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Historical Document

On the Law of Inertia*

Translation of: Ueber das Beharrungsgesetz

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I.

In a systematic investigation of the principles of dynamics, the primary task is to replace the somewhat antiquated Galilei–Newton formulation of the law of inertia by an appropriately modern formulation. Since Carl Neumann[1] and Ernst Mach[2] have conclusively proven the inadequacy of the conventional formulation, there exists no doubt that we have here no imaginary but a real and absolutely legitimate requirement of science. Credit is also due to Streintz[3] for recently pointing anew to the outstanding significance of the above problem. I myself then treated the topic in another place[4] mainly from a methodological viewpoint and will here now examine more closely its mathematical-physical side.

The inadequacy of the usual formulation of the law given by Newton is primarily due to the fact that it is neither said in relation to which coordinate system the motions of points left to themselves are straight nor in relation to what timescale (see below) they are uniform. It is only stated: "they are straight" and "they are uniform". However, disregarding for the moment the second statement, in relation to what are they straight? It seems unnecessary to investigate in detail the generally agreed difficulties which confront an answer to this question. No given material object in the universe is qualified to serve in all cases as reference object for the law of inertia.[5] In particular, a gravitational mechanics of the fixed stars would have nothing in the whole wide universe at its disposal to which the motions of the (only fictitious) points left to themselves and the fixed stars could be related. The same is true for molecular dynamics, this stellar astronomy in miniature.

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[1] Ueber die Principien der Galilei-Newton'schen Theorie, Lpz. 1870.

[2] Die Geschichte und die Wurzel des Satzes von der Erhaltung der Arbeit. Prag, 1872, p. 47.

[3] Die physikalischen Grundlagen der Mechanik. Leipzig 1883.

[4] Philosophische Studien, ed. by W. Wundt, Vol. II. pp. 266-297.

[5] see Neumann, loc. cit. p. 14 f. Mach, loc. cit. pp. 47-50.

Newton suppresses the answer to our question in the formulation of the law itself which precedes his gravitation theory;[6] however, it is by no means the case that he gives no answer. According to his explanations, he bases the law on a certain coordinate system that he calls the "absolute, homogeneous, infinite and immovable" space. This "absolute space" is real and not merely something conceived, though admittedly it is not accessible to our imperfect human sensory perception. It consists of indiscernable absolutely fixed points arranged next to each other; and it is only by comparison of the bodies with these "in themselves" fixed points, not by comparison with other matter, that the nature of the positions and motions of the bodies are to be recognized.[7] In brief, Newton's absolute space is a phantom that should never be made the basis of an exact science. It would not be difficult to demonstrate that Newton's assumption of an absolute space was not independent of his profound religious beliefs. He places absolute space like "absolute time" (see below)[8] in the most intimate relation to the all-present and eternal god, and precisely in this relation he again sees a guarantee for the infallability of these absolute essences, in contrast to their relative images, sensory space and sensory time, which often lead to deception. Already Euler dared to query Newton's foundation of dynamics[9], and Kant completely rejected *real* absolute space.[10] To be sure, no one was able to set something truly better in its place, and so it comes that mathematicians and physicists to this day continue to speak – though not without uneasiness – of absolute space, absolute fixed points and absolute motions because they do not have a fully valid substitute for these objectionable fictions. *To find a fully valid substitute for them is the goal of the following.*

However, before that we may consider Newton's already mentioned assumption of an "absolute time". All time measurement is, as is well known, based on motion: one measures the time by the path through which a point (the tip of a clock's hand, a fixed star) must run. The moving point supplies in a certain way a "timescale" that we use to order events. To judge the temporal relations of a given motion means to compare it spatially with another motion that is adopted as basis once and for all. But which motion, which timescale shall now be the basis for dynamical chronometry? Here, as everybody knows, arise very similar difficulties as before when we asked for the fundamental reference system of dynamics. No given motion in the universe is, in all rigour of theory, adequate for this purpose.[11] In recognizing this, Newton related the judgements of motion in his dynamical theories, and in particular the law of the constant velocity of points left to themselves, to an "absolute time" which – not

[6] Philosophiae naturalis principia mathematica, Amstaelod. 1714 editio II) pag. 12. Lex I.

[7] Loc. cit. pp. 5-11. Scholium ad definitiones.

[8] Compare Newton, Optice, 1740 (lat.) p. 298, and the final scholium of the second and third edition of the Principia (1714. p. 481 f.), where among other things it is stated that "God is eternal and at all places, and precisely through his eternity and omnipresence he creates time and space". That absolute time and absolute space are meant admits no doubt and is fully confirmed by the use of the word *duratio* in the place of *tempus* . Indeed, at another place it is explicitly said of absolute time in contrast to relative time that "Alio nomine dicitur duratio" (loc. cit. p. 5).

[9] Theoria motus (1765) Tom. I. Cap. I. Since Euler does not share Newton's religious-metaphysical basis, he does not hesitate here to criticize the Newtonian postulates for incomprehensibility. In Cap. II, however, he returns to Newton's views in order to avoid even greater "incomprehensibilities". Compare Streintz, loc. cit. pp. 40 ff.

[10] Namely in his critical period. Not only does Newton's real absolute space fade to a pure idea for Kant, but this idea has also not the slightest tinge of dynamics, so that hardly more than the bare word remains. Kant, Complete Works, ed. by Kirchmann, Vol. VII. Part I. p. 191.

[11] Newton, Principia, p. 7. Neumann, loc. cit. pp. 16 f.

perceptible for us humans – "of itself flows uniformly."[12] It has hardly to be pointed out that this absolute time is a similar phantom as absolute space. Precisely for this reason it is rather remarkable that it has largely disappeared from the modern dynamics, whereas the same is by no means true of absolute space. However, the explanation for this fact is not far to seek. We already have a fully valid substitute for the absolute time. We can formulate the time part of the law of inertia completely correctly without the postulate of an absolute time. We have only, following Neumann, to base the measure of time on the following definition: Two time intervals are said to be equal in which *a point left to itself* passes through equal spatial distances.[13] In other words: the fundamental timescale of dynamics is to be defined through the motion *of a point left to itself.* Under this viewpoint, the law of the "uniform" motion of all points left to themselves is, as Thomson and Tait correctly note, a *pure convention* for one such point, and it is more than convention, it is a research result, only insofar as it applies to *any other* points left to themselves.[14] Obviously it is nothing but a characteristic *elimination* process by which Neumann has removed the incomprehensible absolute time from the formulation of the law of inertia.

The question now arises whether it is possible to eliminate also absolute space by a similar procedure. Indeed this is possible. The fundamental coordinate system of dynamics may be characterized as an "inertial system", the fundamental timescale of dynamics as an "inertial timescale". In exactly the same way as the *one-dimensional* inertial timescale could be defined through *one single* point left to itself, the *three-dimensional* inertial system can be defined through *three* points left to themselves. The following consideration, whose kinematical starting points shall be established below, leads to this definition.

For three (or less than three) points P, P', P'' that are moving arbitrarily relative to each other – they need not to be left to themselves – it is always possible to construct a coordinate system, indeed infinitely many coordinate systems, in relation to which these points move rectilinearly. In contrast, for more than three points this possibility is given only under special circumstances, only contingently.

It follows from this that the law of the constant direction of motion of points left to themselves is *pure convention* for three such points, but embodies a noteworthy research result only insofar as it is valid for more than three, for arbitrary many points in relation to one and the same system. The *physical* condition of being free of external influences has just the very remarkable *kinematical* effect that for *arbitrary many* points obeying this condition there exists a coordinate system in which all of them are moving rectilinearly.

Herewith the sought-after definition of the inertial system is given in its essence. As the inertial timescale could be defined as a timescale in relation to which *one* point left to itself moves uniformly, the inertial system will be defined as a coordinate system in relation to which *three* points left to themselves move rectilinearly. In the further development of this thought, the following composite formulation of the complete law of inertia has indeed emerged, and this moreover along a path that we will take in the following section.

The law of inertia

Definition I. An "inertial system" is any coordinate system of the kind that in relation to it *three* points P, P', P'', projected from the same space point and then left to

[12] Newton, loc. cit. p. 5.

[13] Neumann, loc. cit. p. 18.

[14] Thomson-Tait, Treatise on Natural Philosophy, Vol. I. P. I § 246-248.

themselves – which, however, may not lie in one straight line – move on three arbitrary straight lines G, G', G'' (e.g., on the coordinate axes) that meet at one point.[15]

Theorem I. In relation to an inertial system the path of an *arbitrary fourth* point, left to itself, is likewise rectilinear.

Definition II. An "inertial timescale" is any timescale in relation to which *one* point, left to itself (e.g., P), moves uniformly with respect to an inertial system.

Theorem II. In relation to an inertial timescale *any other* point, left to itself, moves uniformly in its inertial path.[16]

II.

We are first of all concerned to obtain an exact statement as to the extent one is able to construct a coordinate system in relation to which one, two, three, but in general not more than three points, moving relative to each other, are moving rectilinearly. Here we are not yet thinking of material points left to themselves, but of moving geometric points.

1. Let an arbitrary number of such points P, P', P'', ... have relative to a for the moment completely arbitrary parallel coordinate system $\Xi H Z$, and for the moment likewise completely arbitrary timescale $0, \ldots t$, the coordinates ξ, η, ζ; ξ', η', ζ'; ξ'', η'', ζ''; ... as functions of t. Further, let another parallel coordinate system $X_1 X_2 X_3$ be moving in some way relative to $\Xi H Z$; then, as is well known, the coordinates of the points with respect to the second system are obtained by transformations of the form:

$$x_i = \alpha_i \xi + \beta_i \eta + \gamma_i \zeta + \delta_i, \qquad x'_i = \alpha_i \xi' + \beta_i \eta' + \gamma_i \zeta' + \delta_i, \ldots$$
$$(i = 1, 2, 3)$$

where α_i, β_i, γ_i, δ_i are twelve functions of t. Under the assumption of orthogonal coordinates there exist between α_i, β_i, γ_i the six independent equations:

$$\begin{cases} \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, & \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = 0 \\ \beta_1^2 + \beta_2^2 + \beta_3^2 = 1, & \gamma_1\alpha_1 + \gamma_2\alpha_2 + \gamma_3\alpha_3 = 0, \\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, & \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0. \end{cases} \tag{1}$$

Now we ask: For how many given points P, P', ... is it possible, without presupposing special dependences between their variable coordinates ξ, η, ζ, ..., to determine α_i,

[15] The definition given here differs from my earlier one, not with respect to the fundamental methodological idea, but with respect to the mathematical realization. Prof. A. Voss was so friendly to call my attention to a kinematical error and in this way to stimulate this second more mathematical attempt.

[16] One sees that, in a certain sense, the space and the time part of the law express the same fact twice, only in one case with respect to three-dimensional space, in the other with respect to one-dimensional time. I have denoted (loc. cit.) the characteristic elimination procedure through which this analogy expresses itself most strikingly by a special name, the principle of particular determination, because it turned out that it clears up some dark points in mechanics and in mathematical physics generally. Indeed it spreads the brightest light on a number of basic doctrines that are related to the law of inertia in some way: e.g., the law of the proportionality between a force and its effect, Ohm's law, etc.

β_i, γ_i, δ_i as functions of t such that all the paths described by the points are *rectilinear* with respect to $X_1X_2X_3$? At some number, as is immediately evident, this possibility will come to an end.

For each point the requirement that it should move rectilinearly with respect to $X_1X_2X_3$ results in two differential equations, e.g., for P the differential equations

$$\frac{d^2\,(x_2)}{dx_1^2} = 0, \qquad \frac{d^2\,(x_3)}{dx_1^2} = 0,$$

which, after transformation to $\varXi HZ$, and after introduction of the argument t in the twelve unknowns $\alpha_i\beta_i\gamma_i\delta_i$, are of second order. For n points this results in $2n$ differential equations which, by the above orthogonality conditions (1), are supplemented to a system of $2n+6$ equations. As long as $n \leq 3$, the number of equations to be satisfied does not exceed the number of unknowns. However, for $n > 3$ this does happen, and there is therefore no solution to the problem unless special dependences between ξ, η, ζ; ξ', η', ζ'; ... are assumed.

In the case of three given points it is therefore *just still* possible to construct a system in relation to which these are all moving rectilinearly. But not only one but ∞^{12} systems, because, as is easily seen, we must include 12 *arbitrary* integration constants. Now, according to a known result, $\infty^{12} = (\infty^4)^3$ is also the manifold of all possible combinations of three straight lines in space. It is therefore natural to conjecture that it is possible, by appropriate choice of these constants, to obtain as the paths of P, P', P'' three arbitrarily *prescribed* straight lines. The problem posed here can also be formulated and solved purely algebraically as follows.

2. The demand that P, P', P'' shall describe three prescribed straight paths in relation to $X_1X_2X_3$ leads to the following nine algebraic equations ($i = 1, 2, 3$):

$$\begin{cases} x_i = \alpha_i\xi + \beta_i\eta + \gamma_i\zeta + \delta_i = a_i + b_i\varphi(t), \\ x_i' = \alpha_i\xi' + \beta_i\eta' + \gamma_i\zeta' + \delta_i = a_i' + b_i'\varphi'(t), \\ x_i'' = \alpha_i\xi'' + \beta_i\eta'' + \gamma_i\zeta'' + \delta_i = a_i'' + b_i''\varphi''(t), \end{cases} \tag{2}$$

where the a and b are the t-independent quantities that fix the spatial positions of the prescribed paths, whereas $\varphi(t)$, $\varphi'(t)$, $\varphi''(t)$, which are unknown functions of time, are added to the already existing 12 unknowns $\alpha_i\beta_i\gamma_i\delta_i$. The system (2) is augmented by the system (1) of independent orthogonality conditions (see above) to a system of 15 equations which leads to the determination of the 15 unknown functions of t if the old coordinates $\xi\eta\zeta$, $\xi'\eta'\zeta'$, $\xi''\eta''\zeta''$ are, as presumed, given as functions of t. By actual solution we find how the coordinate system $X_1X_2X_3$ is to be moved relative to $\varXi HZ$ if in relation to the first one the paths of the points P, P', P'' are to be three *prescribed straight lines.* The answer can sometimes be infinitely ambiguous; we will soon see which assumptions must be made if this is not to happen.

If we set:

$$\begin{vmatrix} \xi & \eta & \zeta \\ \xi' & \eta' & \zeta' \\ \xi'' & \eta'' & \zeta'' \end{vmatrix} = \varDelta,$$

then it follows from the system (2) that

$$\begin{cases} \Delta\alpha_i = \begin{vmatrix} a_i + b_i\varphi - \delta_i & \eta & \zeta \\ a_i' + b_i'\varphi - \delta_i & \eta' & \zeta' \\ a_i'' + b_i''\varphi - \delta_i & \eta'' & \zeta'' \end{vmatrix}, \\ \Delta\beta_i = \begin{vmatrix} \xi & a_i + b_i\varphi - \delta_i & \zeta \\ \xi' & a_i + b_i\varphi' - \delta_i & \zeta' \\ \xi'' & a_i'' + b_i''\varphi'' - \delta_i & \zeta'' \end{vmatrix}, \\ \Delta\gamma_i = \begin{vmatrix} \xi & \eta & a_i + b_i\varphi - \delta_i \\ \xi' & \eta' & a_i' + b_i'\varphi' - \delta_i \\ \xi'' & \eta'' & a_i'' + b_i''\varphi'' - \delta_i \end{vmatrix}. \end{cases} \tag{3}$$

We can now multiply each of the six orthogonality conditions by Δ^2:

$$\begin{cases} (\Delta\alpha_1)^2 + (\Delta\alpha_2)^2 + (\Delta\alpha_3)^2 = \Delta^2, \\ (\Delta\beta_1)^2 + (\Delta\beta_2)^2 + (\Delta\beta_3)^2 = \Delta^2, \\ (\Delta\gamma_1)^2 + (\Delta\gamma_2)^2 + (\Delta\gamma_3)^2 = \Delta^2, \\ (\Delta\beta_1)(\Delta\gamma_1) + (\Delta\beta_2)(\Delta\gamma_2) + (\Delta\beta_3)(\Delta\gamma_3) = 0, \\ (\Delta\gamma_1)(\Delta\alpha_1) + (\Delta\gamma_2)(\Delta\alpha_2) + (\Delta\gamma_3)(\Delta\alpha_3) = 0, \\ (\Delta\alpha_1)(\Delta\beta_1) + (\Delta\alpha_2)(\Delta\beta_2) + (\Delta\alpha_3)(\Delta\beta_3) = 0. \end{cases} \tag{4}$$

If we then substitute here the right hand sides of the equations (3), we obtain a system of six equations in which only the six unknowns δ_1, δ_2, δ_3, φ, φ', φ'' appear. This system should now be examined further, in particular to establish the conditions under which its solution becomes infinitely ambiguous. For the moment, we leave this question open. In any case, from the above we obtain as first *necessary* condition that the coordinate system $X_1X_2X_3$ which satisfies the given prescription is not indeterminate:

$$\Delta \neq 0.$$

It says immediately that P, P', P'' must not lie in a plane with the origin of ΞHZ. Since, however, the presupposed coordinate system ΞHZ, and with it also its origin was completely arbitrary, we can certainly assume that this condition is satisfied if P, P', P'' do not lie on a straight line. Therefore we can also say: P, P', P'' must not lie on a straight line. This result is also intuitively evident. If the points lie in a line, we can rotate the system $X_1X_2X_3$ around this line in an arbitrary manner, so that it is not determinate. In fact, since Δ is a function of t, we need to distinguish between the possibilities that $\Delta = 0$ holds at an instant or that it holds identically. In the first case the indefiniteness is only momentary and of little significance; in the latter case it is, however, persistent.

Let us now consider in more detail the system that arises from (4), with the unknowns δ_1, δ_2, δ_3, φ, φ', φ''. This is, as is easily shown, quadratic, and therefore possesses several real and imaginary solutions; real coordinate systems $X_1X_2X_3$ correspond to the former, to the latter, when they occur, imaginary systems, in relation to which the points nevertheless move on real paths, namely the prescribed ones; I disregard here of course the case that one could also prescribe imaginary paths. We mention in passing that it is obviously possible in the general case that a coordinate system, obeying the given conditions, passes at times from the real to the imaginary domain, and vice versa.

The conditions under which the system that we have to solve possesses no series of solutions would not easily be found along the analytic route taken here. We prefer to translate this question back into the intuitive setting. Assume that P, P', P'' do not lie on a straight line. What conditions have to be added that for a fixed instant t no series of coordinate systems $X_1X_2X_3$ exists in relation to which P, P', P'' lie on the three prescribed straight paths? According to the assumption, the position of the system $X_1X_2X_3$ in relation to $\Xi H Z$ at time t is determinate if the position of the points P, P', P'' in relation to $X_1X_2X_3$ is determined. These points, however, have at the fixed instant certain given distances. The supplementary condition that we seek agrees therefore with the condition that the three points P, P', P'' separated by given distances cannot be placed in infinitely many different ways on three prescribed straight lines G, G', G''. Intuition already suggests that this condition is nothing else than that the prescribed straight lines must not be parallel. A calculation, which will not be given here, confirms this intuition. In order that there exists a series of arrangements of three rigidly connected points on three prescribed straight lines, it is necessary and sufficient that these are parallel; however, it is assumed here that a restriction is made to the prescription of real straight lines.[17]

Let us summarize the obtained kinematical results: The motion of a number of moving points on straight lines is a matter of *convention* as long as this number does not exceed 3. One can let three points move on three prescribed fixed straight lines by constantly adjusting the coordinate system with respect to which the fixed straight lines are referred to the changes of the distances between the points. Generally there are several such adjusted systems moving relative to each other; however, there is no series provided the three points do not lie in a straight line and the three prescribed straight lines are not parallel.

3. From the preceding *purely kinematical* considerations we now come to our true problem as we make the *dynamical* assumption that P, P', P'' are to be material and left to themselves. By itself the question arises: Can one perhaps define an "inertial system" simply as a coordinate system in relation to which three arbitrary points, not lying on a straight line and left to themselves, move on three non-parallel straight lines?

Already due to simple manifold considerations this question is to be answered in the negative. As already Newton and Euler realized[18], inertial systems constitute a manifold of systems that move rectilinearly without rotation and (according to the inertial timescale) uniformly with respect to one of them which can be chosen arbitrarily. The dimension of this manifold is easily specified; for this, let us restrict ourselves here to orthogonal systems. If we put together at any time all inertial systems that are not moving relative to each other and which differ only by origin and direction of the axes into one complex, then there are ∞^3 such complexes. This is because the rectilinear uniform motion of each of them against the chosen reference inertial system can be different according to the twofold variable direction and the single variable velocity. However, each complex again contains ∞^6 systems so that there are in total ∞^9 orthogonal inertial systems.[19] Now the manifold of the orthogonal systems in which P, P', P'' move rectilinearly in some way is ∞^{12} (see above). Therefore the considered systems include non-inertial ones, and we need further conditions under

[17] This follows from the consideration of the condition under which a certain functional determinant with three rows vanishes identically.

[18] Newton, Principia, p. 18 (Coroll. V). Euler, Mechanica, Tom. I § 59. 69. 77. 80. 82. (Cf. Theoria motus.) It may be noted that both authors muddy this finding by the superfluous metaphysical supposition of a real absolute space.

[19] If one regards, as I did earlier (loc. cit.), all inertial systems at rest relative to each other as one system (taking no account of the kinematically arbitrary origins and directions of the axes), the manifold of the three-dimensional inertial systems is then ∞^3. By analogy, the manifold of the one-dimensional inertial timescales is, if one ignores possible shifts of the origin, ∞^1.

which the too wide provisional definition of an inertial system just stated is appropriately restricted. In the completely general case in which the three "fundamental points" P, P', P'' are arbitrary, it seems to me that it would not be easy to find such conditions. It will be different in the special case that we base the construction on three points ejected simultaneously from the same place and then left to themselves. Then the necessary and sufficient restriction is that the three straight lines must meet in one point. In order to show this, I state and prove the following.

Theorem: A system with respect to which three material points that are simultaneously ejected from the same space point, do not lie in a straight line and are left to themselves describe three non-coincident straight lines that pass through one point is an inertial system. I.e., in relation to such a system *any arbitrary fourth* point left to itself also moves on a straight line.

Proof: I may start from the assumption, many times physically confirmed, that an inertial system and an inertial timescale are kinematically possible; i.e. that in relation to some not yet known system $\Xi H Z$ and in relation to some not yet known timescale $0, \dots t$ the motions of arbitrarily many points, left to themselves, are rectilinear and uniform. We now assume that all motions are referred to this system and timescale. For simplicity, we place the origin of the inertial system at the common starting point of the fundamental points P, P', P'', and analogously the origin of the timescale at the moment when these three points coincide. Under these assumptions we can simply write:

$$\begin{aligned} \xi &= \kappa\, t, & \eta &= \lambda\, t, & \zeta &= \mu\, t,\\ \xi' &= \kappa'\, t, & \eta' &= \lambda'\, t, & \zeta' &= \mu'\, t,\\ \xi'' &= \kappa''\, t, & \eta'' &= \lambda''\, t, & \zeta'' &= \mu''\, t, \end{aligned}$$

where κ, λ, μ are certain quantities independent of t.

What now is the motion relative to $\Xi H Z$ of a coordinate system $X_1 X_2 X_3$ in relation to which P, P', P'' move along three prescribed straight lines that emanate from one point? If the intersection point of the straight lines coincides with the origin of $X_1 X_2 X_3$, the equations of the prescribed paths have the form ($i = 1, 2, 3$):

$$x_i = b_i\, \varphi(t), \qquad x_i' = b_i'\, \varphi'(t), \qquad x_i'' = b_i''\, \varphi''(t),$$

where the b are prescribed constants, and the $\varphi(t)$ are unknown functions of t. The system (2) therefore simplifies through the elimination of the constants a.

Furthermore we have

$$\Delta = \begin{vmatrix} \kappa & \lambda & \mu \\ \kappa' & \lambda' & \mu' \\ \kappa'' & \lambda'' & \mu'' \end{vmatrix} t^3 \equiv R\, t^3,$$

where by assumption R is a non-zero constant. The first equation of the system (3) can be written as

$$R\, t\, \alpha_i = \begin{vmatrix} b_i \varphi - \delta_i & \lambda & \mu \\ b_i' \varphi' - \delta_i & \lambda' & \mu' \\ b_i'' \varphi'' - \delta_i & \lambda'' & \mu'' \end{vmatrix},$$

or

$$R\, t\, \alpha_i = A_i\, \varphi + A_i'\, \varphi' + A_i''\, \varphi'' + B_i\, \delta_i,$$

and finally, after division by t:

$$R\, \alpha_i = A_i \frac{\varphi}{t} + A_i' \frac{\varphi'}{t} + A_i'' \frac{\varphi''}{t} + B_i \frac{\delta_i}{t},$$

where here A and B are independent of t. Similar equations follow for $R\,\beta_i$ and $R\,\gamma_i$. From the system of orthogonality conditions, we then obtain a system of six equations, quadratic in the unknowns:

$$\frac{\varphi}{t}, \quad \frac{\varphi'}{t}, \quad \frac{\varphi''}{t}; \qquad \frac{\delta_1}{t}, \quad \frac{\delta_2}{t}, \quad \frac{\delta_3}{t}$$

whose coefficients are all independent of t. According to the foregoing, the solutions of this system form no series. Since they consist of quantities p, p', p'', q_1, q_2, q_3 which are all independent of t, we can write:

$$\varphi = p\,t, \quad \varphi' = p'\,t, \quad \varphi'' = p''\,t; \quad \delta_1 = q_1\,t, \quad \delta_2 = q_2\,t, \quad \delta_3 = q_3\,t.$$

All direction cosines take *constant* values, according to the equations set up for $R\,\alpha_i$, $R\,\beta_i$, $R\,\gamma_i$.[20]

Therefore the coordinate system $X_1X_2X_3$ with respect to which P, P', P'' move along the three prescribed straight lines moves relative to the unknown inertial system ΞHZ rectilinearly without rotation and uniformly according to the unknown inertial timescale $0, \ldots t$. Therefore one does not need an analytic proof that it is an inertial system.

Among the uncountable inertial systems which arise from the given definition, by changing the combination of the three prescribed straight lines there are real and imaginary ones. However, it can be shown algebraically that among these systems is none which at times passes from the real to the imaginary domain or vice versa.

The *ideal construction of an inertial system* would be accomplished in the following way. Three material points are simultaneously projected from the same space point and then left to themselves. After one has assured that they do not lie in a straight line, one connects them by straight lines with a fourth arbitrarily chosen space point whereby a three-sided pyramid-like solid is formed. If one now lets the solid keep its shape unchanged and moves it relative to the fundamental points in such a way that every point moves continuously along an edge, then every coordinate system in which the solid has an unchanging position is an inertial system. The solid itself can also be directly taken as an inertial reference system, only its edges may then not lie in a plane. It deserves mention that this construction always leads to real systems.

III.

Now I have to say a few words about the attempts at a new formulation of the law of inertia which have been made prior to mine by other authors.

Carl Neumann tries, as is known, to give a comprehensible content to the law by the transcendent or (if one prefers) transcendental assumption that the reference system for the spatial part of the law is represented in an unknown way by some matter of the universe; for instance in such a way that at an unknown place of the universe there exists an absolutely rigid body "Alpha", relative to which the paths of points left to themselves are straight.[21] The *hypothesis of existence* thereby proposed would be entirely appropriate if it were *absolutely necessary:* it could and should then be accepted unhesitatingly, like numerous other hypotheses of the natural sciences. In the present case, however, we can get by, as I think I can say has been proven, with a

[20] The assumption, not unimportant for the following, that there is no jump from one solution to another different one can be satisfied by the requirement that the points move continuously on their paths.

[21] loc. cit. p. 15 f.

mere (but useful) *convention* which satisfies our need for understanding much better than any *hypothesis*.[22] I may also point out that the proposed definition of an inertial system is nothing else than the transfer of Neumann's convention for time measurement from the one-dimensional time to three-dimensional space. To the transcendent body Alpha one could have easily set in parallel a transcendent succession "Beta"; if one omits the one hypothesis, it appears as a pure consequence to omit also the other one.

Mach's attempt at a new formulation of the law of inertia amounts to relating the point left to itself to the whole matter of the universe.[23] Since, however, this does not constitute an unchanging complex, one can, as Mach also points out, only speak of a "mean" motion of the point relative to it. However, there is no proof that this motion is rectilinear and uniform with sufficient relative precision that one could use it for reference purposes in eventual stellar-dynamic investigations; on the other hand, for dynamical study of the planetary motions the complex of the fixed stars, treated as unchanging, is adequate. It may also be noted that a reference system based on the distributed and eternally flowing matter of the universe lacks the desirable simplicity and uniformity.

Streintz defines a coordinate system that is thought of as rigidly connected with a "fundamental body" as a "fundamental system". His definition of the fundamental body derives, however, from the following basic idea. Gyroscopes and similar instruments provide us the means to establish whether a body is "free of rotational motions." For instance, the rotation axes of two arbitrary rotating gyroscopes enclose one and the same unchanging angle; they provide us with two "invariable" directions to which one has only to make reference in order to establish whether a given body moves rotationally. The rotation of the body recognized in this way can be said to be absolute because it is independent of the choice of two particular gyroscopes, i.e., it proves to be the same whatever pair of gyroscopes one may use. Streintz now defines a fundamental body as one which is perceived to be non-rotating by gyroscopic observation and which can be considered as completely independent of all surrounding bodies; accordingly, a coordinate system thought of as rigidly connected to it is a fundamental system.[24]

In that Streintz now relates the law of the invariable direction of motion of points left to themselves to a fundamental system, he tries to make absolute space superfluous. One may admit that he succeeded in that in the sense of *practical* physics. But not only do his definitions lack the desirable elegance, but insofar as they should provide the foundations of *theoretical* dynamics they also rest on a *methodological circle.* The law of the relatively invariable inclination of all gyroscopic rotation axes had to be stated in advance of the definition of the fundamental system, and therefore also of the law of inertia. In a textbook of theoretical physics one could not do other than to derive (implicitly) this theorem, now in the converse direction, from the law of inertia. However, this would be a circular derivation, completely comparable to the one in geometry, eliminated long ago, which contains the unreasonable demand to define the straight line as the shortest path between two points, and to prove afterwards that the straight line must be the shortest path between two points. I do not need to demonstrate that the definition of the inertial system proposed above does not suffer from any criticism of this sort. Now Streintz has indeed tried more than once to reject from the outset the objections just raised. It is true that his arguments certainly refute the unjustified criticism of logical circularity but in no way methodological circularity.[25]

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22 Mach, loc. cit. p. 48. Compare in addition Streintz, loc. cit. p. 9.

23 Mach, loc. cit., and "Mechanik,", Leipzig 1883. pp. 217 f.

24 Streintz, loc. cit. pp. 15-25.

25 Streintz, loc. cit. p. 31. Compare Philos. Studien, Vol. II. p. 285.

If accordingly I consider Streintz's formulation of the law of inertia as inappropriate to serve as a basis of *theoretical dynamics,* I am of course far from denying it *secondary* significance at the place where it is intended *to accomplish the important transition from theory to application.*[26]

Moreover, as regards the designation "fundamental system", it seems to me an unfortunate choice. A name for the reference system to be presupposed for all dynamical considerations will surely be *indispensable* in the future, but it will serve its aim all the better, the more precisely it exhibits the characteristic property of that system, namely to be the reference system of the law of inertia. Apart from other merits, the word "inertial" may also recommend itself by its ease of compound formation. I have only to recall the words "inertial rotation, inertial acceleration" and others whose content is immediately intelligible to the mathematician.

Streintz bases a second definition of the fundamental system on an aphoristic remark by Sir W. Thomson and Tait[27] which is here all the more of interest as it has some superficial similarity with the proposed definition of an inertial system. This definition, which, it may be noted, is placed after the one discussed above because it is "less natural", reads: "If several material points are projected from the same position A with arbitrary velocities in different directions, and then each point is left to itself, experience teaches that the angles which are enclosed by the directions of any two points at a time ..., are of invariable magnitude. A coordinate system which also stays directionally invariable against these directions and which has its origin in any of the points shall be called a fundamental system". I must now confess that this definition, as the basic definition of dynamics, appears to me still more natural than the other. Why? Because it is based upon the *basic entity* of dynamical theory, a point left to itself, in a similar way as geometry, i.e., pure geometry, creates its structures from geometric points. Besides that, also this second definition of the fundamental system cannot be freed from the criticism of methodological circularity. For one sees, here too the enunciation of the law of inertia is preceeded by the enunciation of an "experience" which can, conversely, be deduced from the law, indeed constitutes a consequence of the law; namely from the experience that the angles enclosed by the connection lines of the points are constant. It is to be remarked furthermore that *three* points fully suffice, and that each additional point would obscure the essence of the matter. Finally a surely not inconsiderable advantage of the formulation of the law of inertia proposed above consists in the fact that it presents in the most direct way not only the recognition of the *partial convention* lying in the law but also the infinite multiplicity of the inertial systems.

Finally I come to speak of a fourth attempt which has become public more or less simultaneously with my earlier one and which, as the third or fourth attempt within a year, can serve as further evidence that the question dealt with has presently become a vital one.[28] James Thomson dispenses altogether with the definition of an inertial system, or, more correctly, he does not consider it necessary to define the fundamental space system *prior* to the statement of the law. Namely, he expresses the law of inertia approximately in the following way: For a group of material points left to themselves it is *kinematically possible* to have a coordinate system, and not only one but infinitely many, in which all of the points move rectilinearly, and a timescale is *kinematically possible* in relation to which they all move uniformly on their straight lines.

[26] I mention in passing that also the d'Alembert-Poisson definition of a dynamical measure of time, recommended by Streintz (loc. cit.), is based on an analogous methodological circle. Compare Philos. Studien, Vol. II. pp. 292 f.

[27] Treatise, Vol. I. P. I. § 249. Streintz, loc. cit. p. 63. Marginal note.

[28] Proceedings of the R. S. of Edinburgh, Vol. XII. No. 116, pp. 568-578. On the Law of Inertia ...

However, anyone who does not know that it is extremely remarkable if *more than three* points move rectilinearly relative to one and the same coordinate system would not be able to fully appreciate the “great natural truth” just stated. Therefore, if one cannot avoid, at some time or other, from giving expression to this truth, it should be by far the most appropriate to express it, as done above, immediately in the formulation of the law itself. Moreover, Thomson agrees closely in many essential points with my views formulated earlier, as I may state with pleasure.