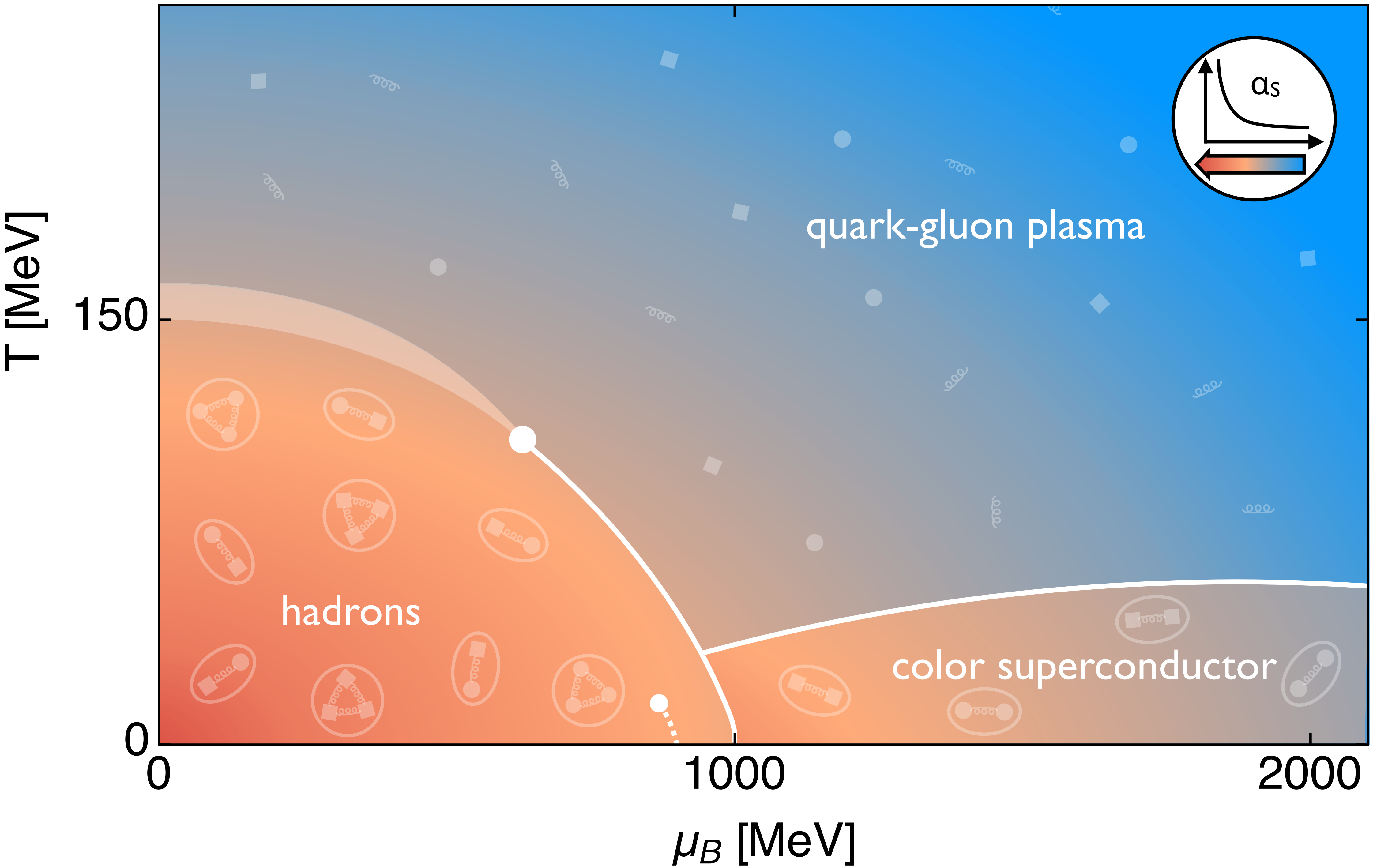
CRITICALITY IN QCD

Fabian Rennecke

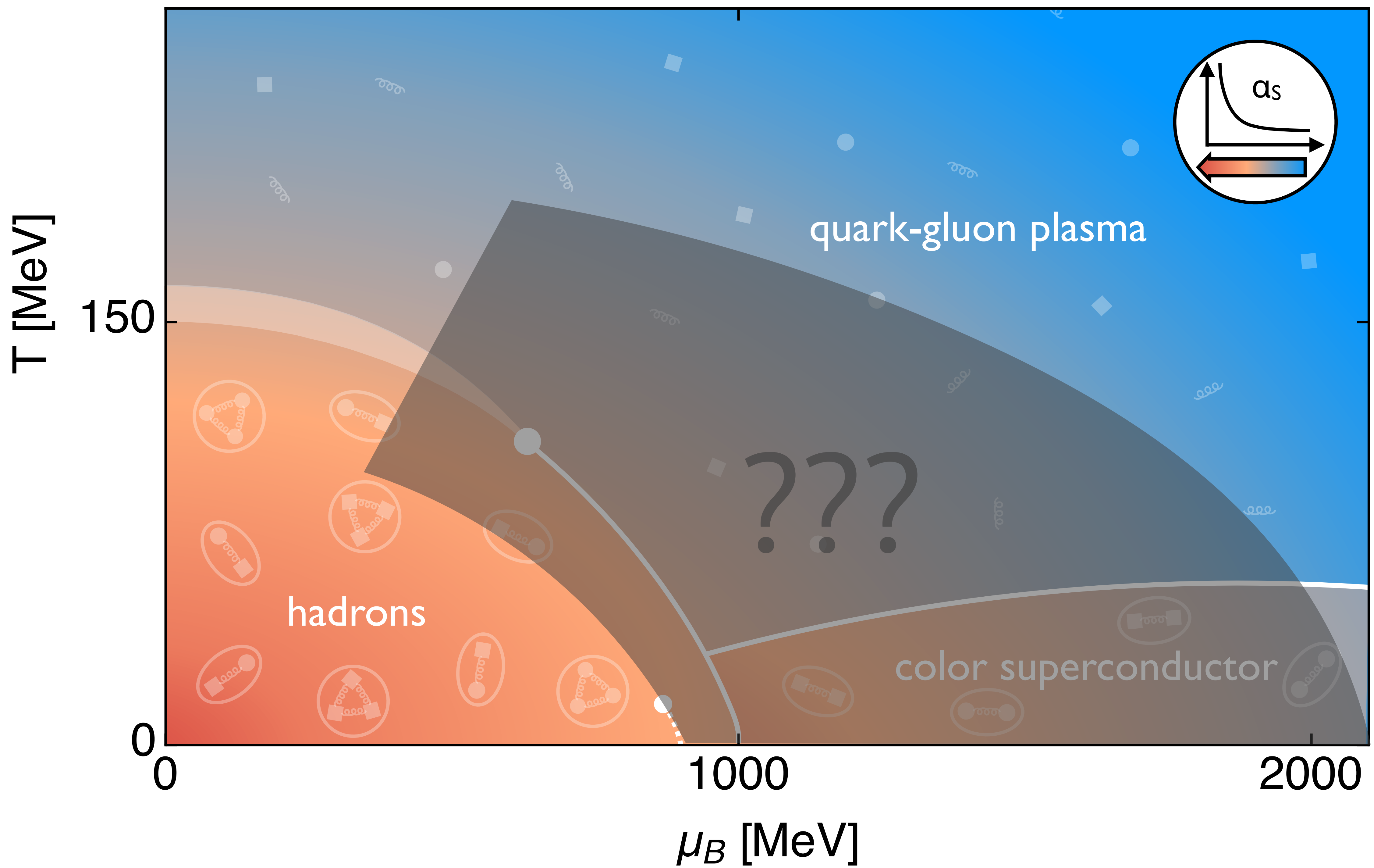
[Haensch, FR, von Smekal, arXiv:2308.16244]
[Herl, FR, Schmidt, von Smekal (in preparation)]

THEORY SEMINAR
BUDAPEST - 21/11/2023

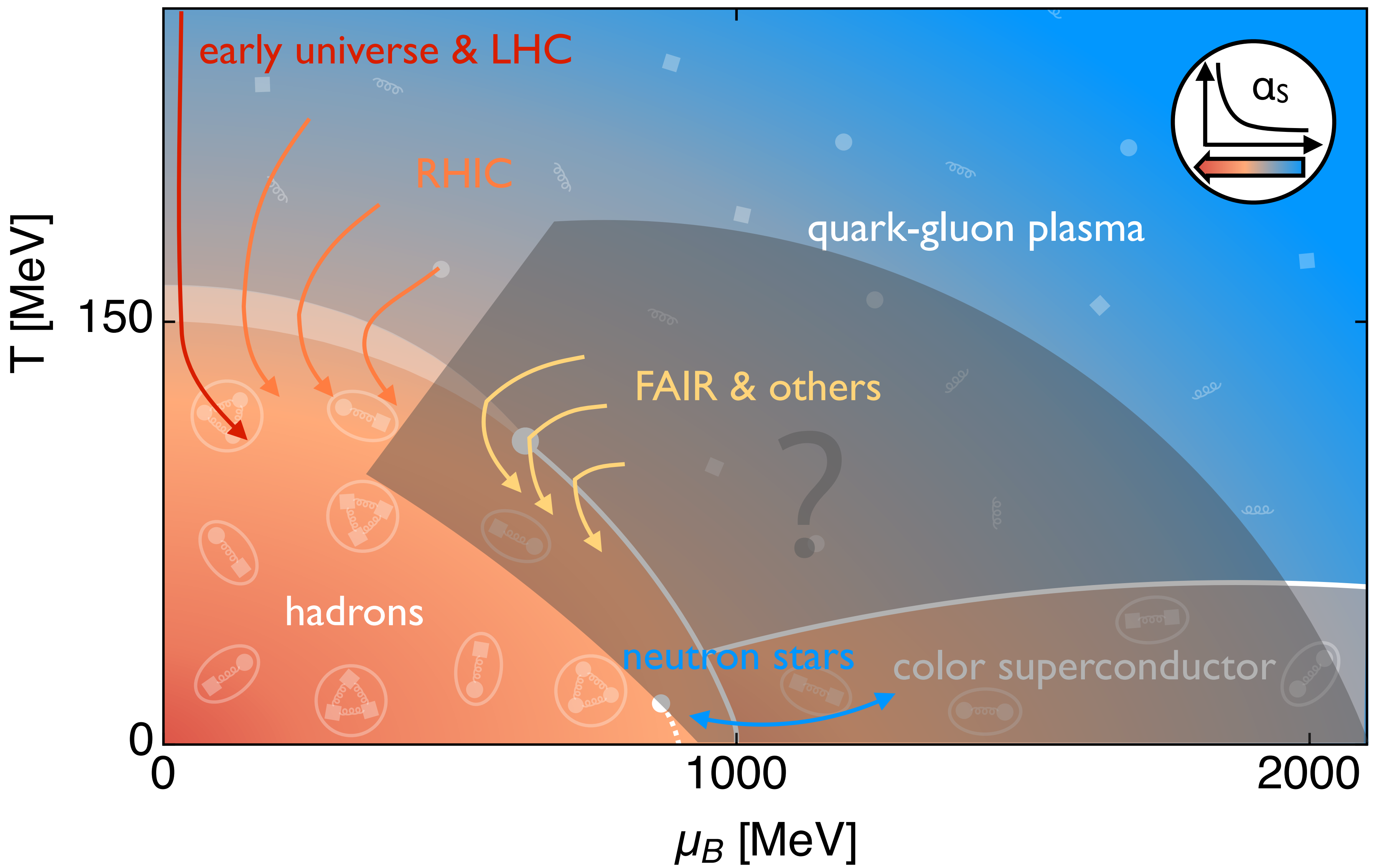
QCD PHASE DIAGRAM



QCD PHASE DIAGRAM

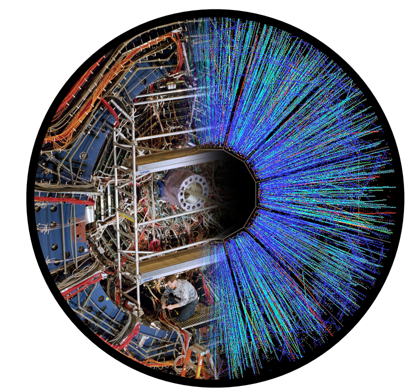


QCD PHASE DIAGRAM

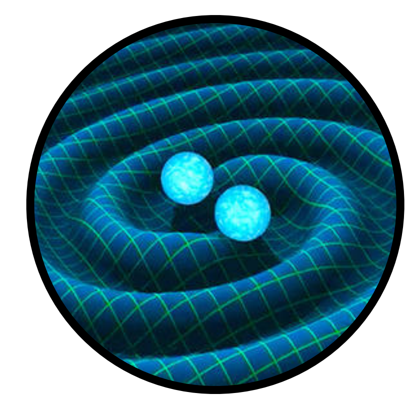


Experiments:

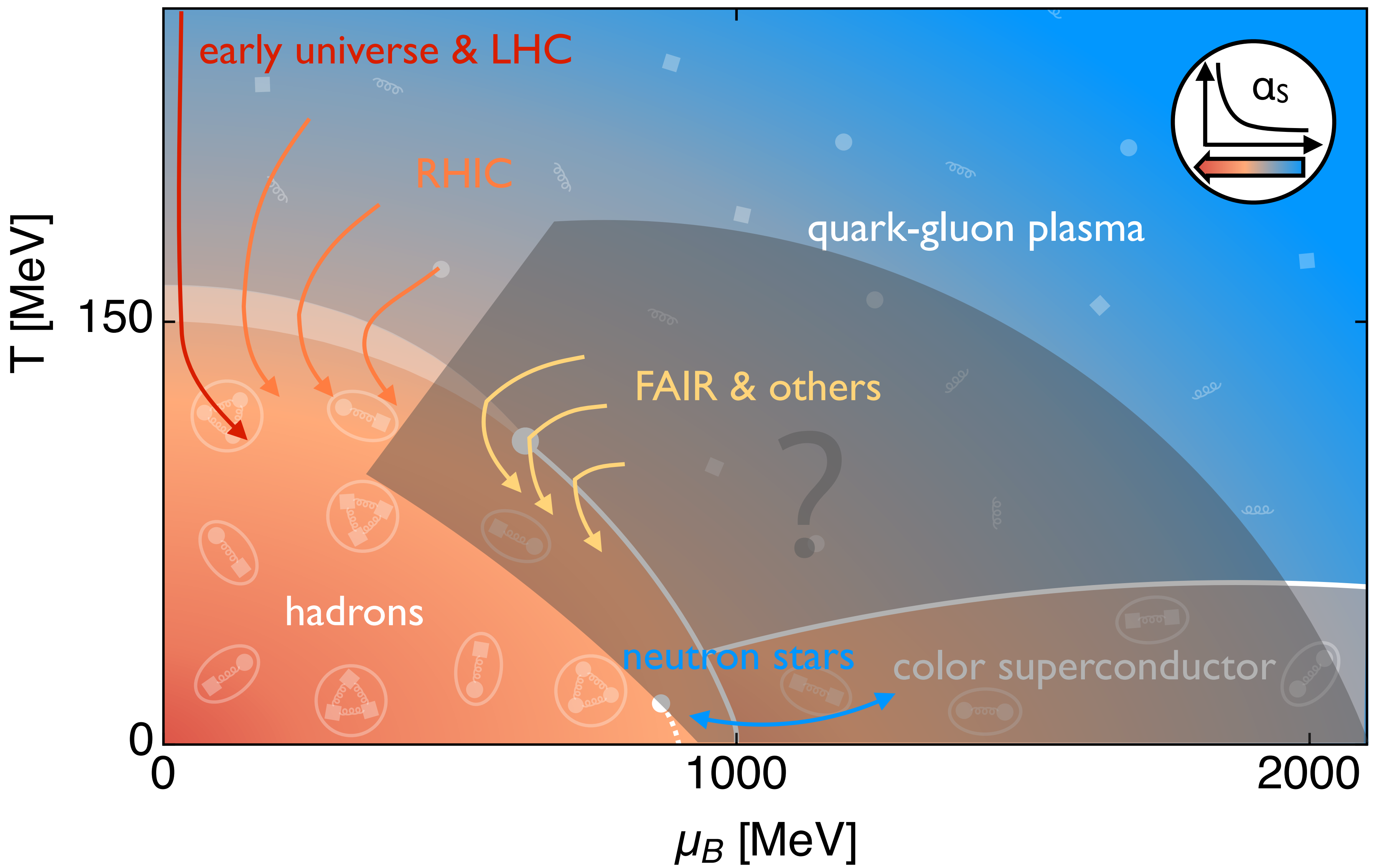
heavy-ion collisions



e.g. gravitational waves

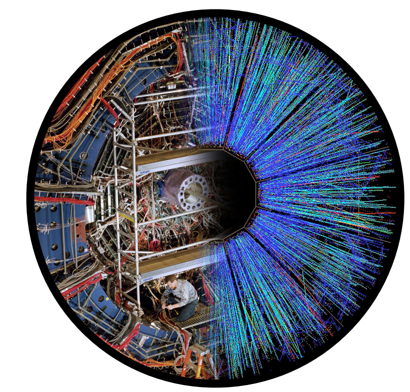


QCD PHASE DIAGRAM

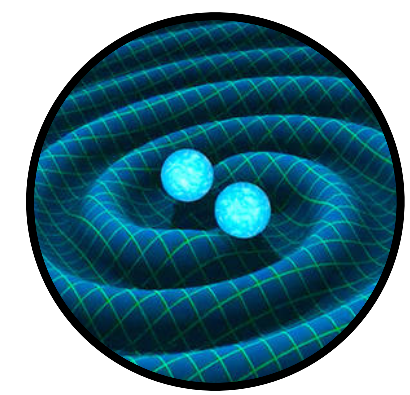


Experiments:

heavy-ion collisions



e.g. gravitational waves



- direct access to the actual transitions difficult both in theory and experiment
- universality and analytic structure near 2nd-order transitions can yield powerful constraints

⟶ can this be leveraged for our understanding of the phase diagram?

OUTLINE

- Yang-Lee edge singularities and 2nd order phase transitions
- medium-induced mixing and critical modes in QCD
- the Columbia plot and YLEs

YANG-LEE EDGE SINGULARITIES & 2ND ORDER PHASE TRANSITIONS

LEE-YANG THEORY

phase structure $\longleftrightarrow$ **analytic structure in the complex plane**

[Yang, Lee (1952)]

Consider a system of N atoms with

- a finite size and a hard core (short-range repulsion)
- finite interaction range (required for well-defined thermodynamic limit)
- the interaction is nowhere $-\infty$ (potential is bounded from below)

The grand canonical partition function is a polynomial of degree N in a finite volume V,

fugacity $z \sim e^{\mu/T}$,
μ: chemical potential, magnetic field, source,...

$$Z_V = \sum_{i=1}^{N} \mathcal{Z}_i(T)\, z^i = \prod_{i=1}^{N} \left(z - z_i\right)$$

canonical partition function
with i particles

Yang-Lee zeros

LEE-YANG THEOREMS

Given these assumptions, Lee and Yang have proven two theorems:

- For all $z > 0$ the free energy density,

$$f(T,z) = -T \lim_{V\to\infty} \frac{1}{V} \ln Z_V(T,z)$$

is a continuous, monotonically increasing function of z. The limit is independent of the shape of V.

- If in a region R in the complex z-plane Z_V is free of zeros, then all thermodynamic quantities are analytic functions of z in R for $V \to \infty$

[Yang, Lee (1952)]

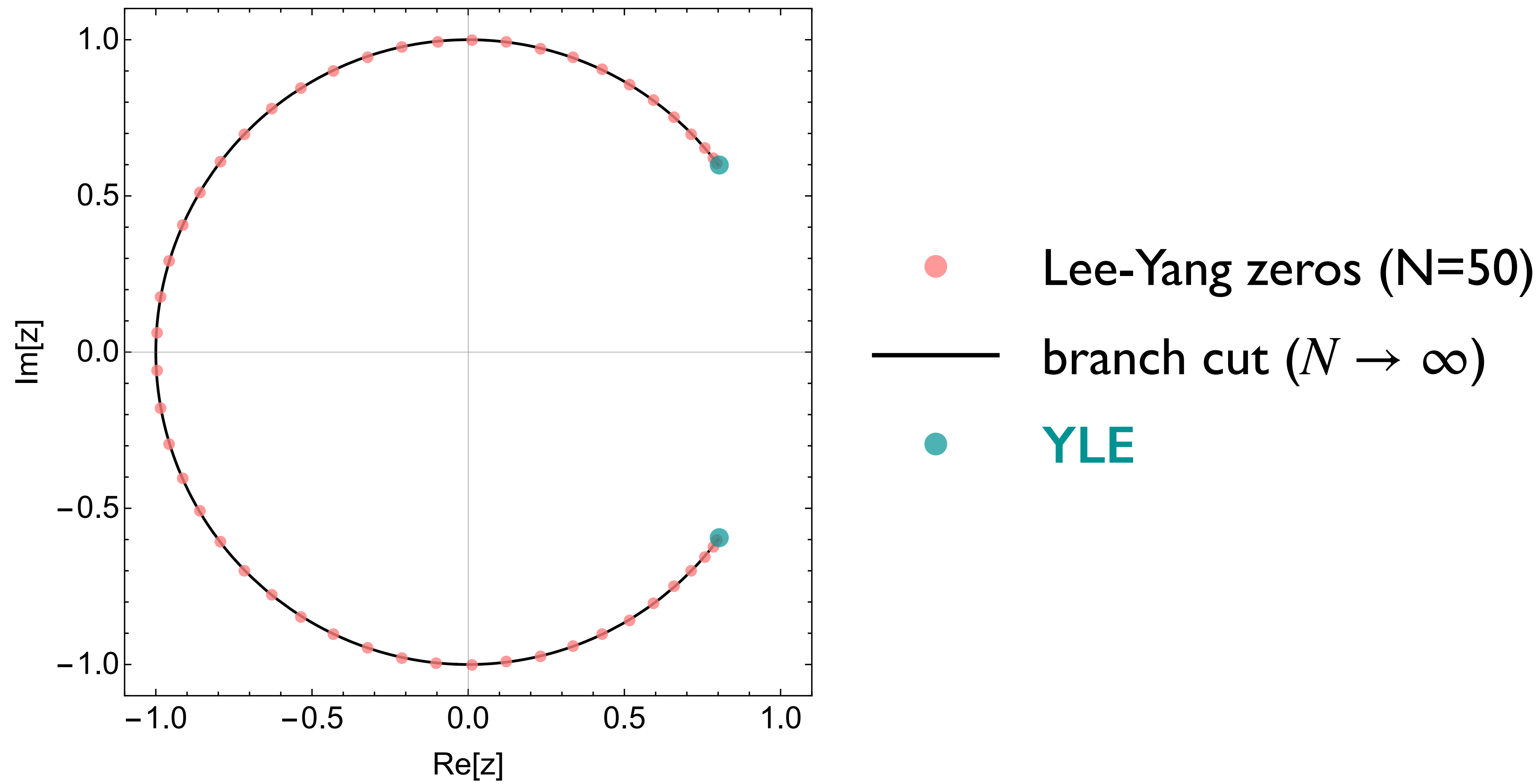
At a phase transition thermodynamic functions can't be analytic

- Lee-Yang zeros are poles of the free energy
- For $V \to \infty$ they coalesce into branch cuts in the complex z-plane
- The branch points (ends of the cuts) are called **Yang-Lee edge singularities (YLE)**

⟶ **Lee-Yang zeros/cuts & YLEs encode phase structure**

YANG-LEE EDGE SINGULARITY

Example: analytic structure of the free energy density of the 1d Ising model, $z = e^{2h/T}$



- no thermal phase transition in 1d Ising: YLE never touches the real, positive axis
- zeros/cut on the unit circle/at purely imaginary h: **Lee-Yang circle theorem**

All zeros/cuts/YLEs are at imaginary magnetic fields

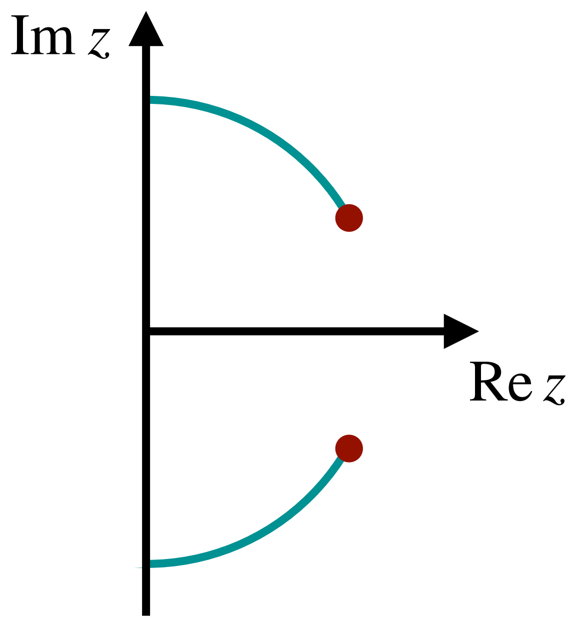
- rigorously proven for ferromagnetic spin-1/2 systems and $O(N = 1,2,3,\infty)$
- systematic results suggest that it holds for all N

[Lee, Yang (1952); Simon, Griffith (1972); Dunlop, Newman (1975); Lieb, Sokal (1981); Kurtze, Fisher (1978); Johnson, FR, Skokov (2020 & 2023)]

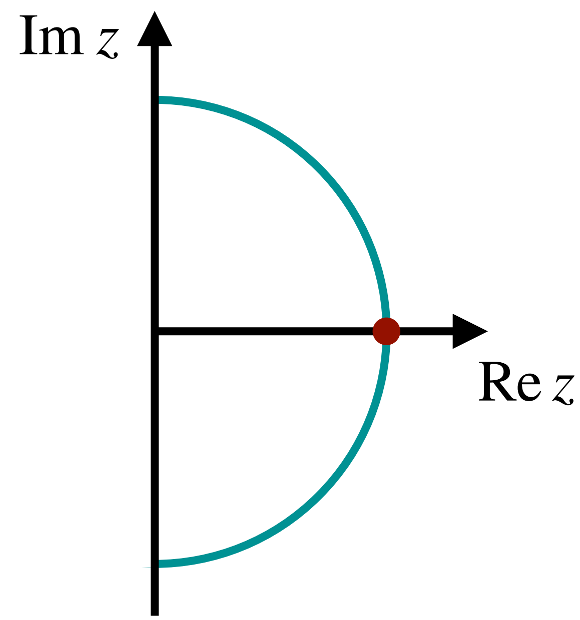
YLE & PHASE TRANSITIONS

Phase transition can be understood from the location of the YLE:

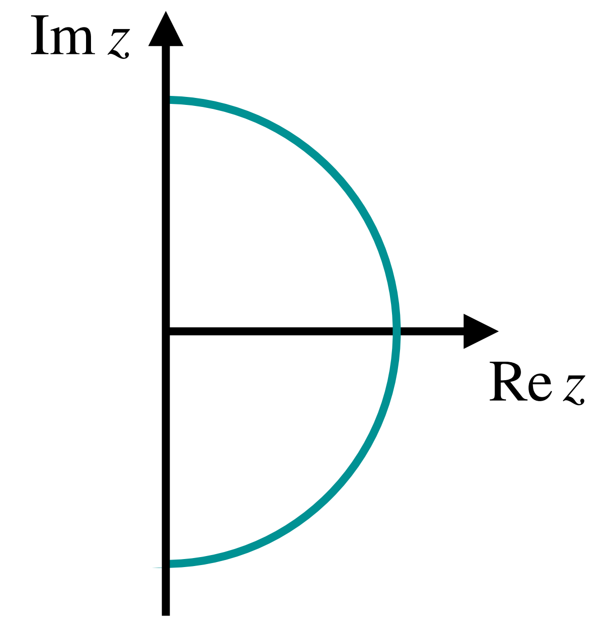
no phase transition
YLE in the complex plane



2nd order transition
YLE pinches real axis



1st order transition
cut cuts real axis

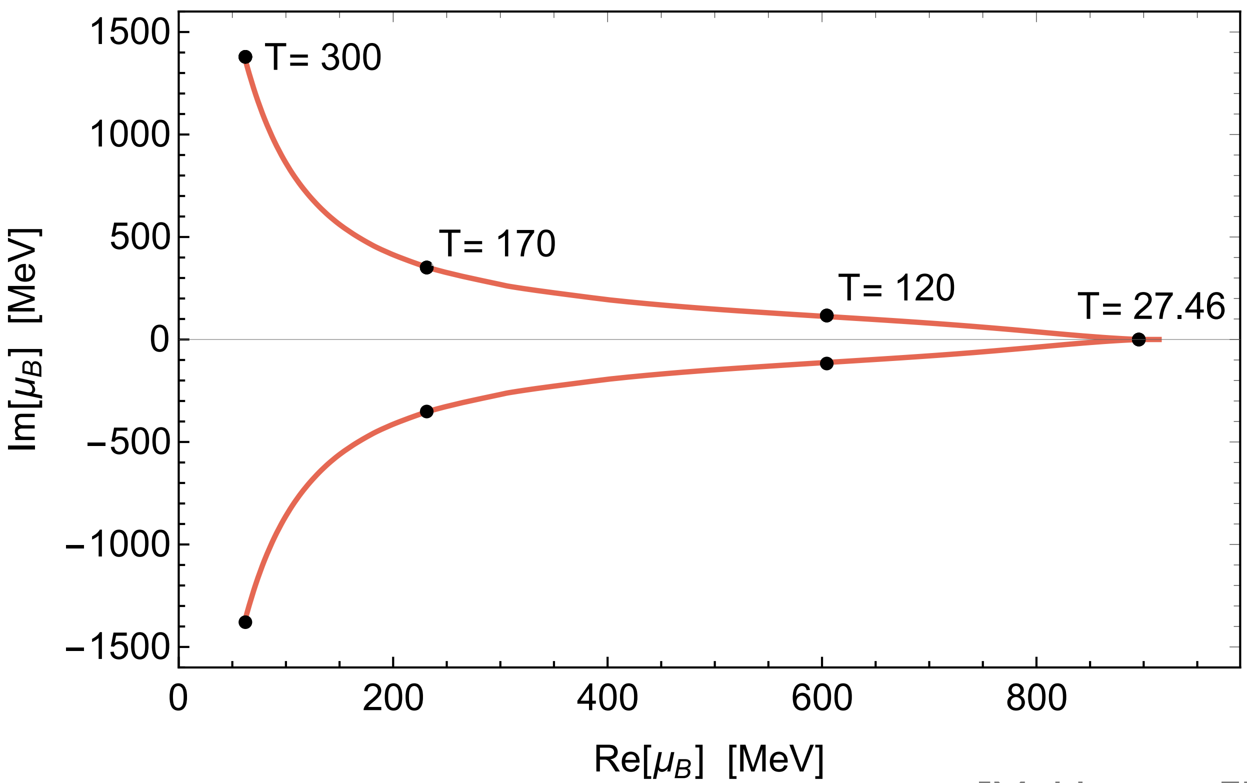


- YLE is a critical point itself: $i\phi^3$-universality with independent critical exponents in the complex plane [Fisher (1987)]
- "close enough" to the real axis, the location of the YLE is also universal

[Johnson, FR, Skokov (2020-2023)]

YLE & THE CRITICAL ENDPOINT

Consider system with a CEP at $(T_{\text{CEP}}, \mu_{\text{CEP}})$ in the complex μ plane



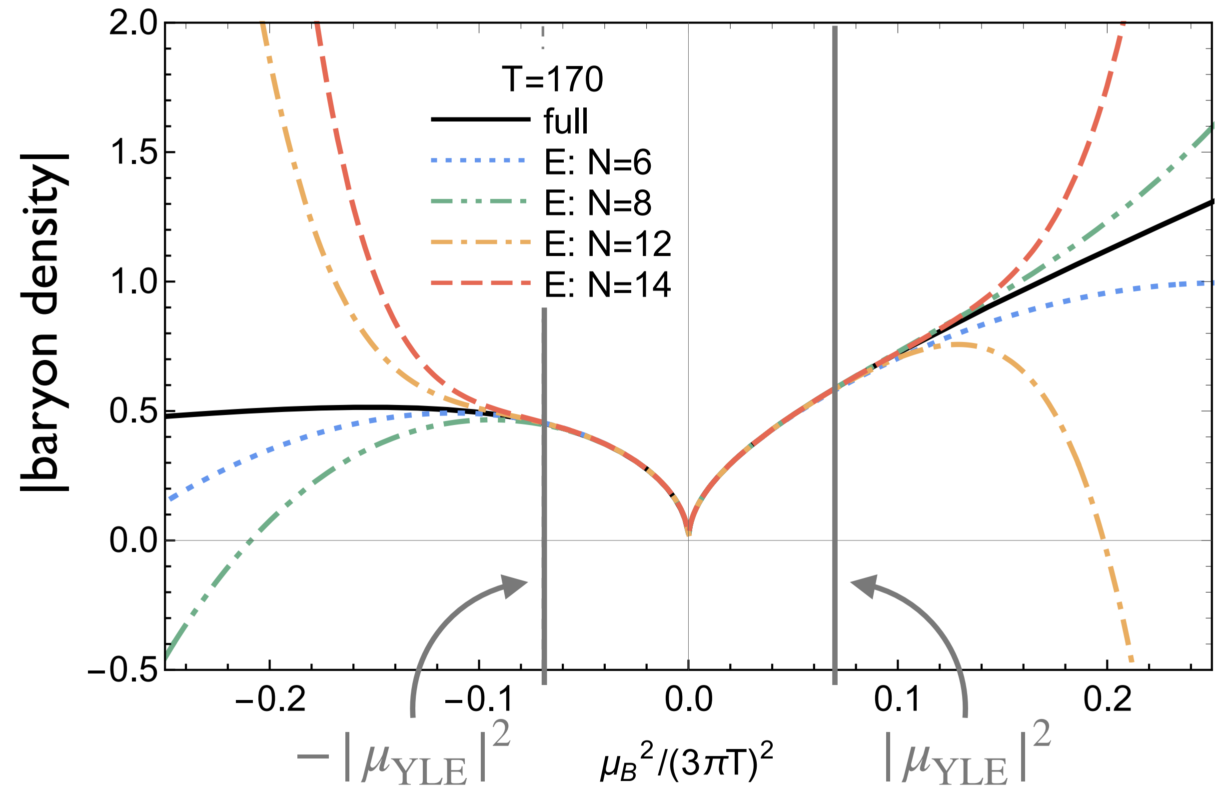
[Mukherjee, FR, Skokov (2021)]

- $T = T_{\text{CEP}} : \mu_{\text{YLE}} = \mu_{\text{CEP}} \in \mathbb{R}$
- $T > T_{\text{CEP}} : \mu_{\text{YLE}} \in \mathbb{C}$

At $\mu = 0$ the YLE is the nearest singularity

→ determines radius of convergence for expansions around $\mu = 0$

CEP inaccessible by Taylor expansions at $\mu = 0$!



IDENTIFYING THE YLE

YLE is a branch point of the free energy as a function of a complex thermodynamic variable z, e.g., $z = \mu, h, \ldots$. It is obtained from the effective effective action as

$$f(z) = \frac{T}{V}\,\Gamma\big[\bar{\phi}(z)\big]$$

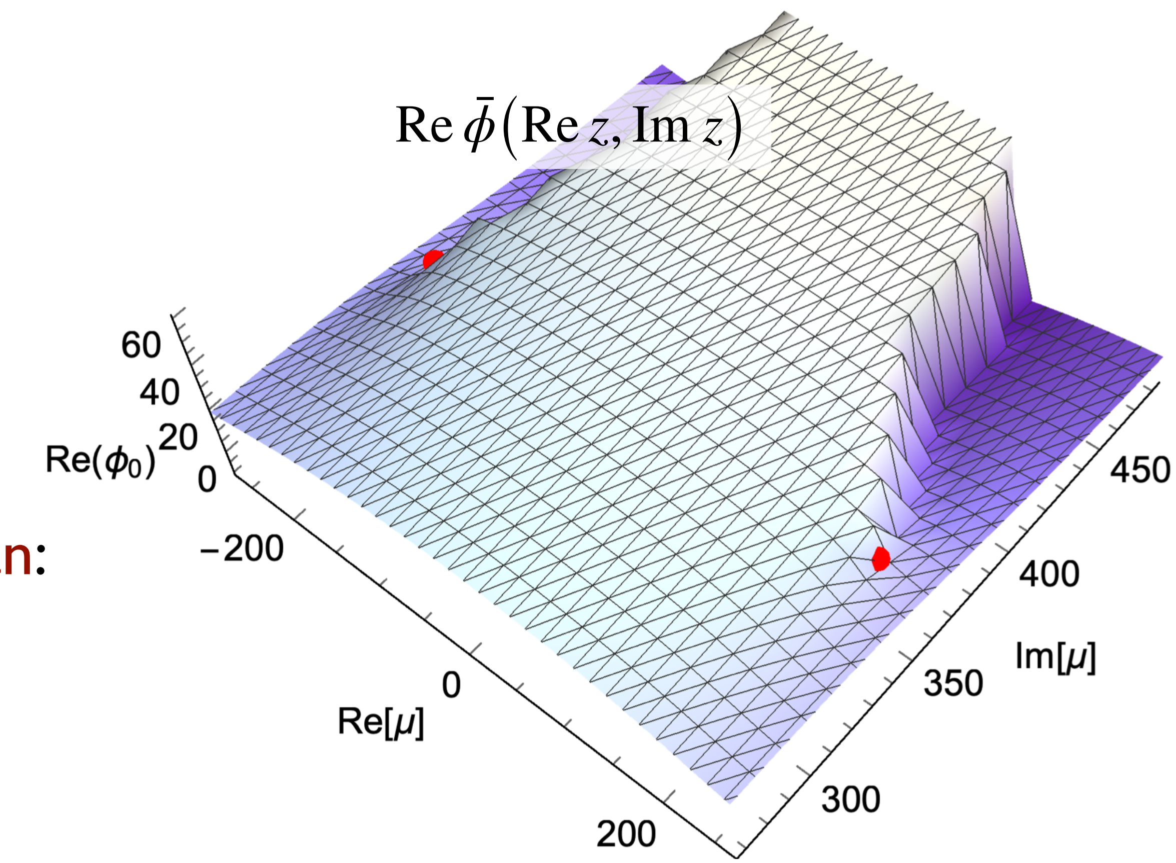
where

$$\Gamma[\phi] = \sup_J \left\{ \int_x J\cdot\phi - \ln Z[J] \right\}, \qquad \ln Z[J] = \int \mathscr{D}\phi\, e^{-S[\phi] + \int_x J\cdot\phi},$$

The effective action is a functional of ϕ. The dependence on z enters in particular through the EoM, which determines the "magnetization" $\bar{\phi}(z)$,

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi}\right|_{\phi=\bar{\phi}(z)} = 0$$

$\longrightarrow$ the magnetization carries the edge singularity



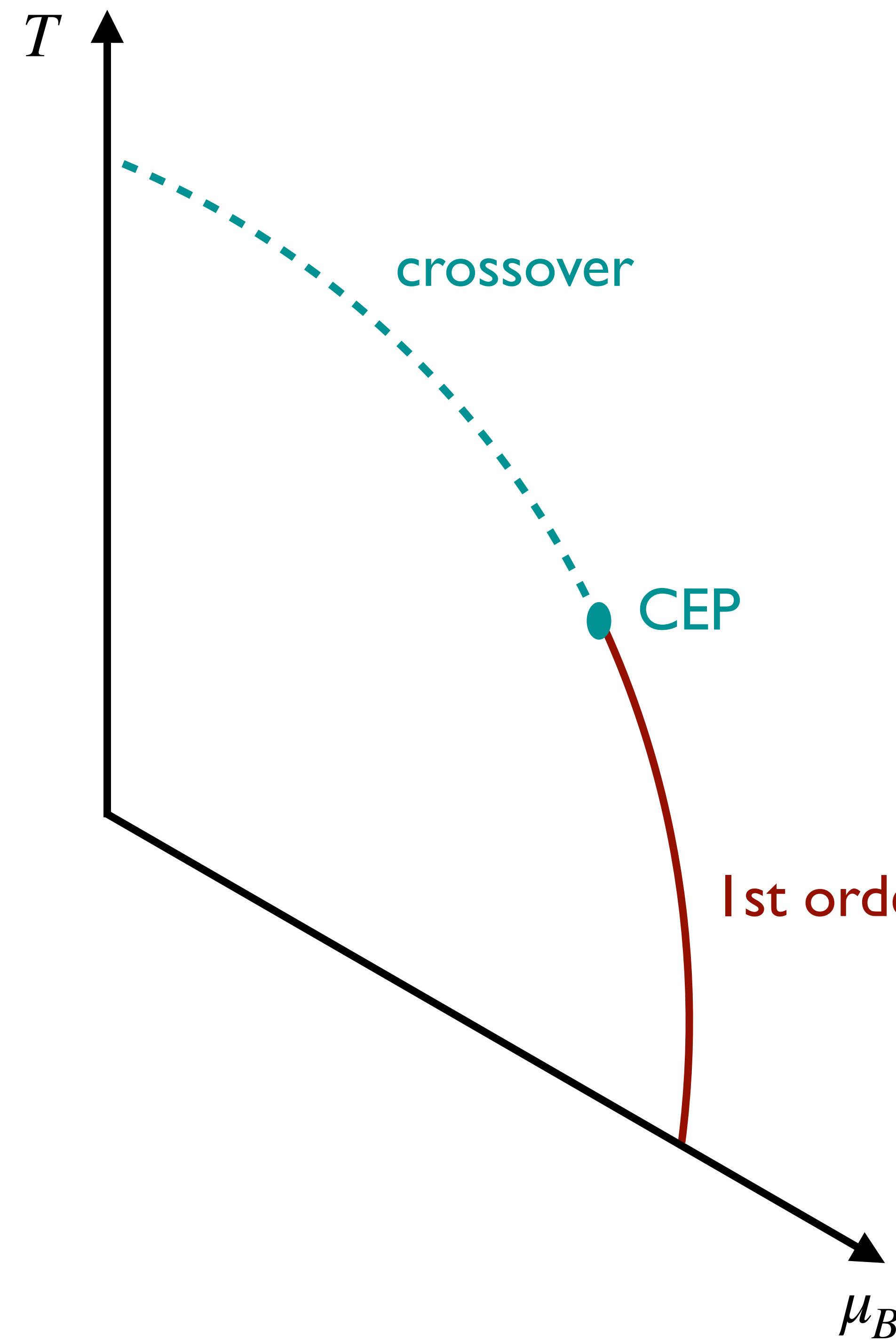
$\bar{\phi}(z)$ is implicitly defined: use the implicit function theorem to identify the singularity z_{YLE} from the Hessian:

$$\det H = \prod_i H_i = \det\left(\left.\frac{\delta^2\Gamma[\phi]}{\delta\phi_i\delta\phi_j}\right|_{\phi=\bar{\phi}(z_{\mathrm{YLE}})} \right) = 0$$

eigenvalues

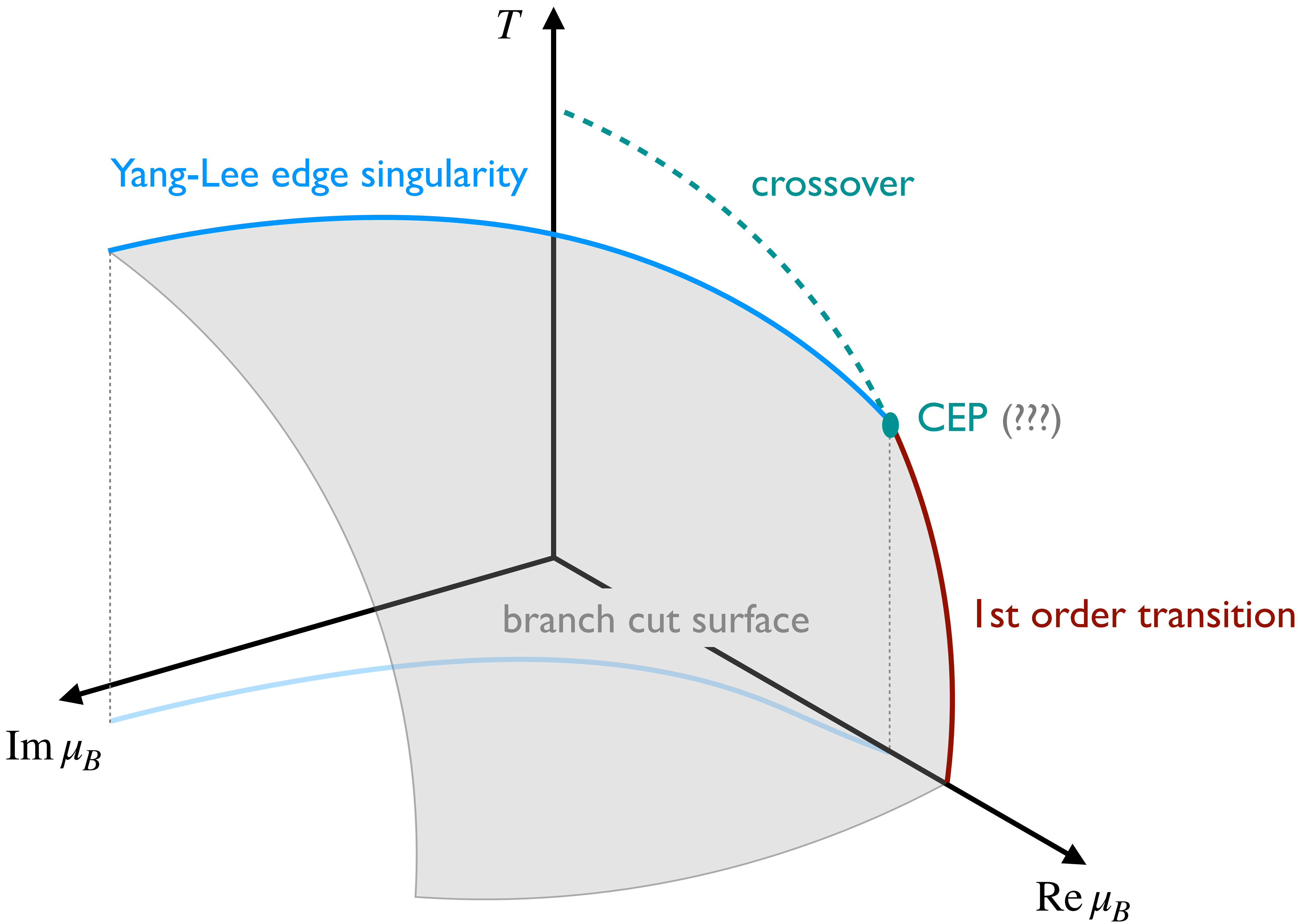
THE CHIRAL PHASE TRANSITION

in the (T, μ_B) plane



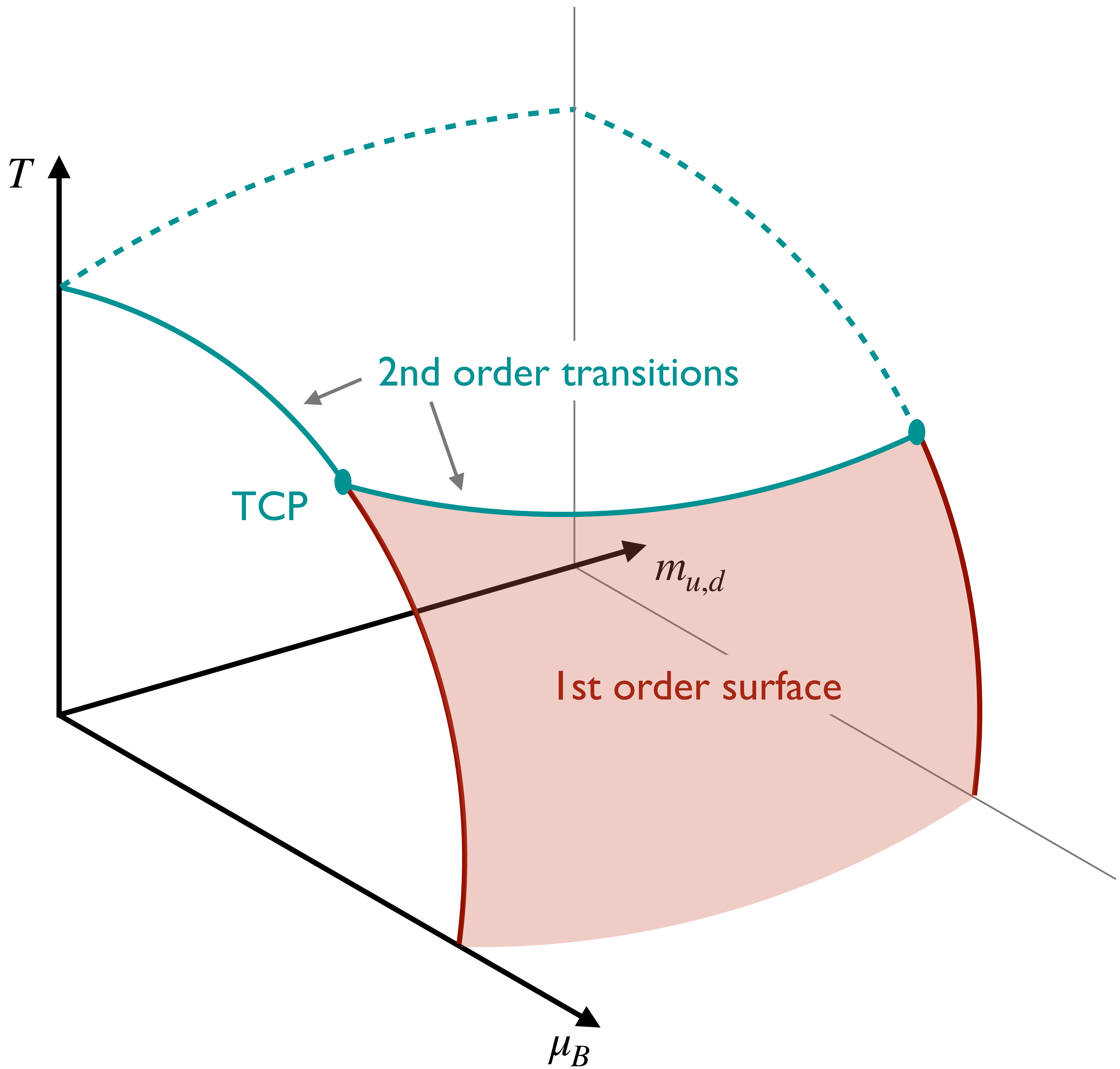
THE CHIRAL PHASE TRANSITION

in the $(T, \mathrm{Re}\,\mu_B, \mathrm{Im}\,\mu_B)$ plane



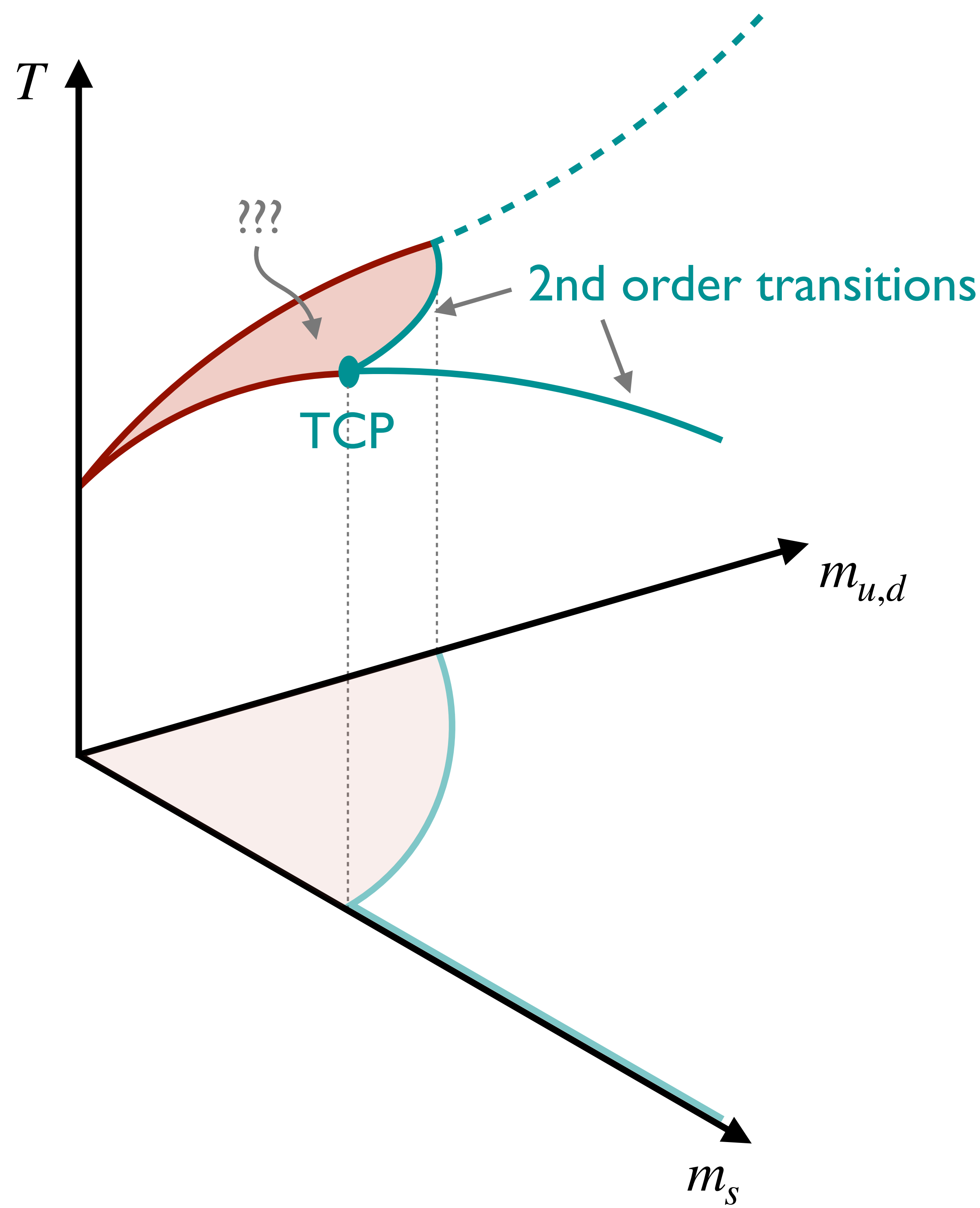
THE CHIRAL PHASE TRANSITION

in the $(T, \mu_B, m_{u,d})$ plane



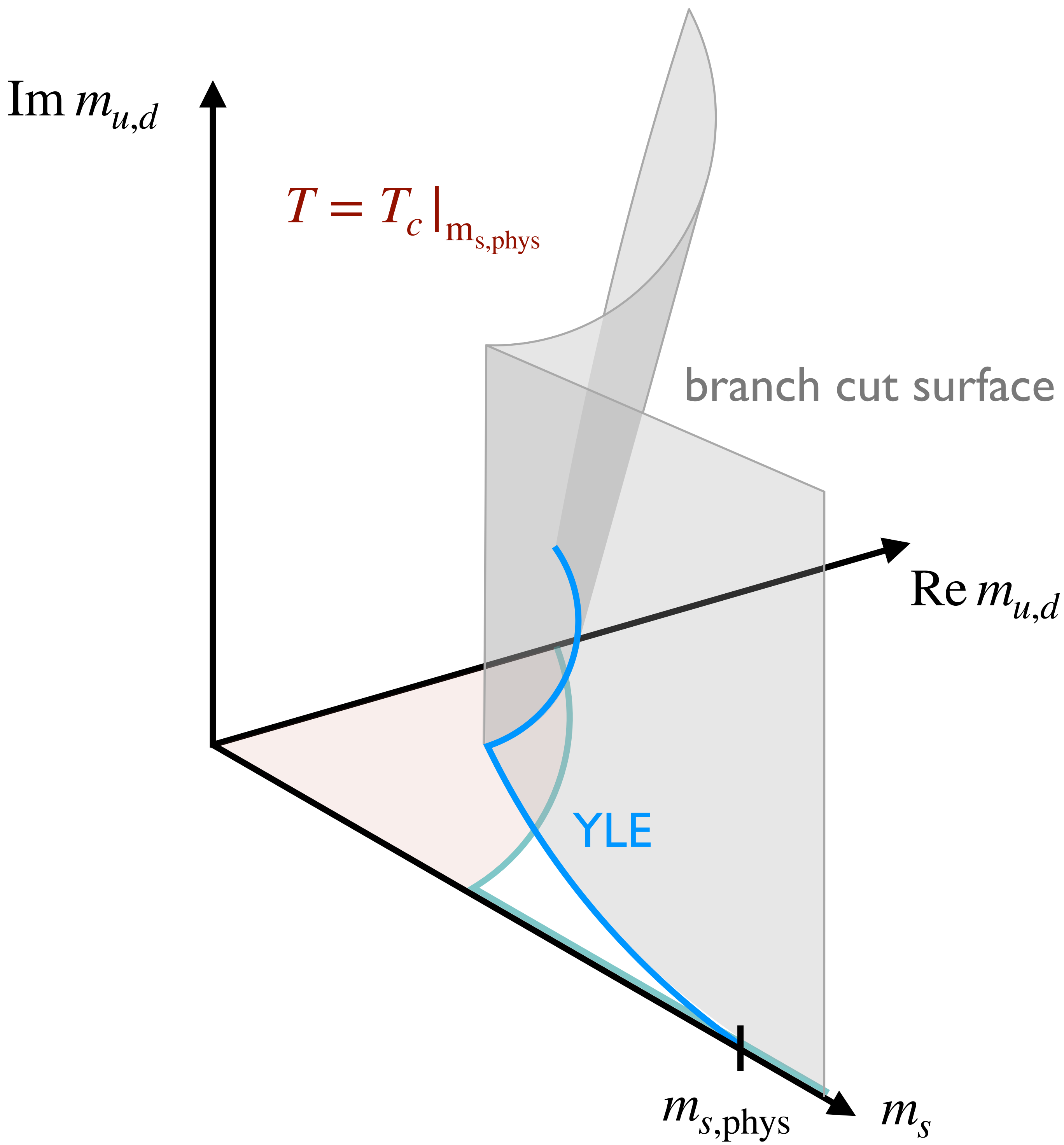
THE CHIRAL PHASE TRANSITION

in the $(T, m_s, m_{u,d})$ plane

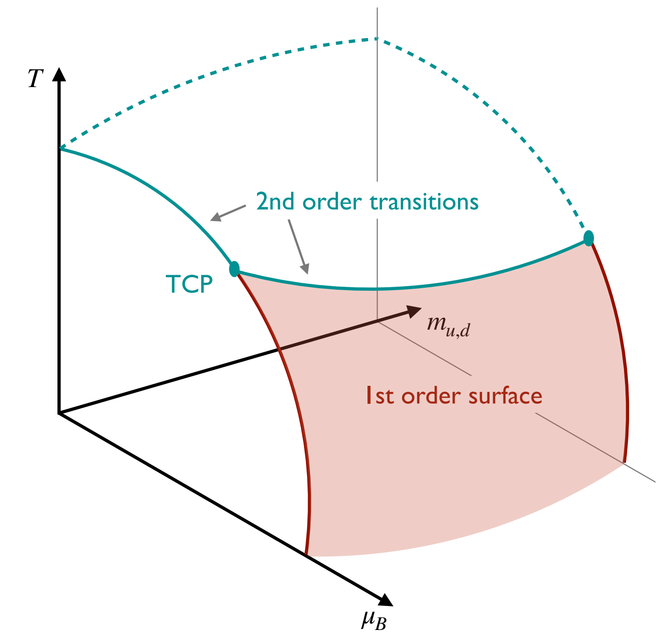


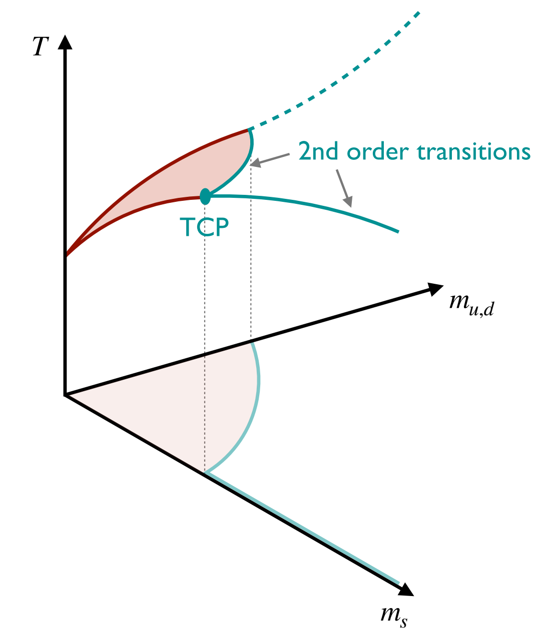
THE CHIRAL PHASE TRANSITION

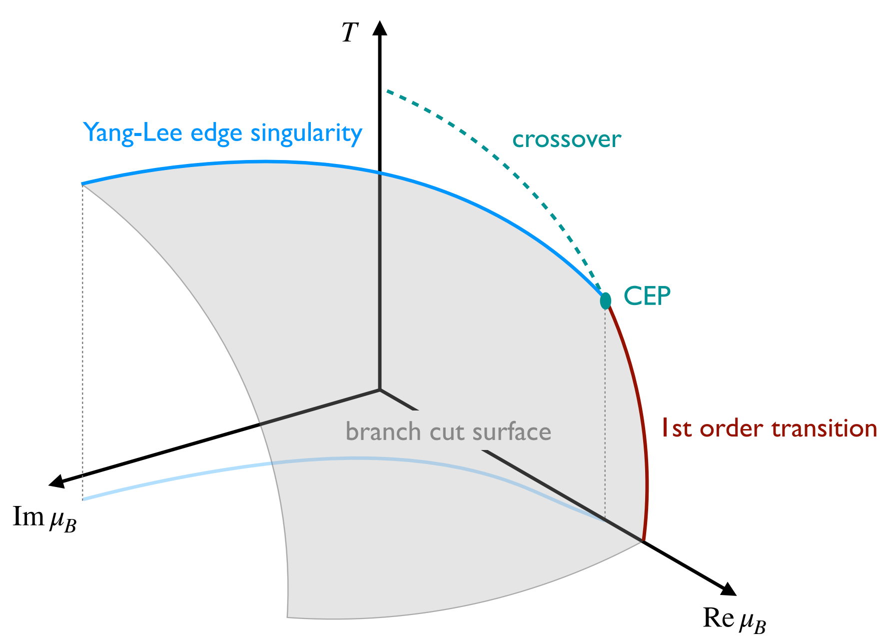
in the $(\mathrm{Im}\, m_{u,d}, m_s\,, \mathrm{Re}\, m_{u,d})$ **plane**

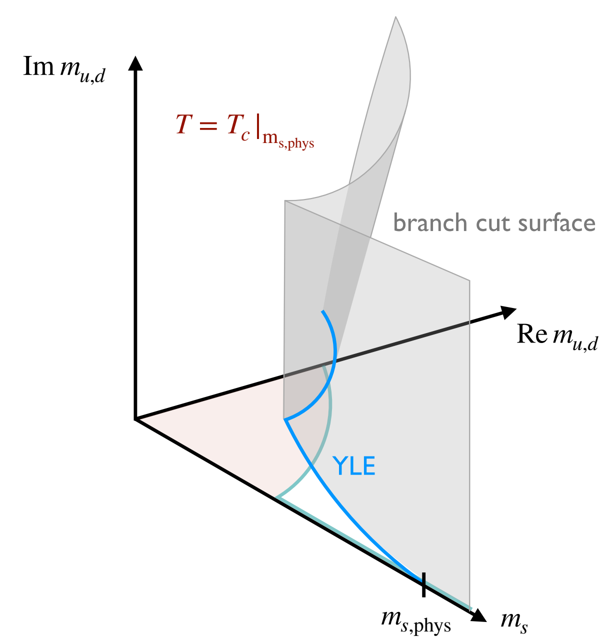


MANY FACES OF THE PHASE TRANSITION







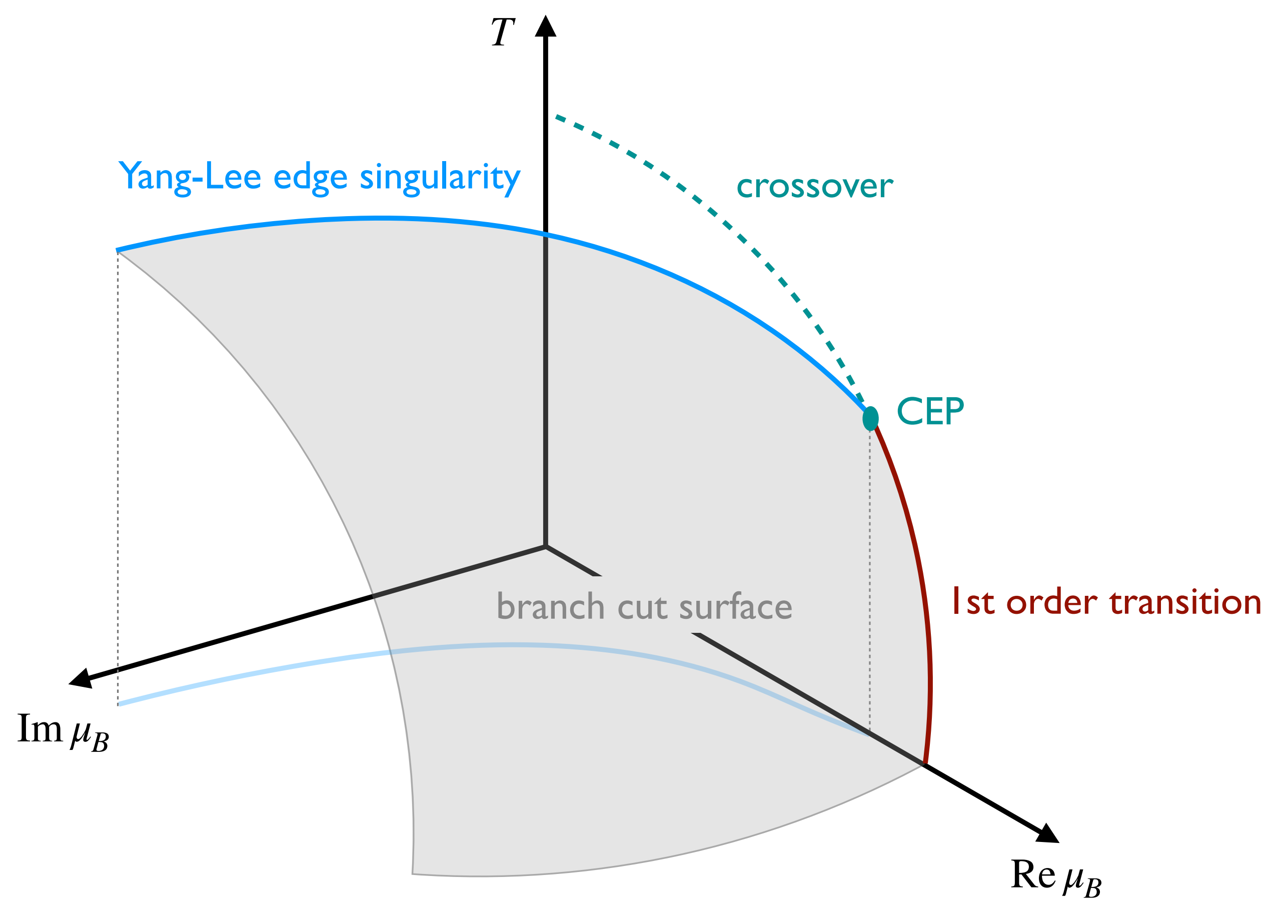


→ a lot of additional information from different directions, including the complex plane

use this to learn something about the physical directions (and vice versa)!

MEDIUM-INDUCED MIXING & CRITICAL MODES IN QCD

[Haensch, FR, von Smekal, arXiv:2308.16244]



QCD PARTITION FUNCTION

$$D_\mu = \partial_\mu - igA_\mu^a T^a$$
$$F_{\mu\nu}^a T^a = [D_\mu, D_\nu]$$
$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

Consider QCD in Euclidean spacetime,

$$\mathscr{L} = \bar{\psi}\big(\gamma^\mu D_\mu + M + \gamma^0\mu\big)\psi + \frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a$$

Since quarks enter quadratically, they can be integrated in the partition function of QCD,

$$Z = \int \mathscr{D}\Phi\, e^{-S[\Phi]} = \int \mathscr{D}A\, e^{-\int_x \frac{1}{4}F^2} \det \mathscr{M}(A)$$

$\Phi = (A, \psi, \bar{\psi})$

The Dirac operator is

$$\mathscr{M} = \gamma^\mu D_\mu + M + \gamma^0\mu$$

It enters the effective action,

$$\Gamma[\Phi] = \sup_J \left\{ \int_x J \cdot \Phi - \ln Z[J] \right\},$$

at finite T as:

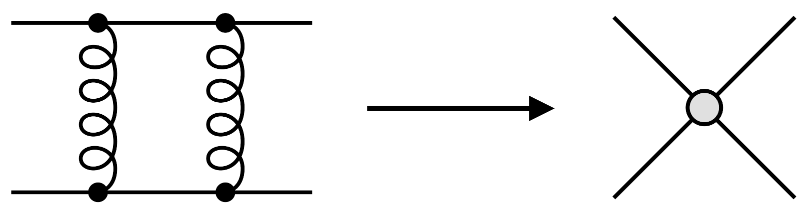
$$\ln \det \mathscr{M} = T \sum_{n\in\mathbb{Z}} \int \frac{d^3p}{(2\pi)^3} \ln \det \left[i\gamma^0\big(\nu_n - gA_0 - i\mu\big) + i\gamma^j\big(p_j - gA_j\big) + M \right]$$

fermionic Matsubara frequencies
$\nu_n = (2n+1)\pi T$

THE QUARK DETERMINANT

The quark determinant $\det\left(\gamma^\mu D_\mu + M + \gamma^0\mu\right)$ is modified through interactions. This gives rise to three crucial contributions regarding the phase structure:

(1) quark-scattering in the scalar-pseudoscalar channel



$\sim \alpha_s^2$ $\qquad$ $\sim (\bar\psi T^\alpha \psi)^2$

with $T^\alpha = \left(1, i\gamma^5 \vec\tau\right)^\alpha$

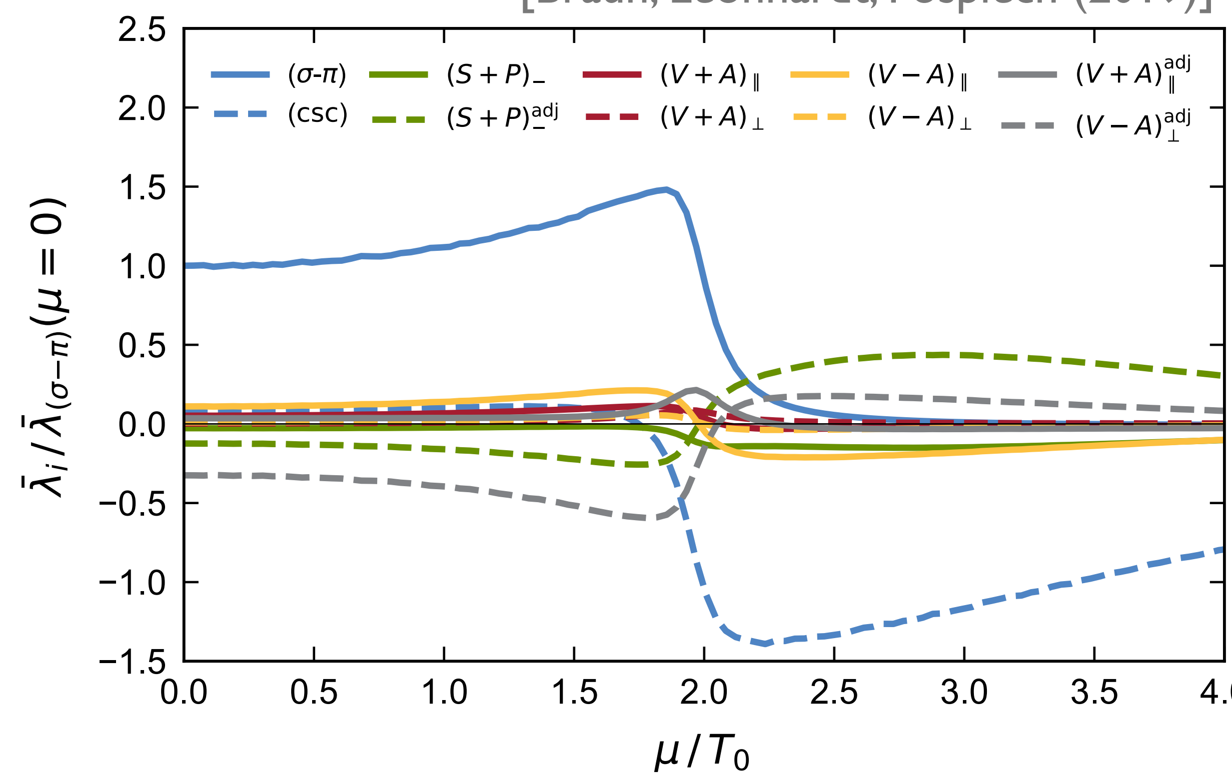
At not too large density this channel becomes resonant and gives rise to a nonzero chiral condensate: **chiral symmetry breaking**

$$\bar\sigma \sim \langle\bar\psi\psi\rangle$$

constituent quark mass $M = m_q + h_\sigma \bar\sigma$

current mass $\qquad$ spontaneously generated mass

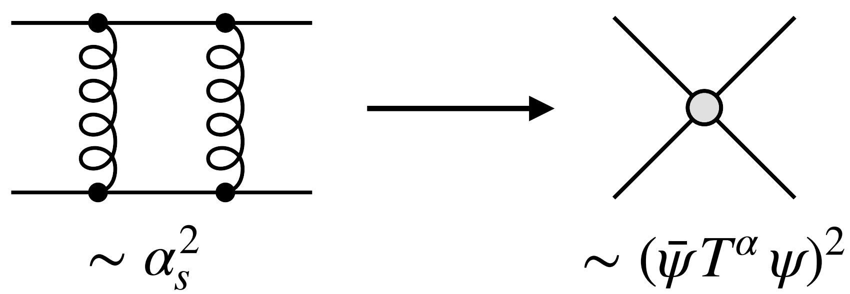
[Braun, Leonhardt, Pospiech (2019)]



THE QUARK DETERMINANT

The quark determinant $\det\left(\gamma^\mu D_\mu + M + \gamma^0\mu\right)$ is modified through interactions. This gives rise to three crucial contributions regarding the phase structure:

(2) quark-scattering in the vector channel



with $T^\alpha = \gamma^\alpha$

Finite density is equivalent to a condensate in this channel,

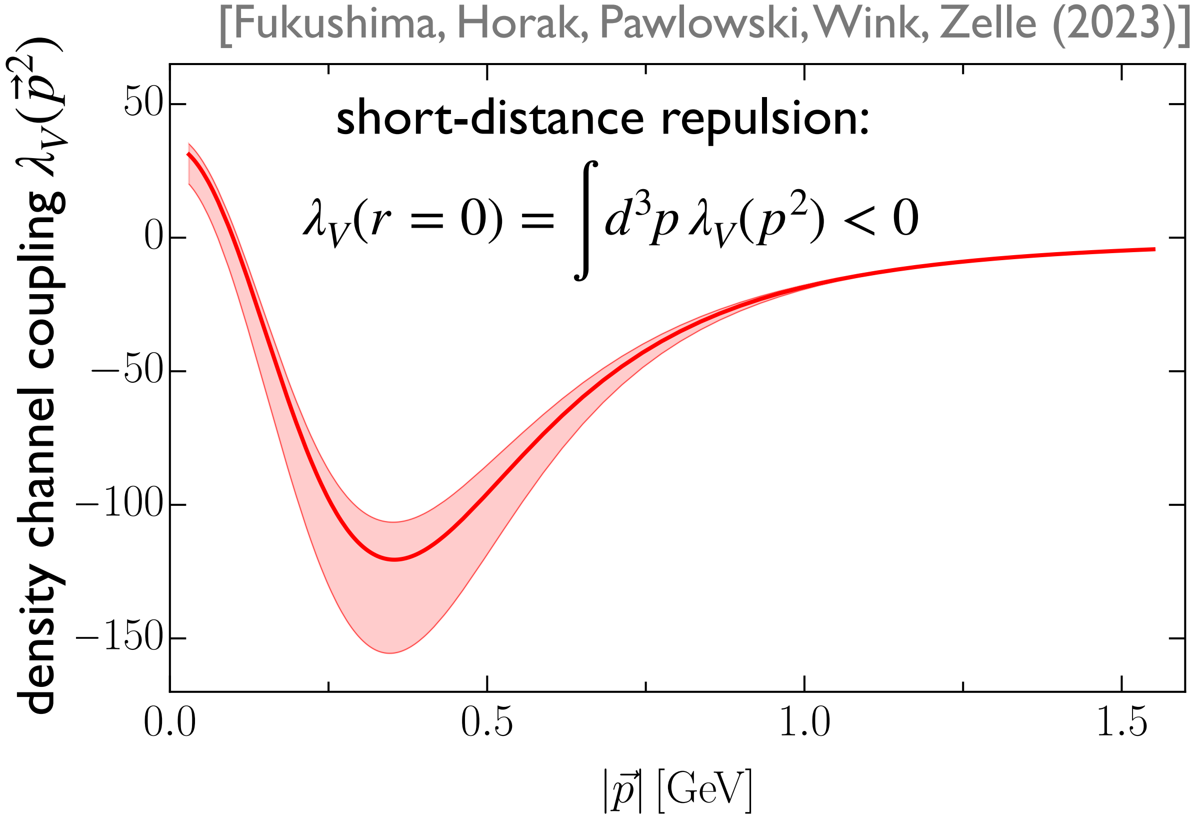
$$-i\bar\omega_0 \sim \langle \bar\psi \gamma^0 \psi \rangle = n_q$$

quark density

→ shifted chemical potential $\mu \to \mu + ih_\omega \bar\omega_0$

short-distance repulsion implies imaginary quark-omega coupling ih_ω and imaginary omega condensate $\bar\omega_0$

[Fukushima, Horak, Pawlowski, Wink, Zelle (2023)]

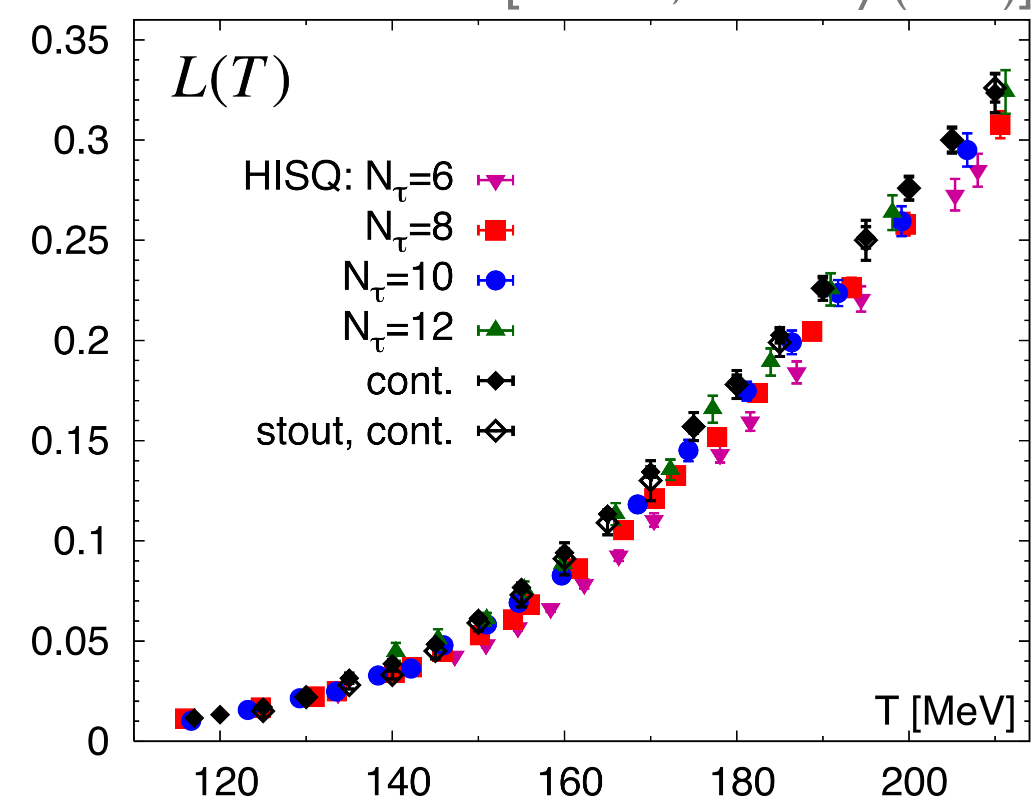


THE QUARK DETERMINANT

The quark determinant $\det\left(\gamma^\mu D_\mu + M + \gamma^0 \mu\right)$ is modified through interactions. This gives rise to three crucial contributions regarding the phase structure:

(3) the Polyakov loop

[Bazavov, Petreczky (2013)]



$Z(3)$ center symmetry of $SU(3)$ Yang-Mills theory is explicitly broken by dynamical quarks and spontaneously by deconfinement. "Order parameter":

$$L = \frac{1}{N_c} \langle \mathrm{tr}\, P \rangle$$

with the temporal Wilson line

$$P(\vec{x}) = \mathscr{P} \exp\left[ig \int_0^\beta dx_0\, A_0(x_0, \vec{x}) \right]$$

$L < 1$ can be described by a static temporal gluon condensate $\bar{A}_0 = \langle A_0 \rangle \neq 0$

$$L \leq \frac{1}{N_c} \mathrm{tr}_c\, e^{ig\bar{A}_0/T} \longrightarrow$$ covariant derivative in A_0 background: $D_\mu \to D_\mu - ig\bar{A}_0$

[Braun, Gies, Pawlowski (2007)]

SADDLE POINT EXPANSION

All these background fields are obtained by solving quantum equations of motion,

$$\frac{\delta\Gamma[\Phi]}{\delta(\sigma, \omega_0, A_0)} = 0$$

$\longrightarrow$ do a saddle-point expansion around the ground state $\bar{\Phi} = (\bar{\sigma}, \bar{\omega}_0, \bar{A}_0)$

The ground state quark determinant is

$$\frac{T}{V}\ln\det\bar{\mathscr{M}} = T\sum_{n\in\mathbb{Z}}\int\frac{d^3p}{(2\pi)^3}\,\mathrm{tr}\ln\Big[i\gamma^0\big(\nu_n - g\bar{A}_0 + h_\omega\,\bar{\omega}_0 - i\mu\big) + i\vec{\gamma}\cdot\vec{p} + m_q + h_\sigma\,\bar{\sigma}\Big]$$

$$= -2TN_f\int\frac{d^3p}{(2\pi)^3}\left\{\ln\Big[1 + N_c\,L\,e^{-(E_p(\bar{\sigma})-\bar{\mu})/T} + N_c\,\bar{L}\,e^{-2(E_p(\bar{\sigma})-\bar{\mu})/T} + e^{-3(E_p(\bar{\sigma})-\bar{\mu})/T}\Big]\right.$$

$$\left. + \ln\Big[1 + N_c\,\bar{L}\,e^{-(E_p(\bar{\sigma})+\bar{\mu})/T} + N_c\,L\,e^{-2(E_p(\bar{\sigma})+\bar{\mu})/T} + e^{-3(E_p(\bar{\sigma})+\bar{\mu})/T}\Big]\right\}$$

quark energy

$$E_p(\bar{\sigma}) = \sqrt{\vec{p}^2 + m_q^2 + h_\sigma^2\,\bar{\sigma}^2}$$

effective chemical potential

$$\bar{\mu} = \mu + ih_\omega\,\bar{\omega}_0$$

Polyakov anti-loop

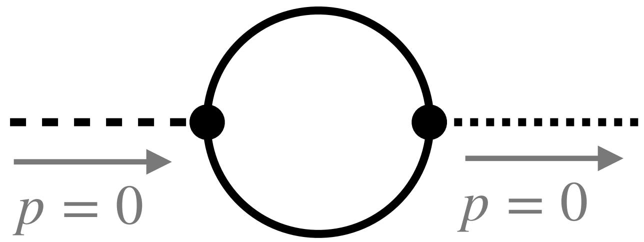
$$\bar{L} = \frac{1}{N_c}\langle \mathrm{tr}\,P^\dagger\rangle$$

$\longrightarrow$ consider fluctuations

$$\sigma = \bar{\sigma} + \delta\sigma$$
$$\omega_0 = \bar{\omega}_0 + \delta\omega_0$$
$$A_0 = \bar{A}_0 + \delta A_0$$

MEDIUM-INDUCED MIXING

The saddle point expansion of the QCD quark determinant reveals linear couplings (= mixing) between effective degrees of freedom. Focus on zero-momentum contributions,



$$H = \begin{pmatrix} \Gamma^{(2)}_{\sigma\sigma} & \Gamma^{(2)}_{\sigma\omega_0} & \Gamma^{(2)}_{\sigma\bar L} & \Gamma^{(2)}_{\sigma L} \\ \Gamma^{(2)}_{\sigma\omega_0} & \Gamma^{(2)}_{\omega_0\omega_0} & \Gamma^{(2)}_{\omega_0\bar L} & \Gamma^{(2)}_{\omega_0 L} \\ \Gamma^{(2)}_{\sigma L} & \Gamma^{(2)}_{\omega_0 L} & \Gamma^{(2)}_{L\bar L} & \Gamma^{(2)}_{LL} \\ \Gamma^{(2)}_{\sigma\bar L} & \Gamma^{(2)}_{\omega_0\bar L} & \Gamma^{(2)}_{\bar L\bar L} & \Gamma^{(2)}_{L\bar L} \end{pmatrix}$$

Polyakov anti-loop

$$\bar L = \frac{1}{N_c}\langle \mathrm{tr}\, P^\dagger\rangle$$

⟶ nontrivial Hessian/mass matrix

Off-diagonal terms are in general nonzero at finite T and μ!

⟶ **mixing of chiral condensate $\bar\sigma$, density ω_0 and Polyakov loops/$\bar A_0$**

- couplings linear in σ vanish in the chiral limit (chiral symmetry)
- $\bar\omega_0$ is imaginary because of repulsive vector interaction, but $\delta\omega_0$ must be real because this is the steepest descent path/Lefshetz thimble!
- $L \neq \bar L \in \mathbb{R}$ at $\mu \neq 0$

⟶ off-diagonal couplings involving ω_0 are imaginary and $\Gamma^{(2)}_{(\sigma,\omega_0,L)\,L} \neq \Gamma^{(2)}_{(\sigma,\omega_0,\bar L)\,\bar L}$

⟶ **QCD has a non-Hermitian mass matrix at finite density!**

THE CRITICAL MODE OF QCD

If QCD has a CEP at $(T_{\mathrm{CEP}}, \mu_{\mathrm{CEP}})$, we need to find a YLE for real T and μ

⟶ implicit function theorem: zero eigenvalue of Hessian H

- eigenvalue of H determines the curvature mass of the eigenmode χ,

$$m_{\mathrm{curv}}^2 = G_\chi^{-1}\left(p_0 = 0, \vec{p}^2 = 0\right)$$

Euclidean propagator

- relevant for the phase transition is the spacelike screening mass,

$$G_\chi\left(p_0 = 0, \vec{p}^2 = -m_{\mathrm{scr}}^2\right) = 0$$

- smallest screening mass determines the (largest) correlation length ξ in the system,

$$\lim_{|\vec{x}_1 - \vec{x}_2| \to \infty} \left\langle \chi(t, \vec{x}_1)\chi(t, \vec{x}_2) \right\rangle \sim e^{-|\vec{x}_1 - \vec{x}_2|/\xi}, \qquad \xi = \frac{1}{m_{\mathrm{scr}}}$$

- CEP: $\xi \to \infty$, i.e. $m_{\mathrm{scr}} = m_{\mathrm{curv}} = 0$

⟶ the eigenmode with zero eigenvalue is the critical mode

⟶ **the critical mode of the CEP is a mixture of the chiral condensate, the density/ω_0 and the Polyakov loops/A_0**

THE CRITICAL MODE OF QCD

Is it even relevant to know what the critical mode is?

- divergence of susceptibilities is insensitive to it,

$$\chi_{ab} = \frac{d^2\Omega}{dz_a\, dz_b} = \frac{\partial^2\Omega}{\partial z_a \partial z_b} + \frac{\partial^2\Omega}{\partial z_a \partial \phi_i} H^{-1}_{ij} \frac{\partial^2\Omega}{\partial \phi_j \partial z_b}\Bigg|_{\mathrm{EoM}}$$

→ static critical physics are insensitive to the nature of the critical mode

But in-medium mixing is a general feature that needs to be taken into account for a consistent description of QCD

→ **physical degrees of freedom are mixtures**

Furthermore, the dynamic critical behavior depends crucially on the nature of the critical mode

- dynamic universality not only determined by symmetry and dimensionality, but by all slow modes in the system and whether or not they are conserved [Halperin, Hohenberg (1977)]
- mixing between the chiral condensate and the density has been recognized before in nuclear matter (e.g. [Wolf, Friman, Soyeur (1998)]). Crucially, the density is conserved

→ dynamic universality of CEP is model B, not A [Son, Stephanov (2004)]

- can the admixture of A_0 lead to different dynamic universal behavior?

THE HESSIAN AND $\mathscr{CK}$-SYMMETRY

The mass matrix of QCD is non-Hermitian at finite μ. This is related to the breaking of charge conjugation symmetry $\mathscr{C}$ ant finite μ

- vector fields change sign under charge conjugation: $\mathscr{C}\omega_0 = -\,\omega_0\,, \mathscr{C}A_0 = -\,A_0\,, \mathscr{C}L = \bar{L}$
- $\mu \neq 0$ leads to $L \neq \bar{L}$

⟶ mixing reflects $\mathscr{C}$ breaking and renders H non-Hermitian

However, the system remains invariant under charge + complex conjugation, $\mathscr{CK}$-symmetry

$$\begin{aligned} \mathscr{K}\bar{\omega}_0 &= -\,\bar{\omega}_0 \\ \mathscr{K}L &= \bar{L} \end{aligned} \quad \longrightarrow \quad \mathscr{CK}\,H = H$$

The Hessian obeys the relation

$$H = CH^*C$$

[Nishimura, Ogilvie, Pangeni (2014)]

with an orthogonal matrix C. This implies that H and H^* share the same eigenvalues,

$$\det(H - \lambda I) = \det(CH^*C - \lambda I) = \det[C(H^* - \lambda I)C] = \det C^2 \det(H^* - \lambda I) = \det(H^* - \lambda I)$$

⟶ **all eigenvalues of H are either real or come in complex-conjugate pairs**

EXAMPLE: PQM MODEL

Simplest model with the basic features of the QCD quark determinant: Polyakov-Quark-Meson model (PQM) in mean-field approximation

$$S_{\mathrm{PQM}} = \int_0^\beta dx_0 \int d^3x \left\{ \bar\psi\left[\gamma^\mu\partial_\mu + \gamma^0(\mu + ih_\omega\omega_0 + iA_0) + h_\sigma\sigma\right]\psi \right.$$

$\longleftarrow$ $N_f = 2$ quark contribution

$$+\frac{\lambda}{2}\left(\sigma^2 - \nu^2\right)^2 - j\sigma + \frac{1}{2}m_\omega^2\omega_0^2$$

$\longleftarrow$ mean-field meson potential $V(\sigma, \omega_0)$

$$\left. -\frac{b_2(T)}{2}L\bar L - \frac{b_3}{3}\left(L^3 + \bar L^3\right) + \frac{b_4}{4}(L\bar L)^2 \right\}$$

$\longleftarrow$ $Z(3)$ symmetric Polyakov loop potential $U(L, \bar L)$ from [Ratti, Thaler, Weise (2006)]

This yields the effective potential,

$$\Omega(\sigma, \omega_0, L, \bar L) = V(\sigma, \omega_0) + U(L, \bar L) - \frac{T}{V}\left[\ln\det\mathscr{M}\left(\sigma, \omega_0, L, \bar L\right) + \ln\det\mathscr{M}_{\mathrm{vac}}(\sigma)\right]$$

same as for the saddle point expansion above

vacuum contribution; with dim. regularization

$$= N_f N_c \frac{h_\sigma^4\sigma^4}{2^8\pi^2}\ln\left(\frac{h_\sigma^2\sigma^2}{4\Lambda^2}\right)$$

We solve this model,

$$\frac{\delta\Omega}{\delta(\sigma, \omega_0, L, \bar L)} = 0\,,$$

and study the eigenvalues of the Hessian in the vicinity of the CEP at finite T and μ

MASS MATRIX

The Hessian in terms of Polyakov loops cannot really be interpreted as a mass-matrix, because they aren't fields. We therefore parametrize them in terms of "eigenvalue fields"

$$L = \frac{1}{N_c}\sum_c \left\langle e^{ig\theta_c/T}\right\rangle \longrightarrow \begin{aligned} L &= \frac{1}{3}\exp\left(\frac{ia_8}{2\sqrt{3}T}\right)\left[2\cos\left(\frac{a_3}{2T}\right) + \exp\left(-\frac{3ia_8}{2\sqrt{3}T}\right)\right] \\ \bar{L} &= \frac{1}{3}\exp\left(\frac{-ia_8}{2\sqrt{3}T}\right)\left[2\cos\left(\frac{a_3}{2T}\right) + \exp\left(\frac{3ia_8}{2\sqrt{3}T}\right)\right] \end{aligned}$$

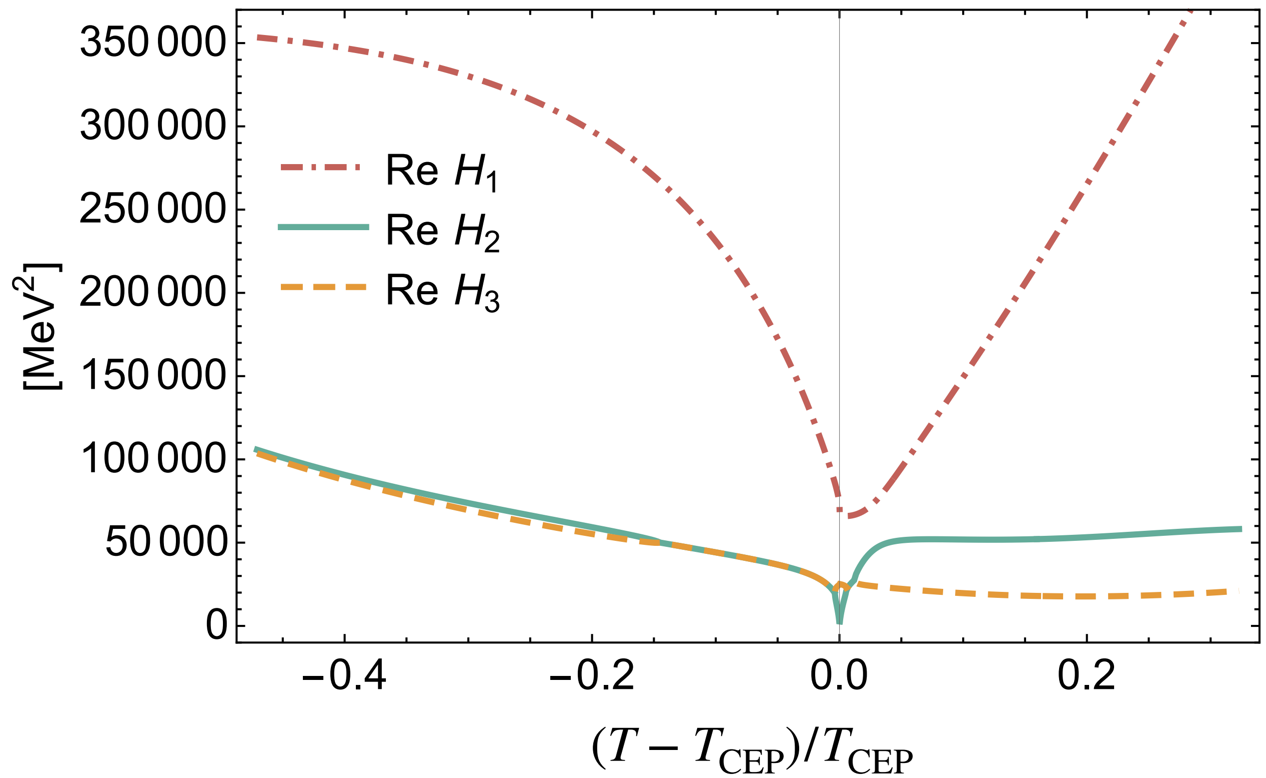
- $L \neq \bar{L} \in \mathbb{R}$ at $\mu \neq 0$ implies a nonzero $a_8 \in i\mathbb{R}$
- the Hessian is then

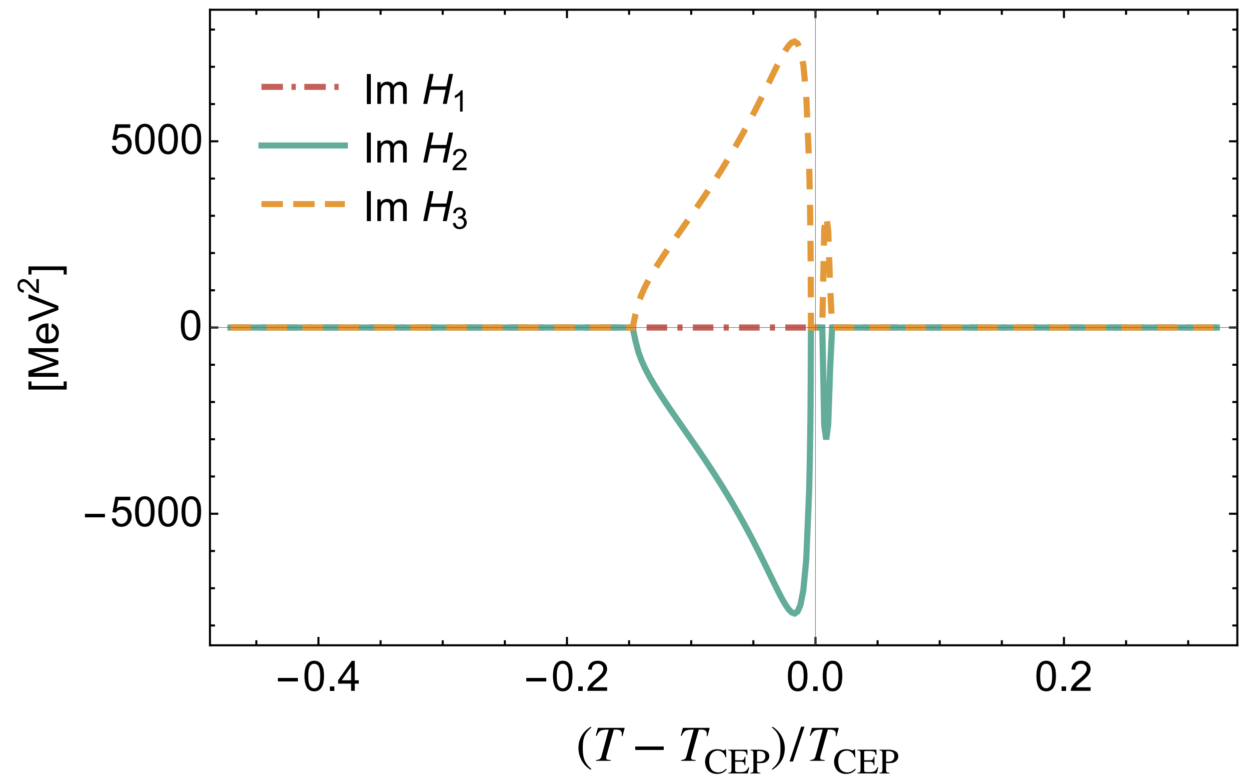
$$H = \begin{pmatrix} \Omega^{(2)}_{\sigma\sigma} & \Omega^{(2)}_{\sigma\omega_0} & \Omega^{(2)}_{\sigma a_3} & \Omega^{(2)}_{\sigma a_8} \\ \Omega^{(2)}_{\sigma\omega_0} & \Omega^{(2)}_{\omega_0\omega_0} & \Omega^{(2)}_{\omega_0 a_3} & \Omega^{(2)}_{\omega_0 a_8} \\ \Omega^{(2)}_{\sigma a_3} & \Omega^{(2)}_{\omega_0 a_3} & \Omega^{(2)}_{a_3 a_3} & \Omega^{(2)}_{a_3 a_8} \\ \Omega^{(2)}_{\sigma a_8} & \Omega^{(2)}_{\omega_0 a_8} & \Omega^{(2)}_{a_3 a_8} & \Omega^{(2)}_{a_8 a_8} \end{pmatrix}$$

- $\mathscr{C}$-symmetry breaking is also reflected in imaginary off-diagonal elements involving a_8

PQM MODEL WITHOUT ω_0

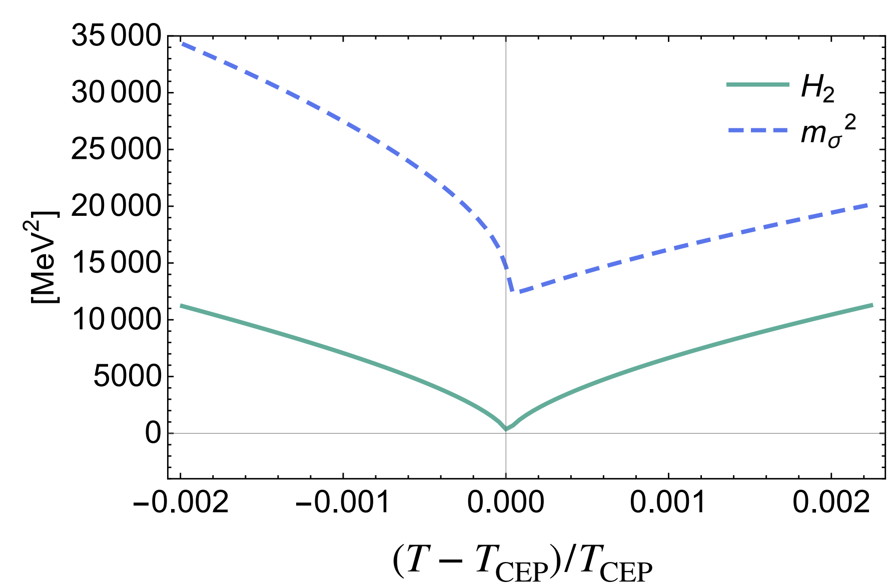
Let's first ignore the ω_0 and the vacuum term for simplicity. Hence, H has three eigenvalues $H_{1,2,3}$.
Go to $\mu = \mu_{\mathrm{CEP}}$ and consider temperatures around T_{CEP} :





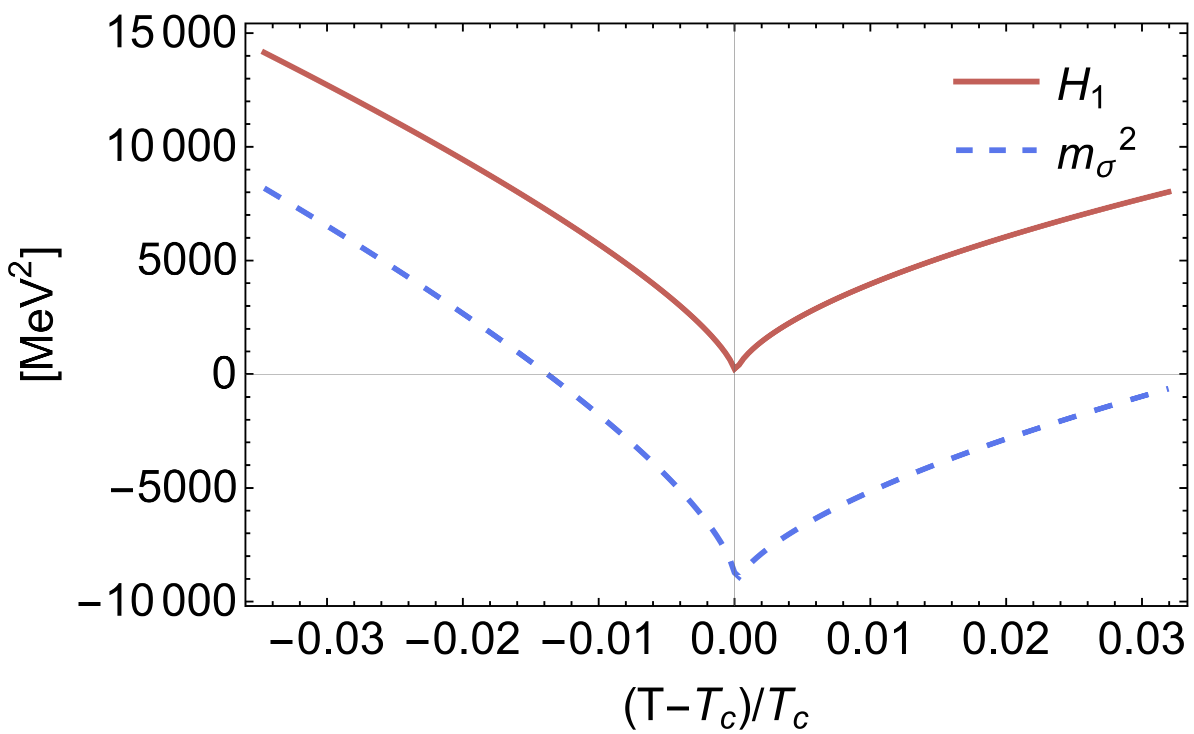
- one real eigenvalue H_1 ($\to \Omega^{(2)}_{\sigma\sigma} = m_\sigma^2$ without mixing)
- complex conjugate eigenvalue pair $H_{2,3}$ ($\to \Omega^{(2)}_{a_{3,8}\,a_{3,8}}$ without mixing)
- H_2 defines the critical mode: mixture of a_3, σ and a_8
- system seems to avoid complex critical mode

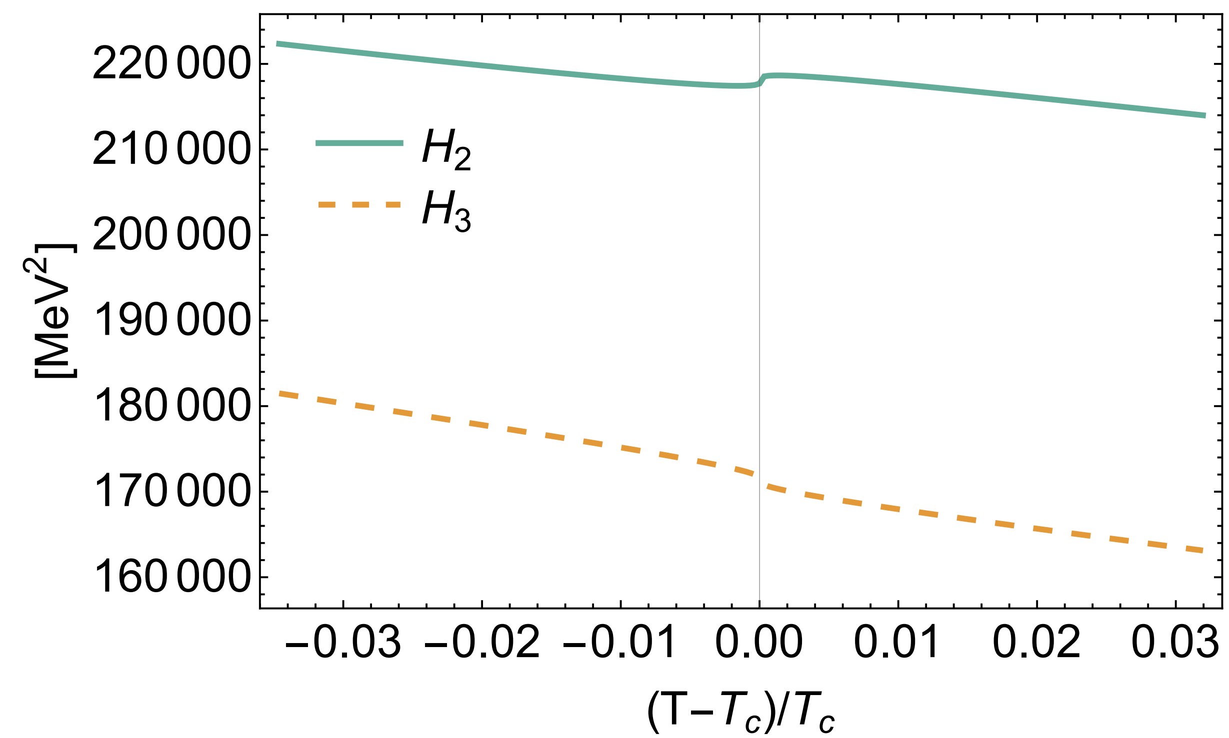
maybe there is a no-go theorem?

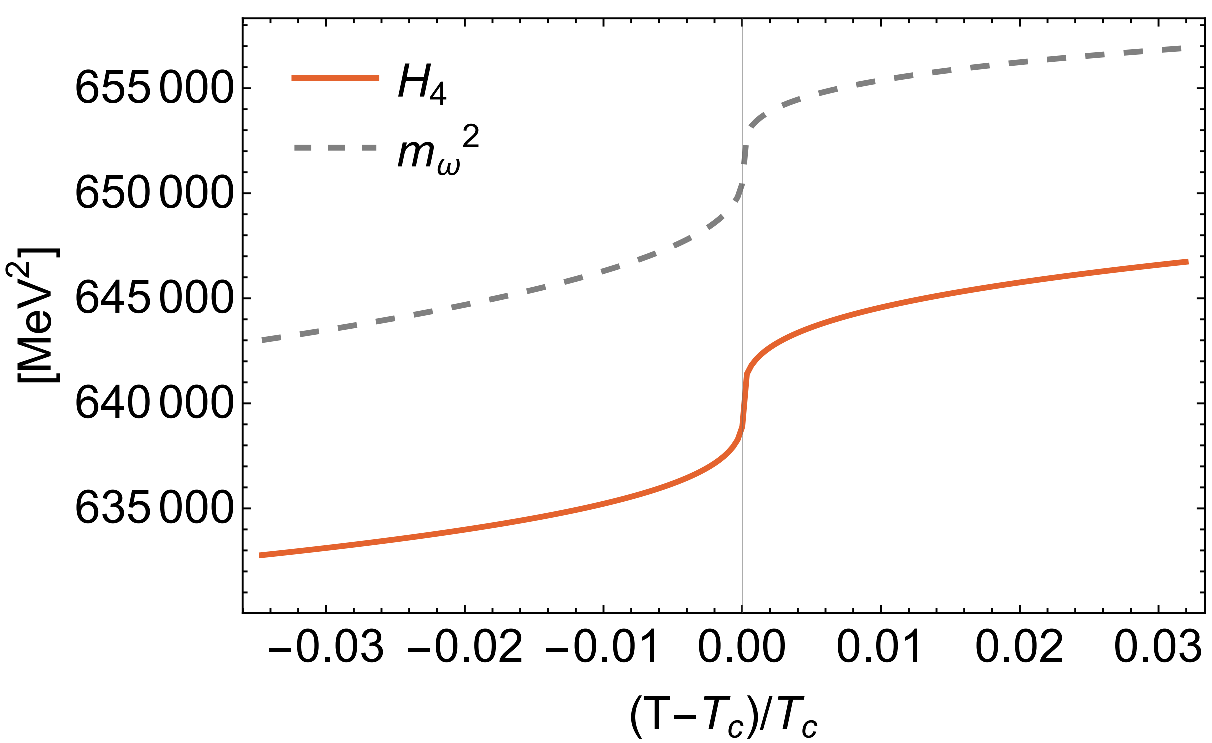


PQM MODEL WITH ω_0

Now consider the full model, including vacuum term and a repulsive ω_0. The four eigenvalues are







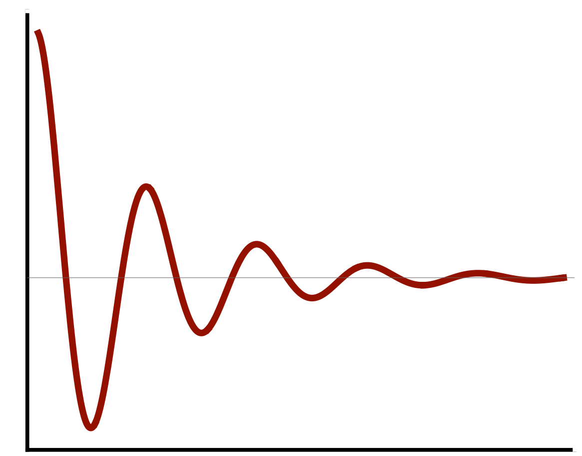
- all eigenvalues are real around the CEP
- H_1 defines the critical mode
- $\Omega^{(2)}_{\sigma\sigma} = m^2_\sigma$ becomes negative, physically irrelevant
- complex eigenvalues still present, e.g., at larger T

the critical mode is a mixture of σ, ω_0, L and $\bar{L}$ at finite T and μ

COMPLEX EIGENVALUES

Eigenvalues of mass matrix related to screening masses of eigenmodes

$$\langle \chi_i(r)\,\chi_i(0)\rangle \xrightarrow{r\to\infty} \sim e^{-r\sqrt{H_i}}$$



Complex eigenvalues,

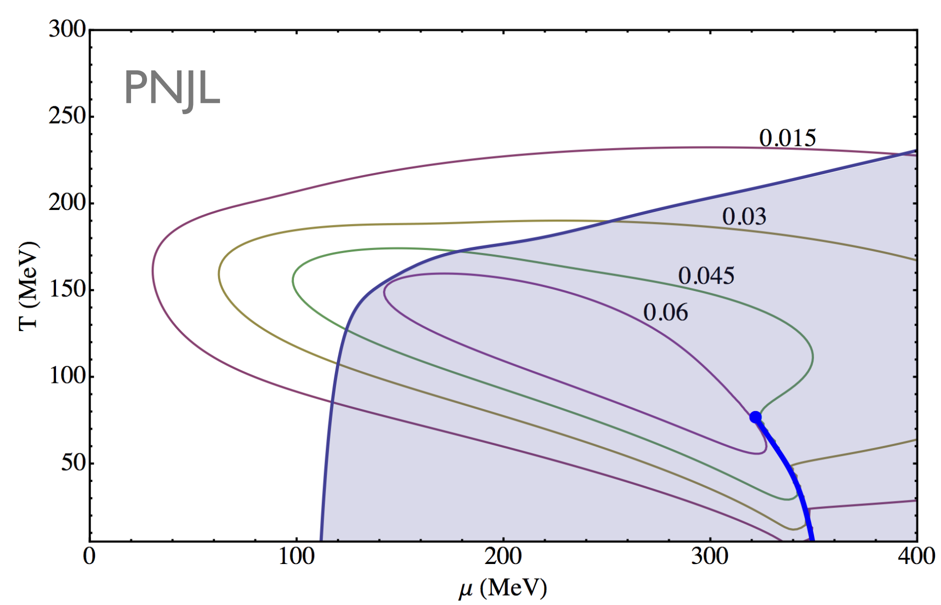
$$\sqrt{H_i} = m_R + i m_I$$

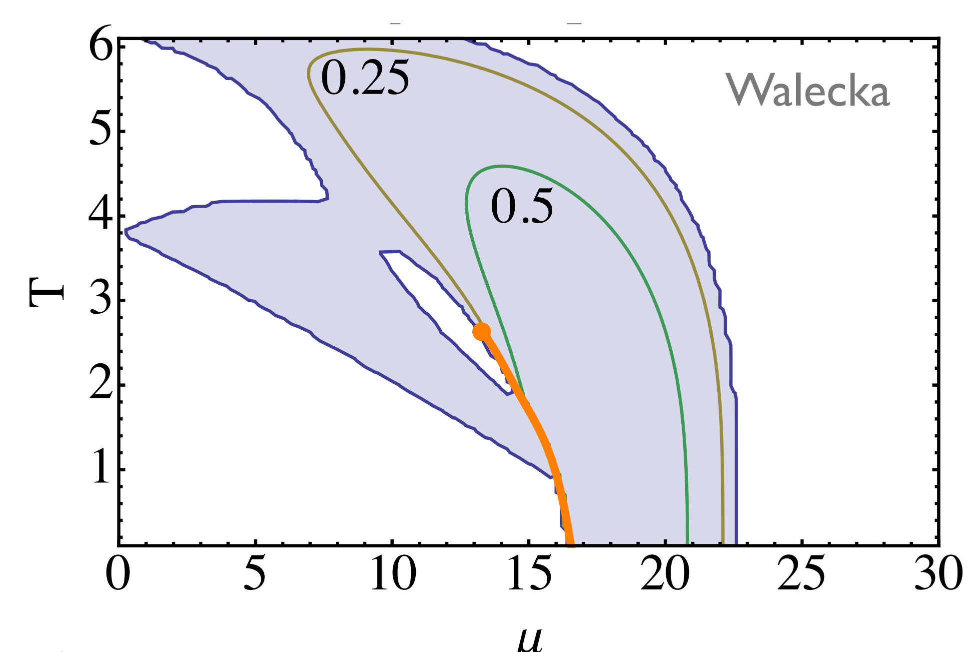
lead to spatially modulated correlations

$$\langle \chi_i(r)\,\chi_i(0)\rangle \xrightarrow{r\to\infty} \sim e^{-m_R r}\sin(m_I\, r)$$

Complex eigenvalues imply the existence of **disorder lines** in the phase diagram, which separate regions with spatial modulations from regions without.

This appears to be a common feature of systems with $\mathscr{C}$-symmetry breaking and a competition between repulsive and attractive interactions

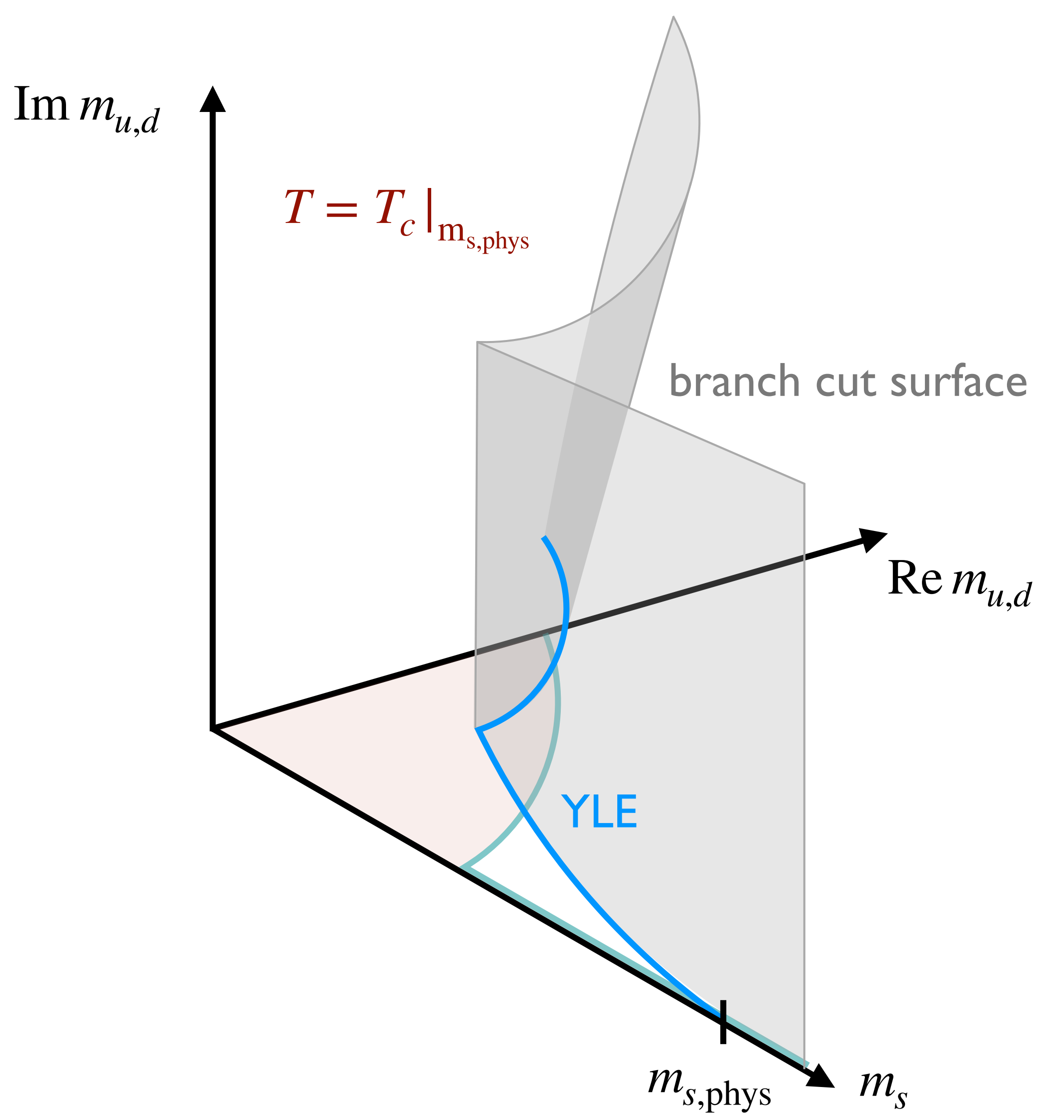




for detailed discussions see:
[Nishimura, Ogilvie, Pangeni (2014-2017); Schindler, Schindler, Medina, Ogilvie (2020); Schindler, Schindler, Ogilvie (2021)]

THE COLUMBIA PLOT AND EDGE SINGULARITIES

[Herl, FR, Schmidt, von Smekal (in preparation)]

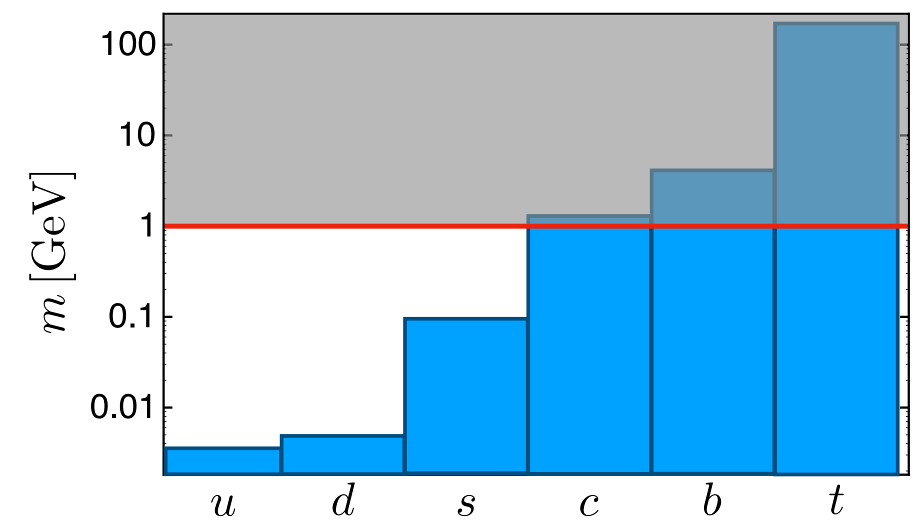


THE COLUMBIA PLOT

How does the order of the chiral phase transition depend on the quark mass?

- distinct mass hierarchy of quarks ($2\pi T_c \approx 1\,\mathrm{GeV}$)

→ what if u, d were even lighter?



- relevant flavor symmetry:

→ any "remnants" at physical quark masses?

$$U(3)_L \times U(3)_R \approx SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$$

↓ axial anomaly

$$SU(3)_V \times SU(3)_A \times U(1)_V \times Z(3)_A$$

↓ chubby strange quark

$$\underbrace{SU(2)_V \times SU(2)_A}_{\sim O(4)} \times U(1)_V \times Z(3)_A$$

↓ light quark masses

$$SU(2)_V \times U(1)_V \times Z(3)_A$$

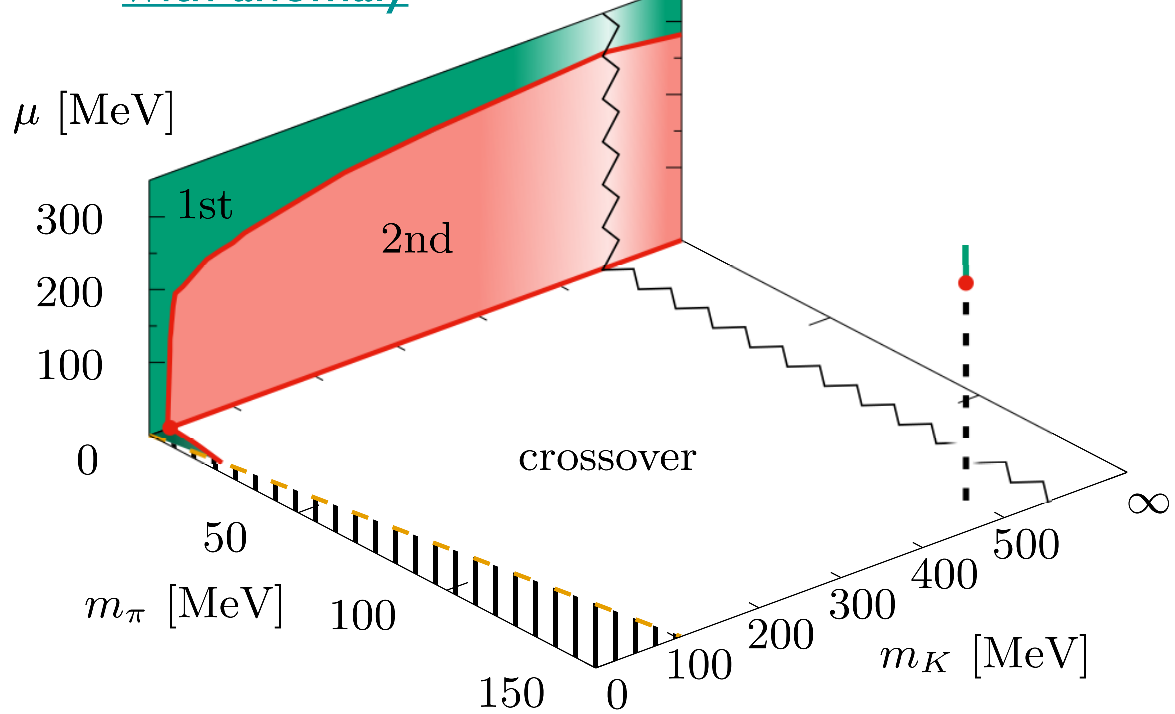
THE COLUMBIA PLOT

Expectation from Pisarski & Wilczek (1983) (perturbative RG analysis of a linear sigma model):

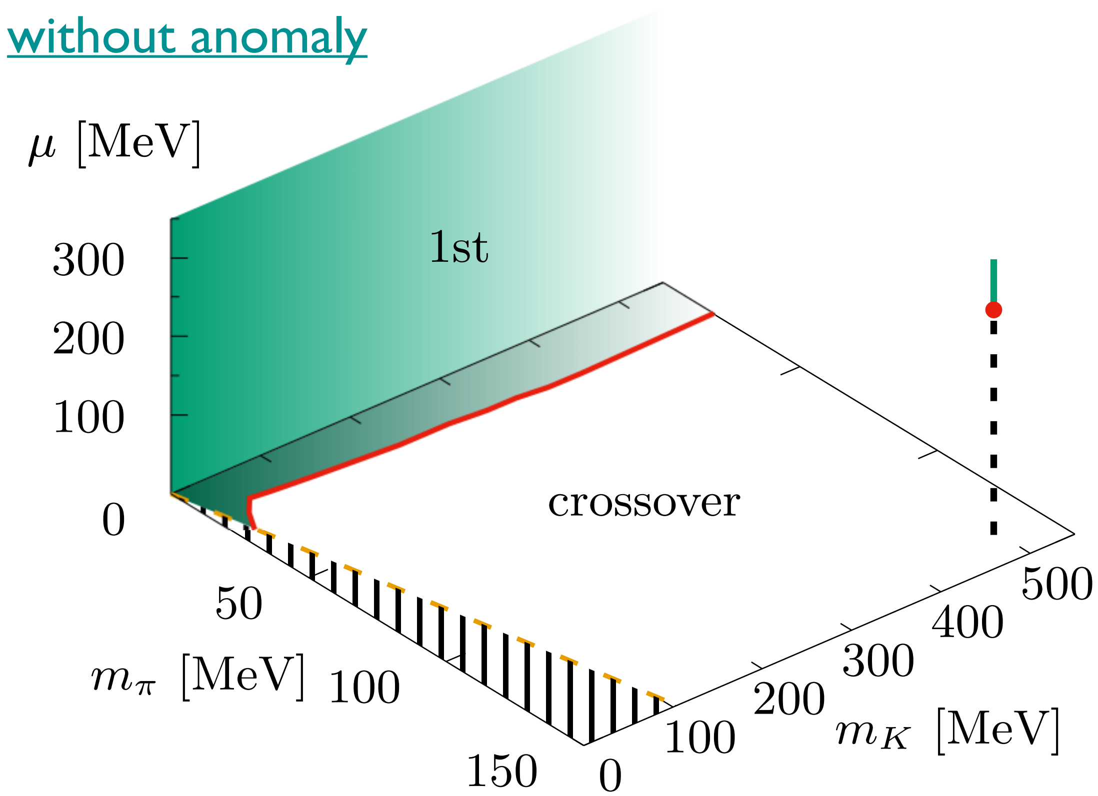
- $N_f = 3$ chiral quarks: 1st order transition
- $N_f = 2$ chiral quarks: depends on the fate of the axial anomaly

Nonperturbative RG analysis: [Resch, FR, Schaefer (2017)]

with anomaly



without anomaly

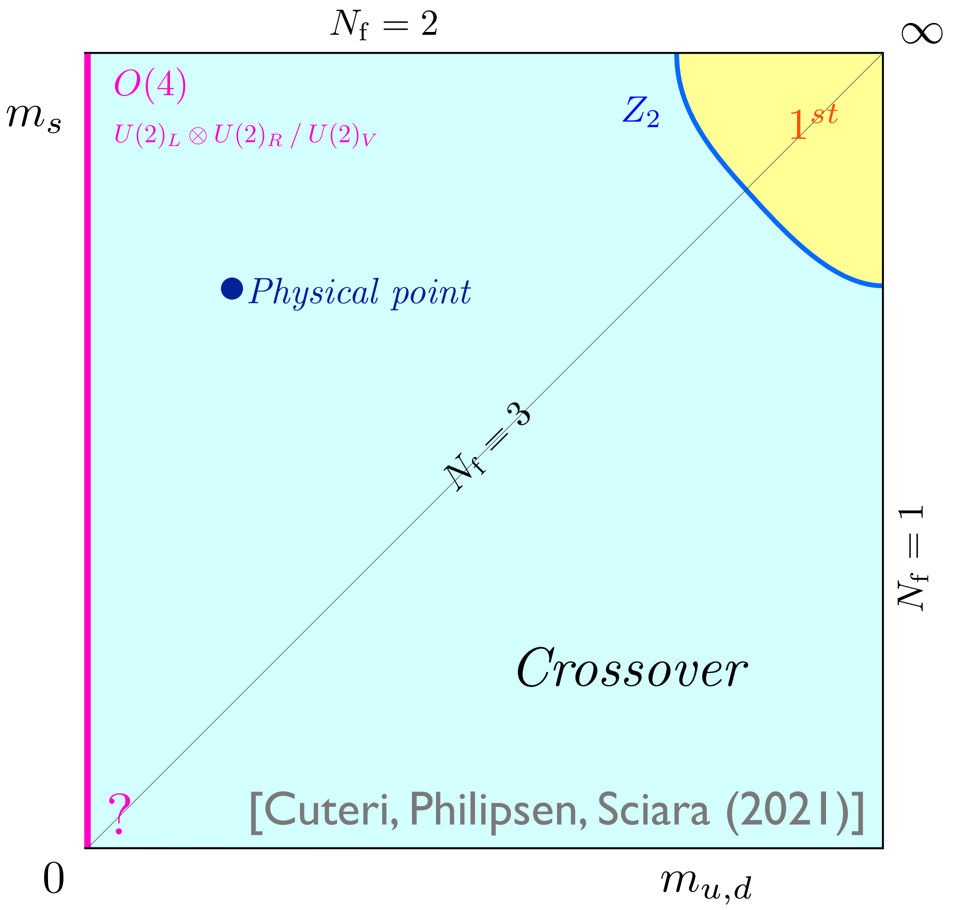


→ suggests very small 1st order region in the 3-flavor chiral limit
(triggered by bosonic fluctuations - much larger in mean-field)

Also: no **stable** fixed point from recent FRG analysis in the 3-flavor chiral limit [Fejos (2022)]

THE COLUMBIA PLOT

Could there even be a 2nd order transition in the 3-flavor chiral limit?



- generic prediction of mean-field studies of models without 't Hooft determinant [e.g. Resch, FR, Schaefer (2021)]
- fixed-point analyses: only possible if $U(1)_A$ is restored at T_c? [Fejos (2022), Kousvos and Stergiou (2023)]
- cannot be excluded from lattice computations [Aarts et al. (2023) & references therein]
- detailed lattice study suggests 2nd order transition even for $N_f \leq 6$ massless quarks [Cuteri, Philipsen, Sciara (2021)]
- suggested by recent DSE study [Bernhardt, Fischer (2023)]

Can YLEs help us here?

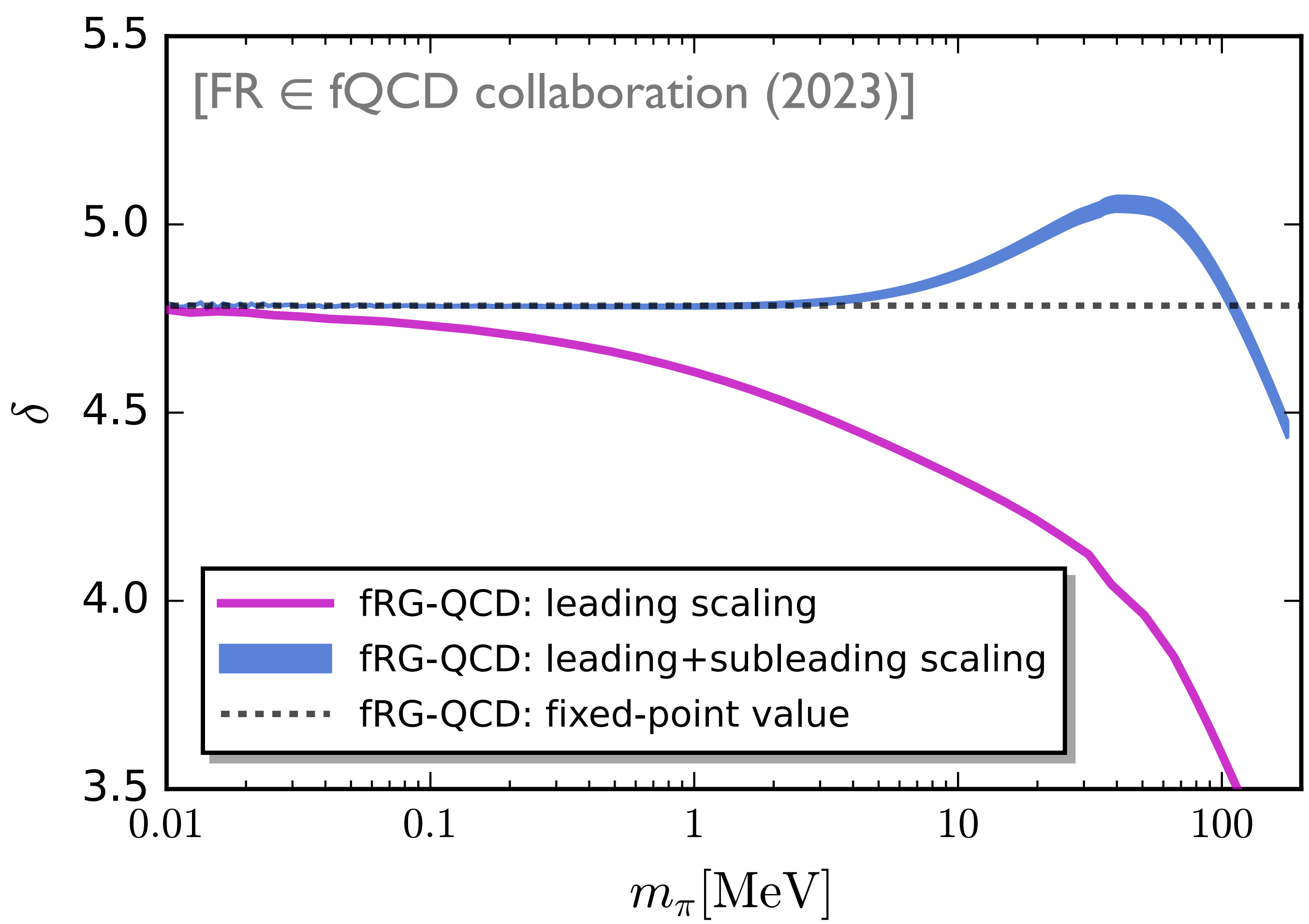
EXTRAPOLATIONS USING LATTICE DATA

Available lattice data still far away from any chiral limit

→ extrapolations are necessary

But how to extrapolate?

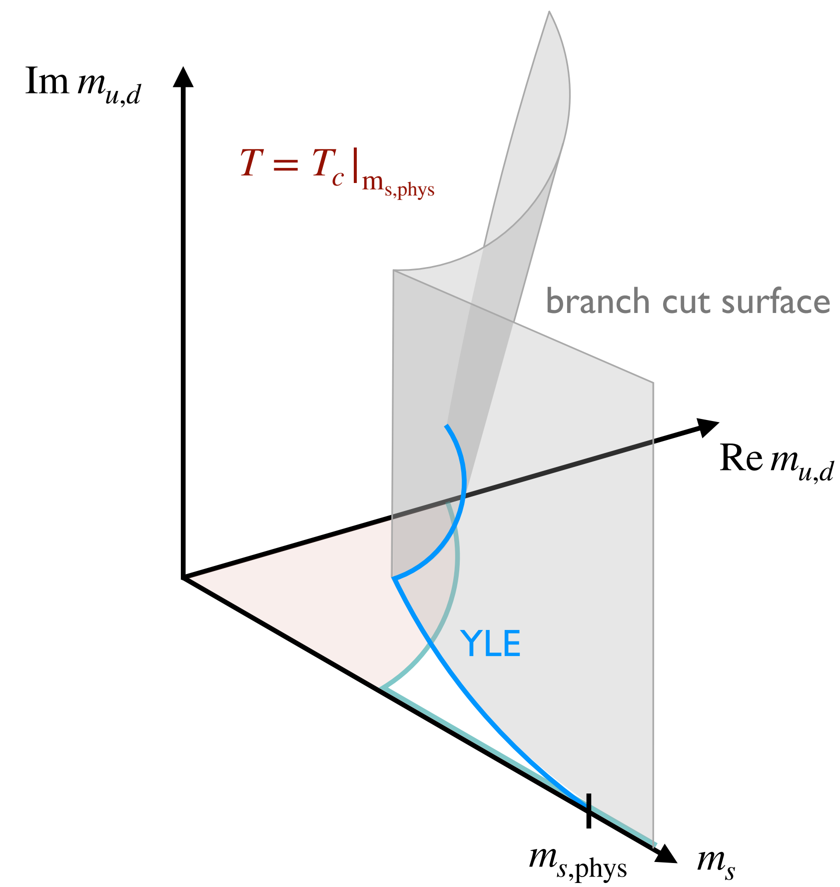
- **If** the data reaches into the scaling region, one can exploit universality
- this is difficult, because scaling regions are generically very small, e.g.,



→ scaling region for $m_\pi \lesssim 5\,\mathrm{MeV}$ at physical strange quark mass

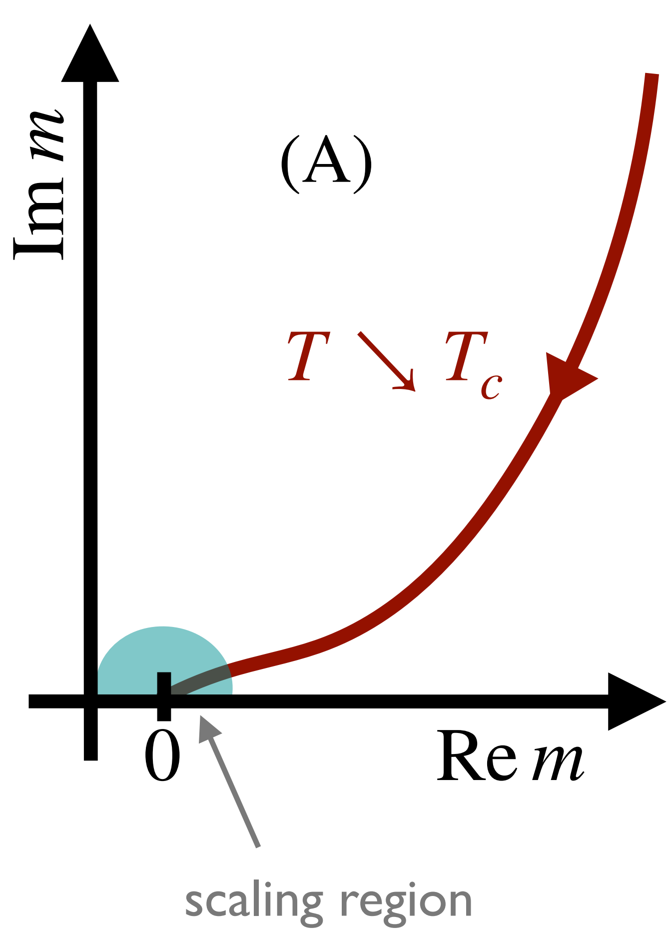
→ even with a lot of precise data, for $m_\pi \gtrsim 25\,\mathrm{MeV}$no signs of scaling

YLE AND THE COLUMBIA PLOT



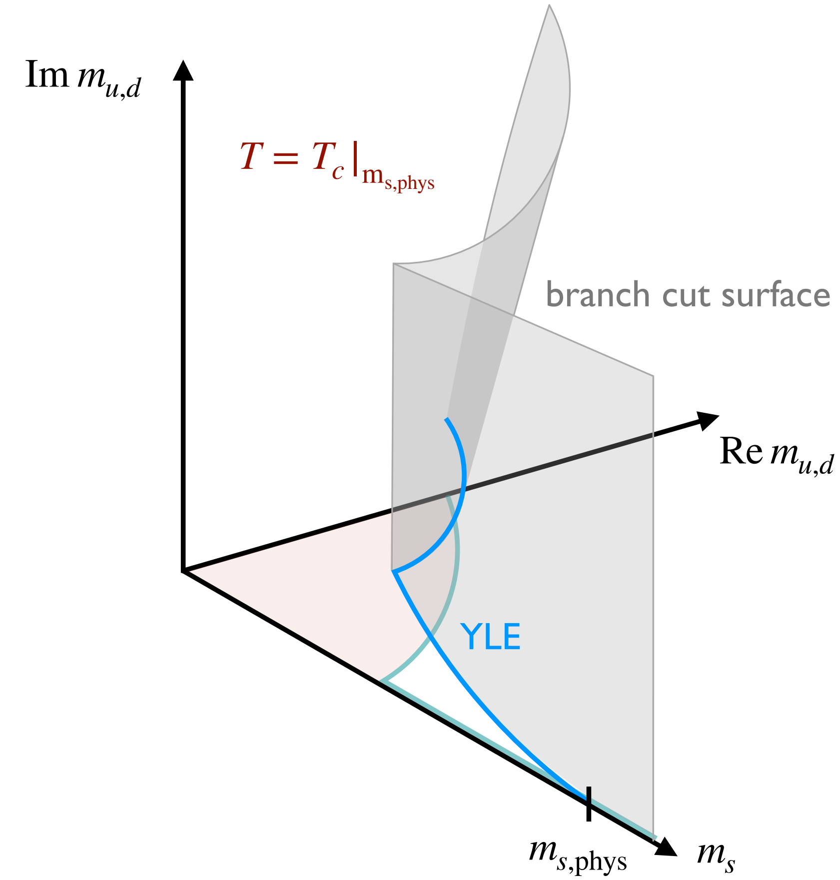
- consider quark mass as thermodynamic control parameter
  (acts like magnetic field in $O(N)$ models)
- search for 2nd order transition at some (T_c, m_c)
- YLE in the complex-mass plane at $T > T_c$

There are in general 3 different scenarios:



- 2nd order transition at zero mass
- no further restriction on the transition
- requires reconstruction + extrapolation for various T in the continuum limit

YLE AND THE COLUMBIA PLOT



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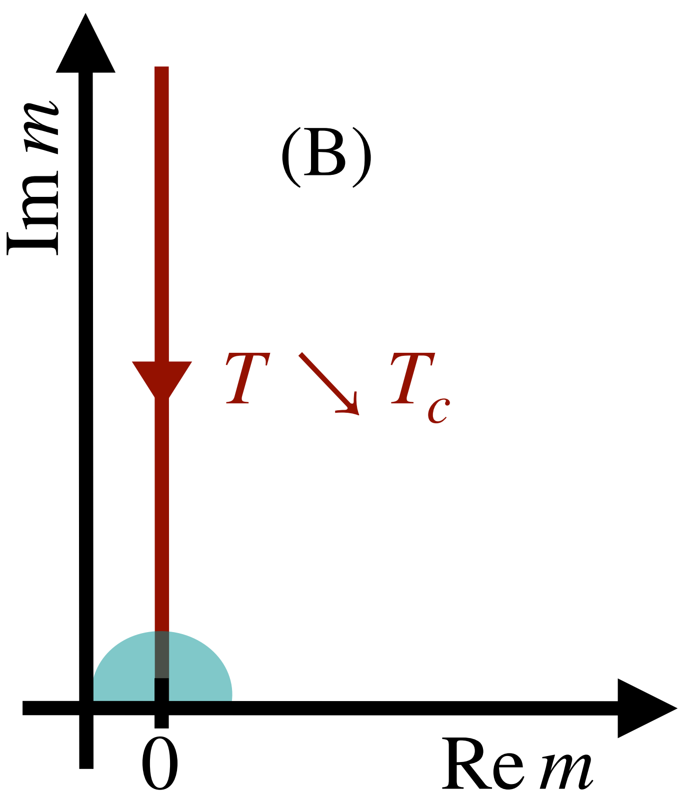
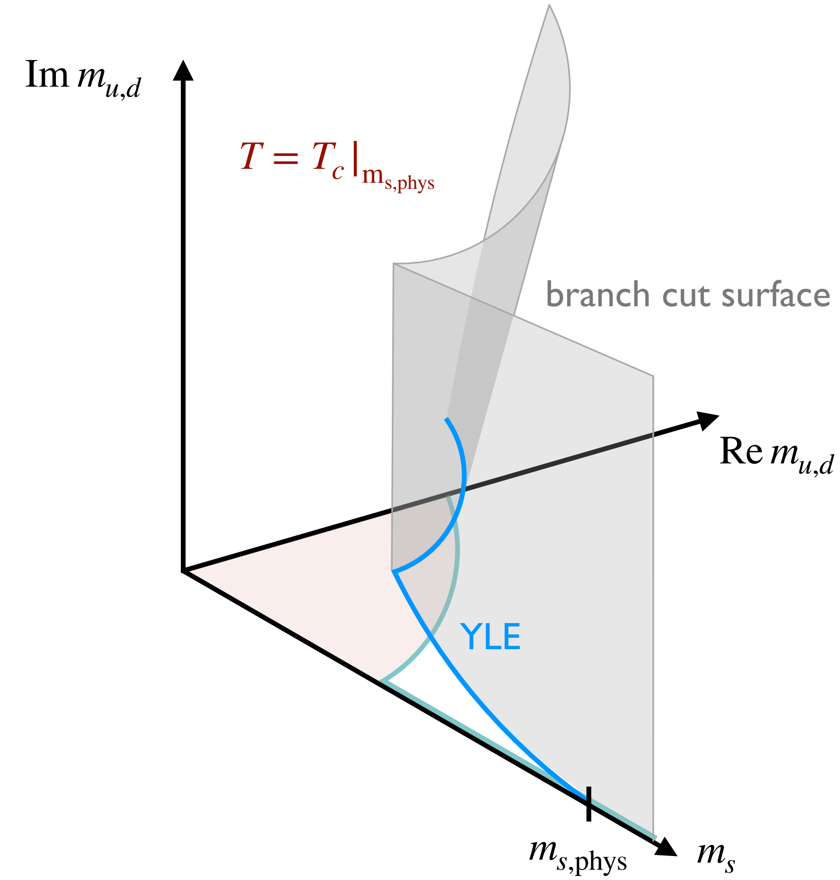


- 2nd order transition at zero mass
- **Lee-Yang circle theorem applies**
- YLE must lie on the imaginary mass axis
- also applies to Yang-Lee zeros

→ **infer that transition must be at zero mass without any extrapolation, neither to small T, m or the continuum**

- reconstruction of YLE still necessary

YLE AND THE COLUMBIA PLOT



- consider quark mass as thermodynamic control parameter
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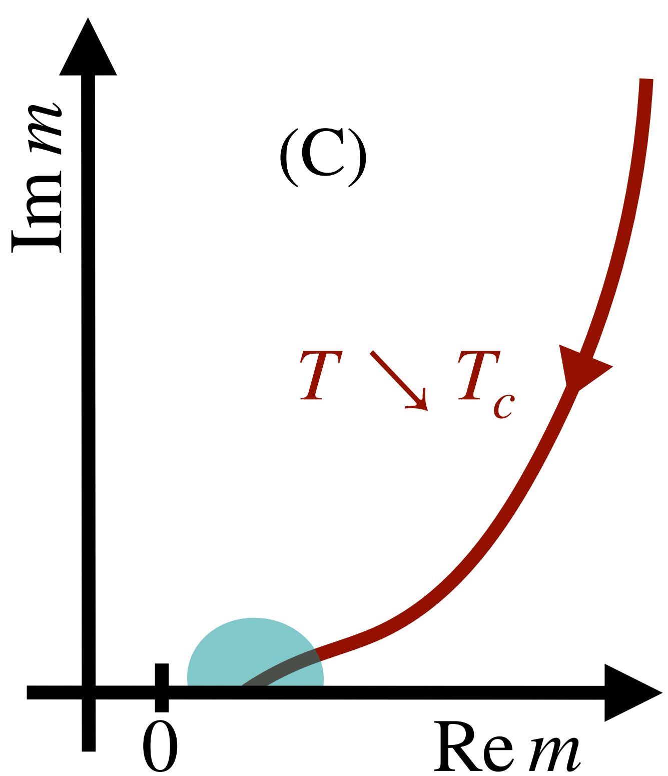


- 2nd order transition at nonzero mass
- circle theorem irrelevant, as map from m to critical magnetic field is nonlinear
- requires reconstruction + extrapolation for various T in the continuum limit

RECONSTRUCTING THE YLE

Adapt the strategy used for finite μ in [Dimopoulos et al. (2022)] to finite m:

$\longrightarrow$ Multi-point Padé reconstruction

- assume that analytic structure of the free energy is captured by a rational function

$$f(z) \approx R_n^m(z) = \frac{P_m(z)}{1 + Q_n(z)} = \frac{\sum_{i=0}^m a_i\, z^i}{1 + \sum_{j=1}^n b_j\, z^j}$$

- consider $f(z)$ at N nodes z_k $(k = 1, \ldots, N)$ and assume we know its derivatives up to order L_k at each node

$\longrightarrow$ we can fix $n + m + 1 = \sum_{k=1}^{N} (L_k + 1)$ Padé coefficients

$$\begin{aligned}
P_m(z_1) - f(z_1)\, Q_n(z_1) &= f(z_1) \\
P'_m(z_1) - f'(z_1)\, Q_n(z_1) - f(z_1)\, Q'_n(z_1) &= f'(z_1) \\
&\vdots \\
P_m(z_N) - f(z_N)\, Q_n(z_N) &= f(z_N) \\
P'_m(z_N) - f'(z_N)\, Q_n(z_N) - f(z_N)\, Q'_n(z_N) &= f'(z_N) \\
&\vdots
\end{aligned}$$

RECONSTRUCTING THE YLE

- rational functions can only have isolated poles (zeros of the denominator)
- branch cuts are indicated by arcs of poles, accumulating at branch points for large N, [Stahl (1997)]
- identify the YLE as the closest pole to the real axis that is stable under variation of the Padé order [m/n]

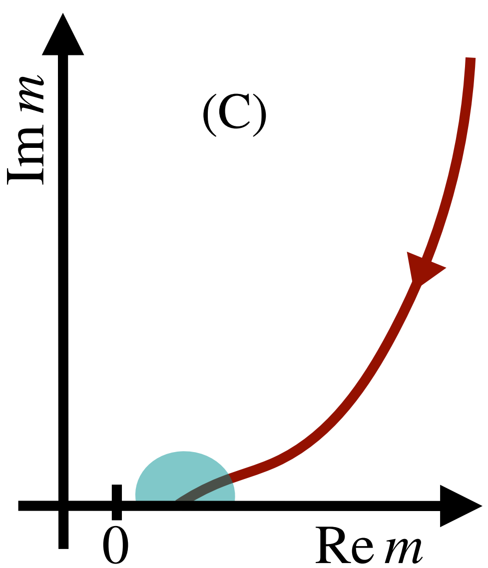
Test this in a simple $N_f = 2$ QM model (PQM with $A_0 = 0$), where parameters can be tuned such that scenario (B) and (C) are realized.

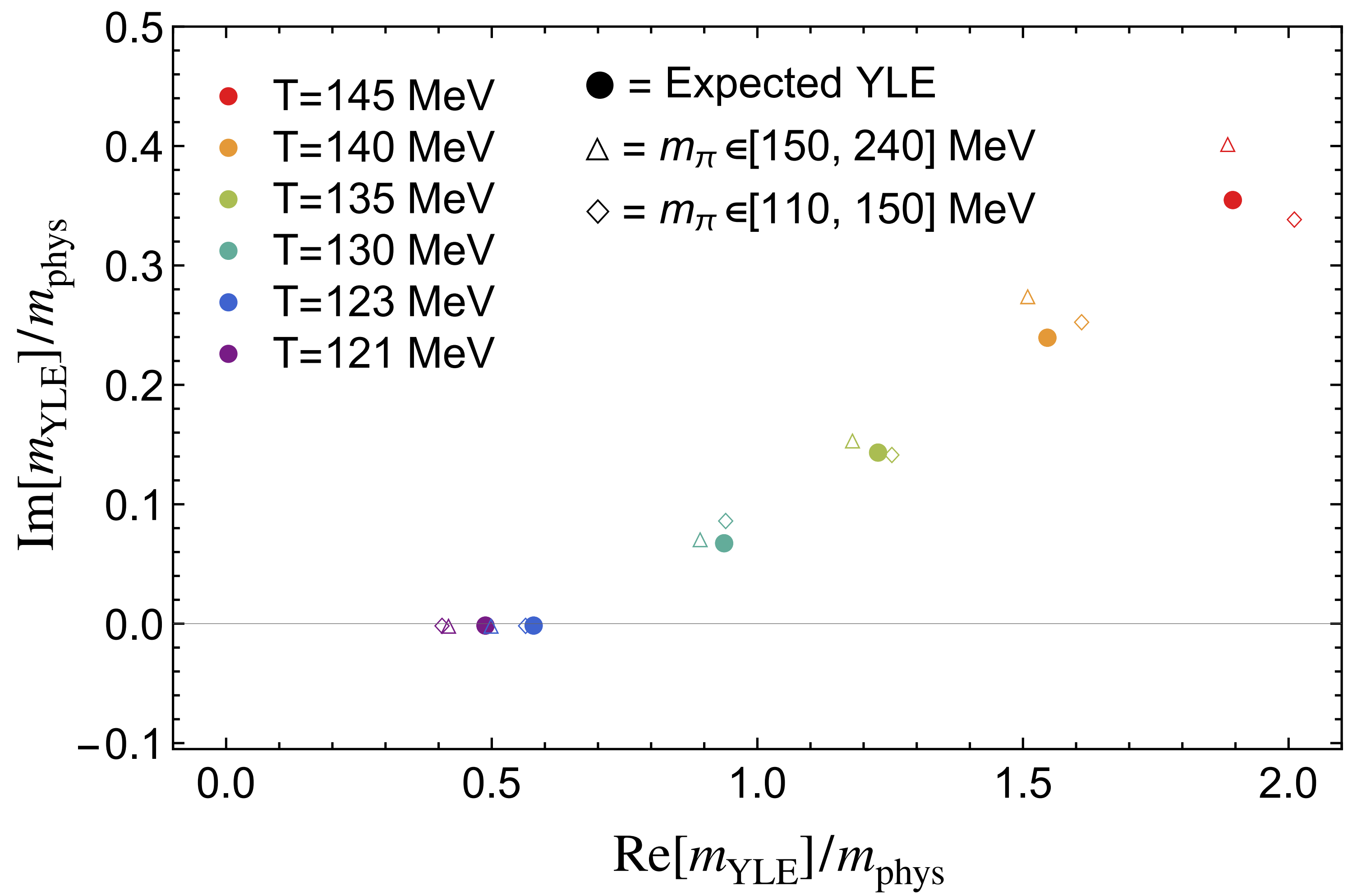
- Use 6 nodes for the chiral susceptibility $\chi_m \sim \dfrac{\delta\sigma}{\delta m}$
- 2 known derivatives at each node
- susceptibility is an even function of m

$\longrightarrow$ use [16/18] Padé in m

SCENARIO C

In this model: Ising transition at $m > 0$

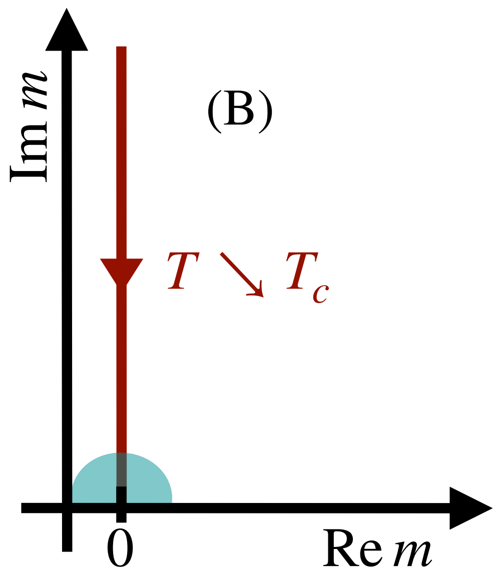


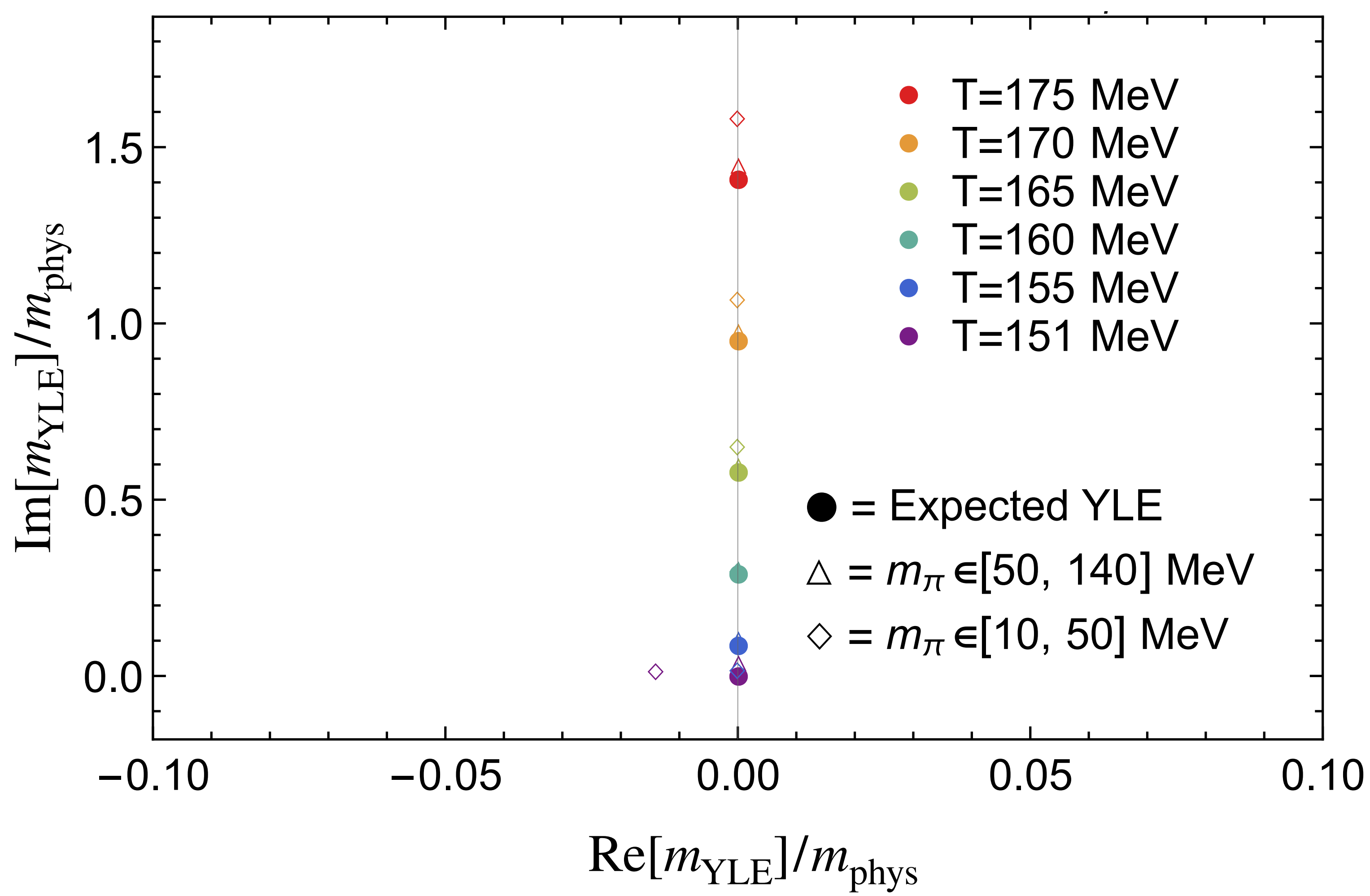


→ reconstruction works well, but extrapolation is required if data at smaller T not available

SCENARIO B

In this model: $O(4)$ phase transition at $m = 0$





→ reconstruction works well, no extrapolation required to infer m_c

TOWARDS THE APPLICATION TO QCD

So far, we did a successful proof-of-principle based on a simple model.
We are currently working on:

- improving the reconstruction, e.g., using conformal Padé [Basar (2021)]
- a conjecture regarding the application of the Lee-Yang circle theorem to the $SU(3) \times SU(3)$ transition relevant for the 3-flavor chiral limit
- applying our idea to lattice data

Note that if the circle theorem holds also in the 3-flavor chiral limit, our method can be very powerful as there is no need for any extrapolation

In any case, analysis of YLEs in the complex mass plane will add another layer of useful information to this unsolved problem

SUMMARY

Analytic structure in the complex plane can shed new light onto open problems in QCD

We demonstrated this on two examples:

- we identified the critical mode of the CEP based on in-medium mixing and the resulting branch point

→ it is a mixture of the chiral condensate, the density and the Polyakov loops

- we proposed a new method to study the chiral phase transition based on the YLE in the complex mass plane

→ the circle theorem can provide powerful constraints, circumventing the need for extrapolations