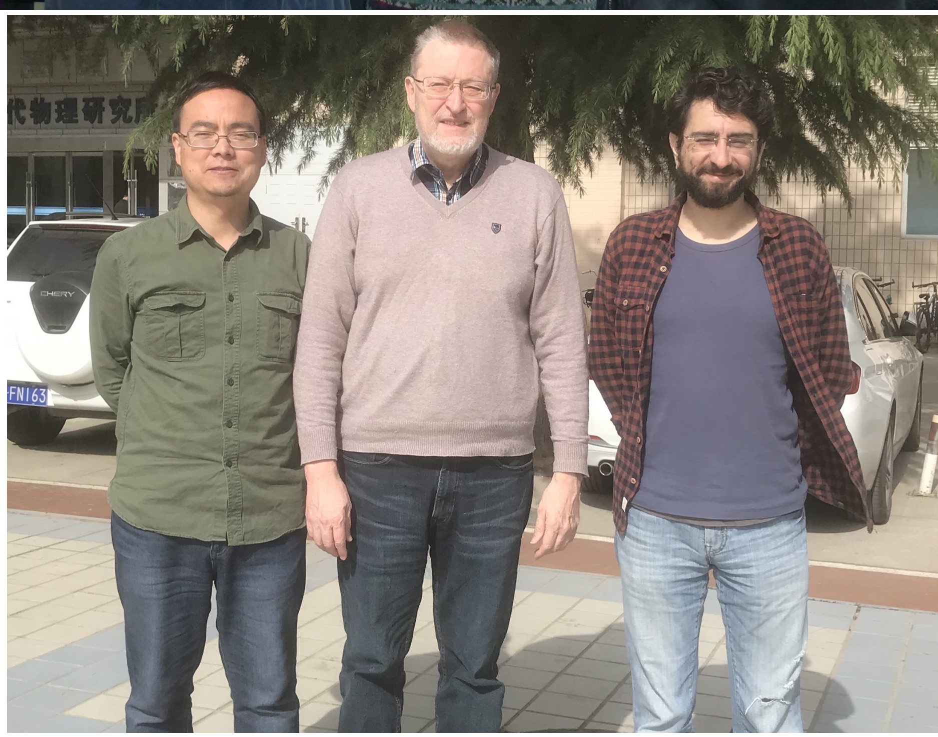
Memory Effect & Carroll Symmetry for gravitational waves



Budapest - Wien April 2024

Abstract: Observing the displacement of particles by a Gravitational Wave could provide a way to detect the latter. Following Souriau (1973), the geodesic equations can be integrated by using the 5-parameter isometries of plane gravitational waves, identified more recently as Lévy-Leblond's (1965) "**Carroll**" group in 2+1 dimensions with no rotations, which acts as a symmetry. The associated conserved quantities determine the trajectories.

Joint work with C. Duval, M. Elbistan, G. W. Gibbons, P. M. Zhang :

- *"Carroll symmetry of gravitational plane waves,"* Class. Quant. Grav. **34** (2017) 175003 [arXiv:1702.08284 [gr-qc]]

- *"The Memory Effect for Plane Gravitational Waves,"* Phys. Lett. B **772** (2017) 743. [arXiv:1704.05997 [gr-qc]].

- *"Soft gravitons and the memory effect for plane gravitational waves,"* Phys. Rev. D **96** (2017) no.6, 064013 [arXiv:1705.01378 [gr-qc]].

- *"Sturm-Liouville and Carroll: at the heart of the Memory Effect,"* Gen. Rel. Grav. **50** (2018) no.9, 107 [arXiv:1803.09640 [gr-qc]].

Road map:

Memory effect

Ya. B. Zel'dovich and A. G. Polnarev, *"Radiation of gravitational waves by a cluster of superdense stars,"* Astron. Zh. **51**, 30 (1974)

> *. . . two noninteracting bodies (such as satellites). [. . .] the distance should change, and in principle this effect might possibly serve as a nonresonance detector. [. . .] although distance between free bodies will change, their relative velocity will become vanishingly small as* **flyby** *concludes.*

CLAIM – NO PROOF

A. displacement effect DM

Elaborated by V.B. Braginsky & L. P. Grishchuk

"Kinematic resonance and the memory effect in free mass gravitational antennas," Zh. Eksp. Teor. Fiz. **89** *744-750 (1985)*

> *"distance between a pair of bodies is different from the initial distance in the presence of a gravitational radiation pulse. . . . possible application to detect gravitational radiation . . ."*

$$\left(|\dot{\boldsymbol{X}}_1 - \dot{\boldsymbol{X}}_2|\right) \to 0 \;\Leftrightarrow\; |\boldsymbol{X}_1 - \boldsymbol{X}_2| \to \text{const} \qquad (1)$$

Christodoulou : non-linear theory $\rightsquigarrow$ displacement effect *"Nonlinear nature of gravitation and gravitational wave experiments,"* Phys. Rev. Lett. **67** (1991) 1486.

flyby captured by Hubble Space Telescope)

B. Velocity effect VM

J. Ehlers and W. Kundt (1962)

"Exact solutions of the gravitational field equations," in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, London, 1962).

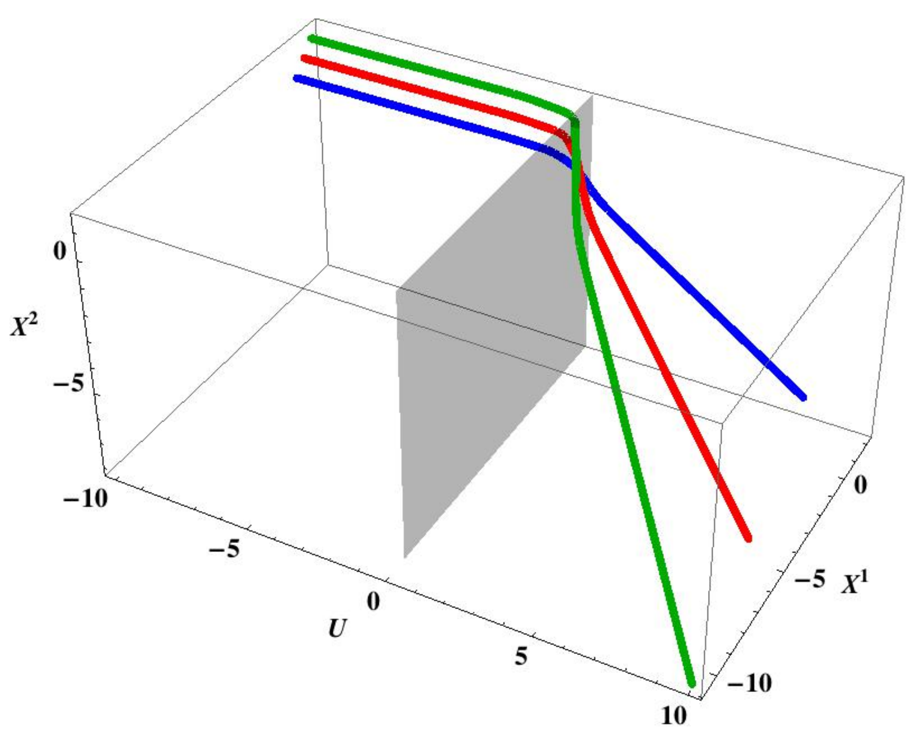
V B Braginsky and K S Thorne

"Gravitational-wave burst with memory and experimental prospects," Nature (London) **327** 123 (1987).

L. P. Grishchuk and A. G. Polnarev

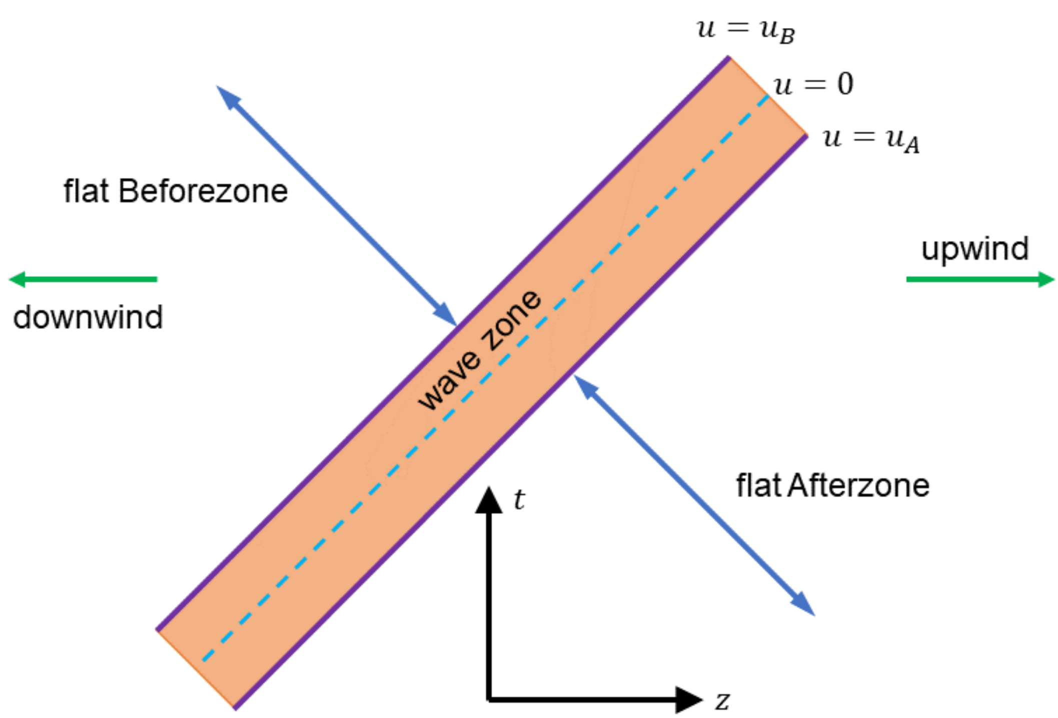
"Gravitational wave pulses with 'velocity coded memory'," Sov. Phys. JETP **69** (1989) 653 [Zh. Eksp. Teor. Fiz. **96** (1989) 1153].

$$\dot{X} \to \text{const} > 0 \qquad |\dot{X}^{(1)} - \dot{X}^{(2)}| > 0 \qquad (2)$$



G. W. Gibbons S. W. Hawking *"Theory of the detection of short bursts of gravitational radiation,"* Phys. Rev. D **4** (1971) 2191.

Sandwich wave: burst of gravitational wave. Spacetime non-flat only in short interval $u_B \leq u \leq u_A$ of retarded time [Wavezone]. Flat both in Beforezone $u < u_B$ that the wave has not reached yet, and in Afterzone $u_A < u$ where has already passed, see fig.



u flows from left to the right, whereas wave advances from right to left.

Geodesics in Brinkmann* coordinates

plane GWs

$$\delta_{ij} dX^i dX^j + 2dUdV + K_{ij}(U) X^i X^j dU^2 \qquad (3a)$$

profile $K_{ij}(U)X^iX^j =$

$$\tfrac{1}{2}\mathcal{A}_+(U)\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_\times(U)\, X^1 X^2 \qquad (3b)$$

where $\mathcal{A}_+$ and $\mathcal{A}_\times$ $+$ and $\times$ polarization-state amplitudes. $\boldsymbol{X} = (X^i)$ transverse, U, V light-cone coords. $(X^\mu) = (U, \boldsymbol{X}, V)$ global

Vacuum Einstein solutions : Ricci flat

$$R_{\mu\nu} = 0 \Leftrightarrow \mathrm{Tr}(K_{ij}) = 0. \qquad (4)$$

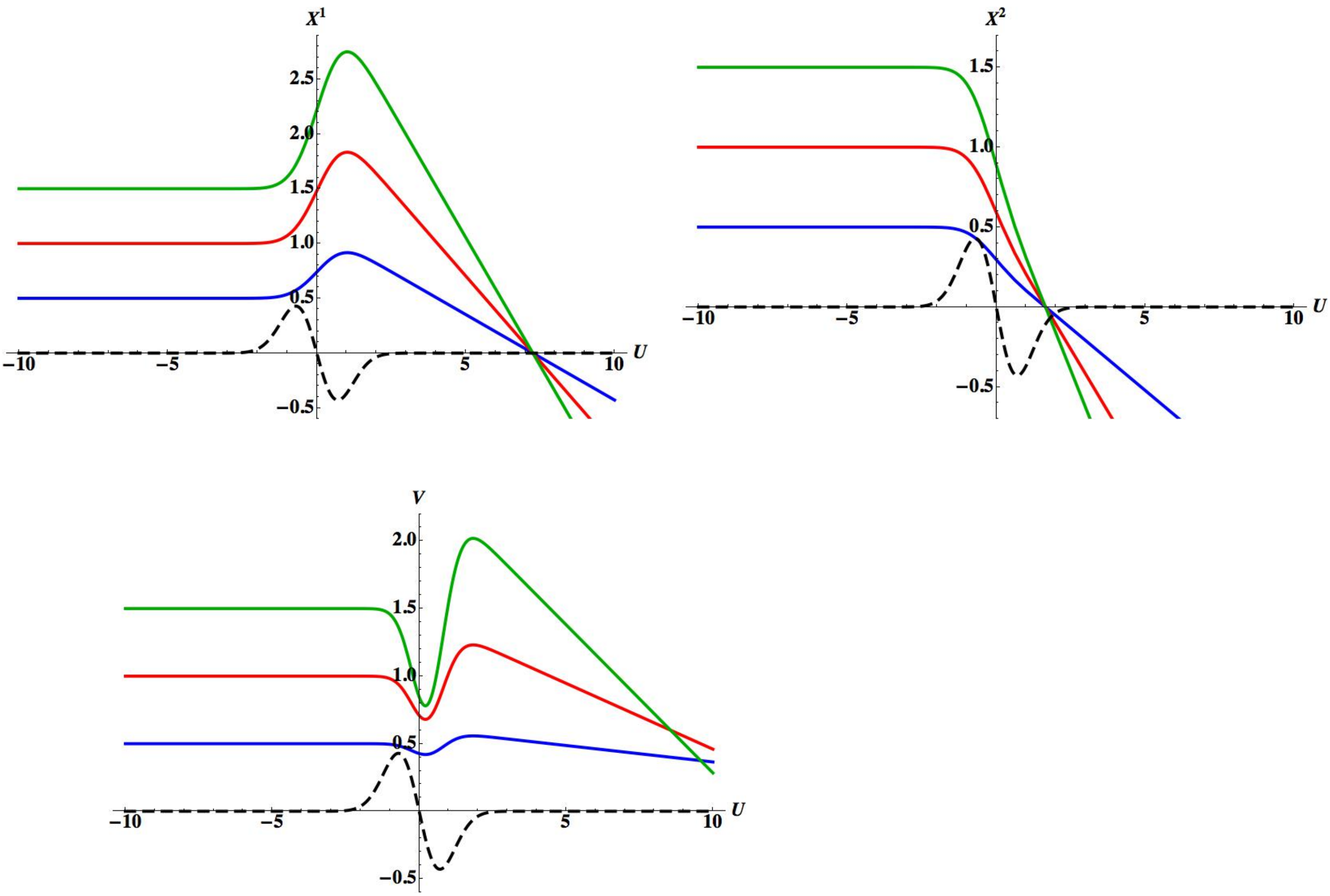
Sandwich wave: $K(U) \neq 0$ only in "wave zone" $U_B < U < U_A$. Assumption : metric Minkowski in "Beforezone" $U < U_B$, flat in "Afterzone" $U_A < U$.

* M. W. Brinkmann, "Einstein spaces which are mapped conformally on each other," Math. Ann. **94** (1925) 119–145.

Gibbons-Hawking: Brinkmann profile for **flyby**

$$\mathcal{A}(U) = \frac{1}{2}\,\frac{d(e^{-U^2})}{dU}\,. \tag{5}$$

Numerically found geodesics :



velocity effect – DM of Zel'dovich-Polnarev (1) ??

Eisenhart-Duval framework

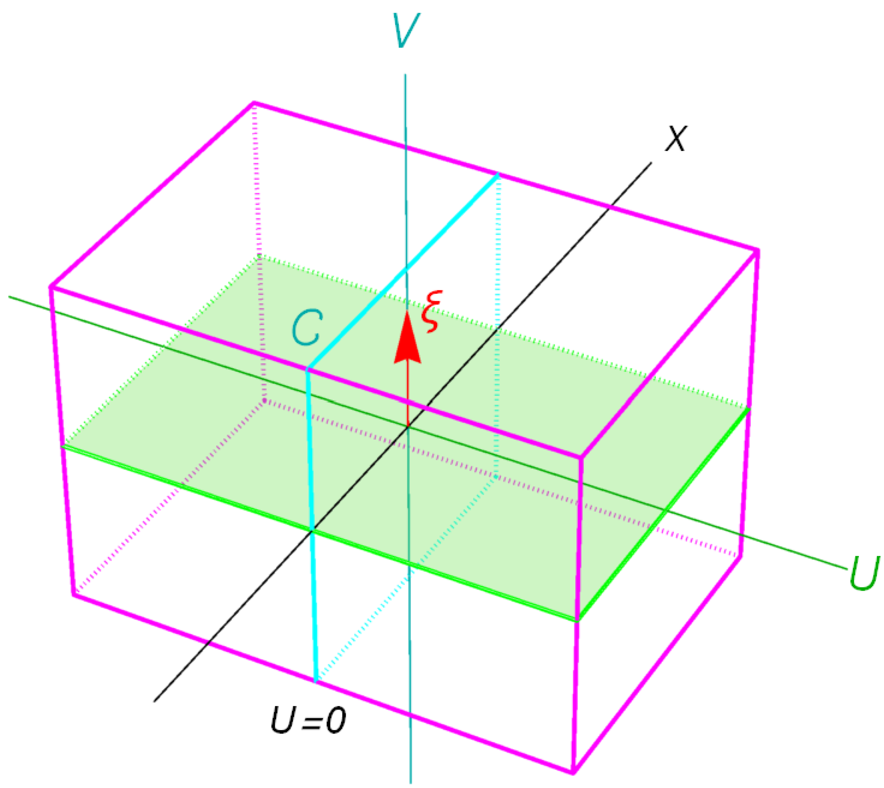
Duval et al 1984: GW plane wave ~ relativistic ("Kaluza-Klein") desription for non-relativistic physics

Bargmann manifold:

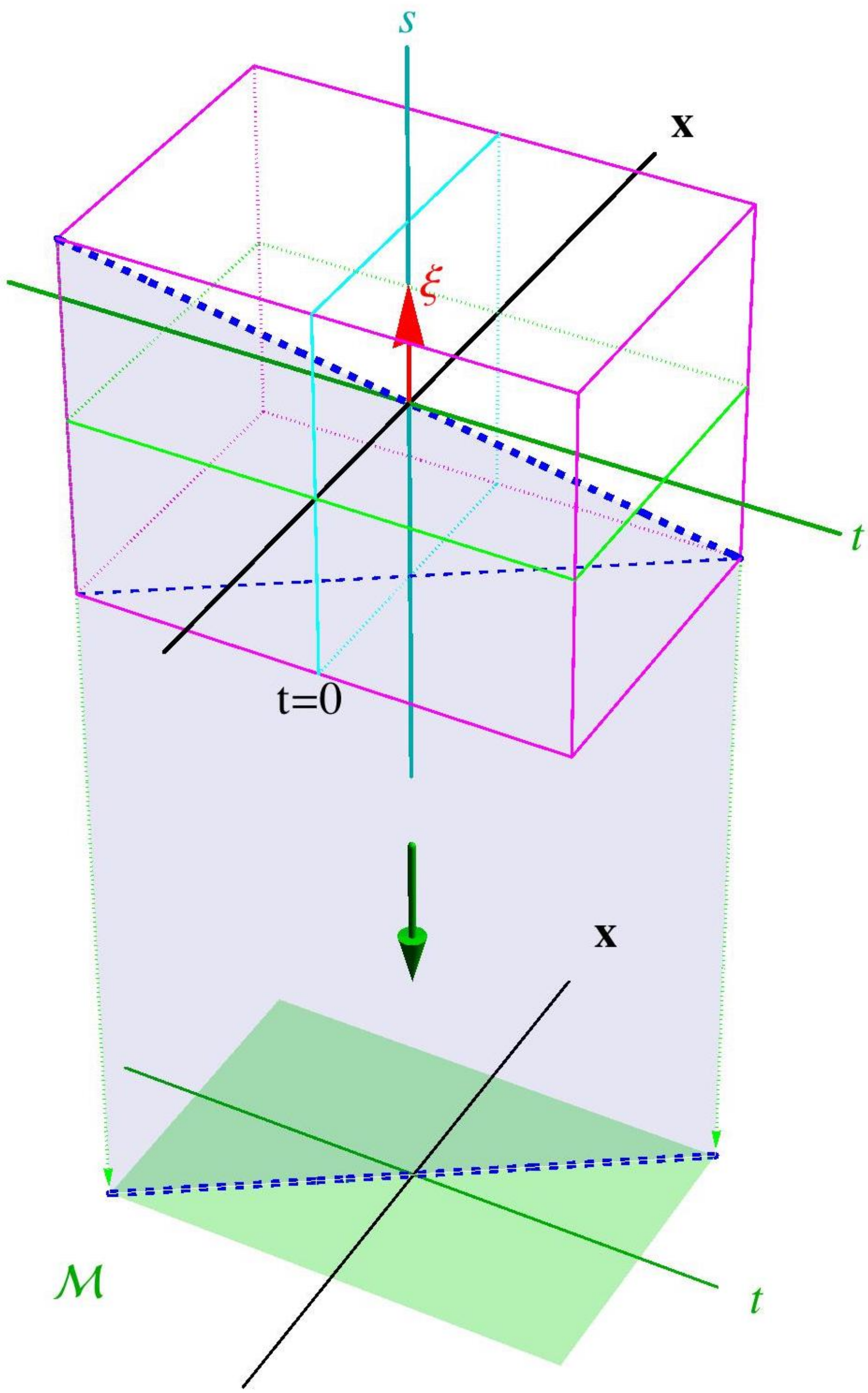
(i) a $(d+2)$-dim manif

(ii) endowed with metric of signature $(d+1,1)$

(iii) carries nowhere vanishing, complete, null "vertical" vector ξ, parallel-transported by Levi-Civita



connection, ∇.
projects to $\{(t,x)\}$ NR spacetime.



NULL GEODESICS in Bargmann

project to

NR MOTIONS in space-time

$$\delta_{ij} dX^i dX^j + 2dU dV + \underbrace{K_{ij}(U) X^i X^j}_{-2(\text{Newton potential})} \quad dU^2$$

Framework first proposed by

L. P. Eisenhart, *"Dynamical trajectories and geodesics"*, Annals. Math. **30** 591-606 (1928).

- forgotten - then rediscovered, independently :

J. Gomis and J. M. Pons, *"Poincare Transformations and Galilei Transformations,"* Phys. Lett. A **66** (1978) 463.

C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, *"Bargmann Structures and Newton-Cartan Theory,"* Phys. Rev. D **31** (1985) 1841.

A. P. Balachandran, H. Gomm and R. D. Sorkin, *"Quantum Symmetries From Quantum Phases: Fermions From Bosons, a $Z(2)$ Anomaly and Galilean Invariance,"* Nucl. Phys. B **281** (1987) 573.

see

C. Duval, G. W. Gibbons, and P. A. Horvathy, *"Celestial Mechanics, Conformal Structures and Gravitational Waves,"* Phys. Rev. **D43**, 3907 (1991)

Geodesics (for linearly polarized $\mathcal{A}_\times = 0$ cf. (3b)) :

$$\frac{d^2X^1}{dU^2} - \frac{1}{2}\mathcal{A}_+ X^1 = 0, \tag{6a}$$

$$\frac{d^2X^2}{dU^2} + \frac{1}{2}\mathcal{A}_+ X^2 = 0, \tag{6b}$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}_+}{dU}\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_+\Big(X^1\frac{dX^1}{dU} - X^2\frac{dX^2}{dU}\Big) = 0\,. \tag{6c}$$

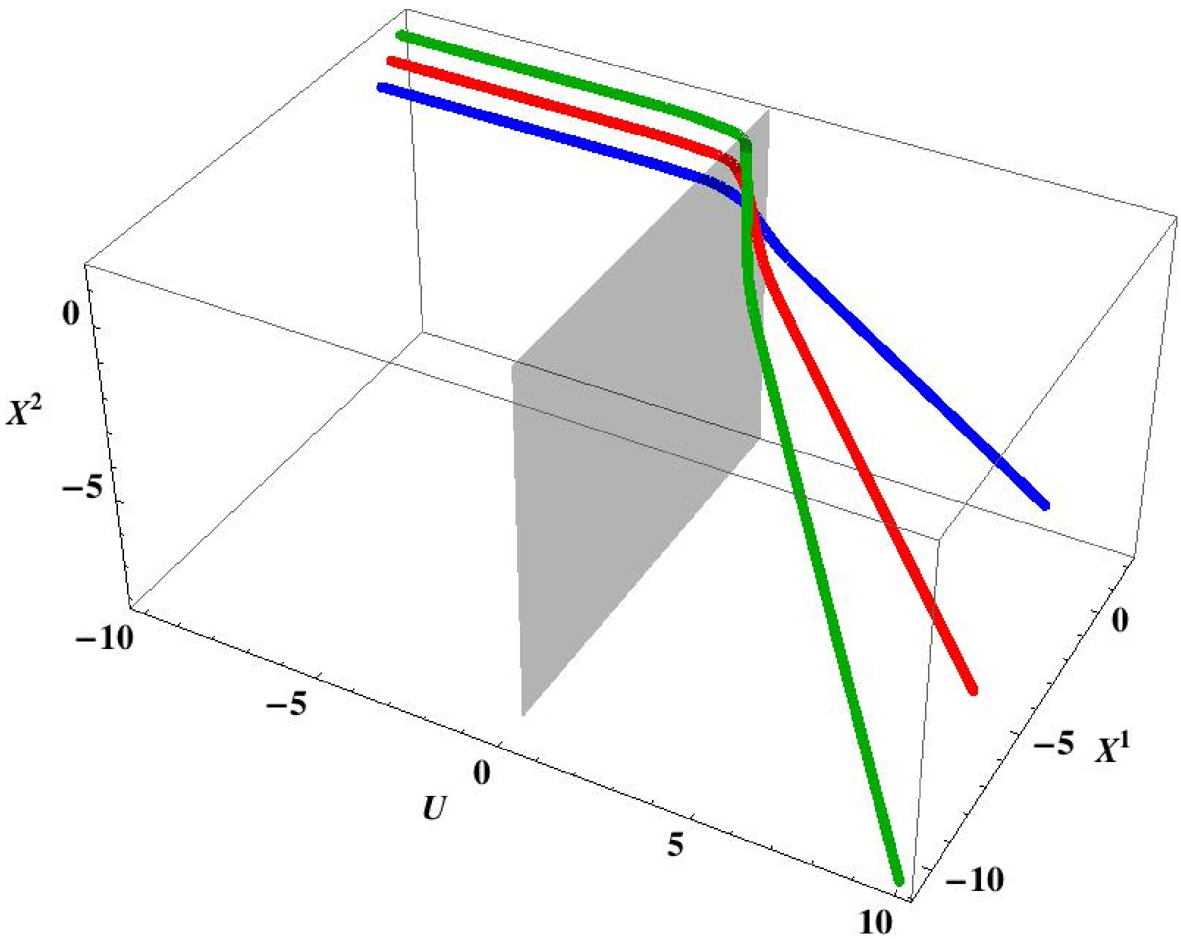
$X^{1,2}$-components decoupled from V. Projection of $4D$ worldline to transverse ($X^1 - X^2$) plane independent of $V(U_0)$ & $\dot{V}(U_0)$. NB: eqn (6c) $\sim$ horizontal lift

N.B. : (6a)-(6b) with

$$K_{ij}(U)X^iX^j = \tfrac{1}{2}\mathcal{A}_+(U)\Big((X^1)^2 - (X^2)^2\Big)$$

$\sim$ harmonic force with possibly time-dependent frequency.

typical scenario for sandwich wave:

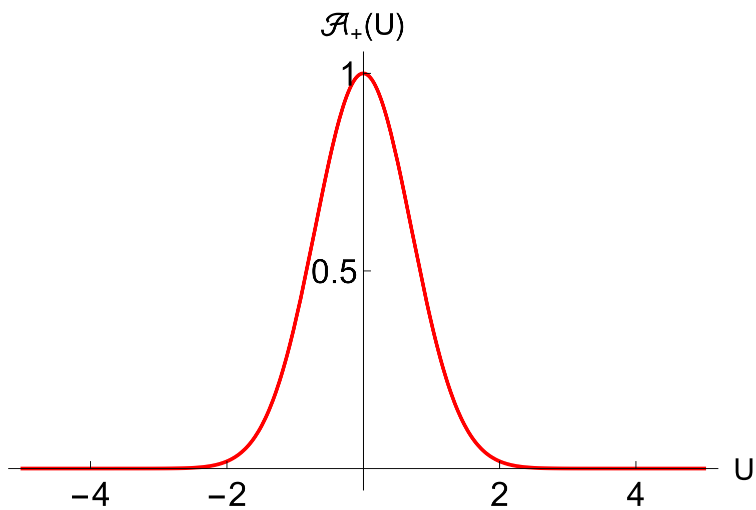


Particle at rest before the wave arrives follow, after the wave has passed, DIVERGING straight trajectories with non-zero constant velocity.

Examples Gibbons-Hawking'71

- Linearly polarized $\mathcal{A}_\times = 0$ with Gaussian profile

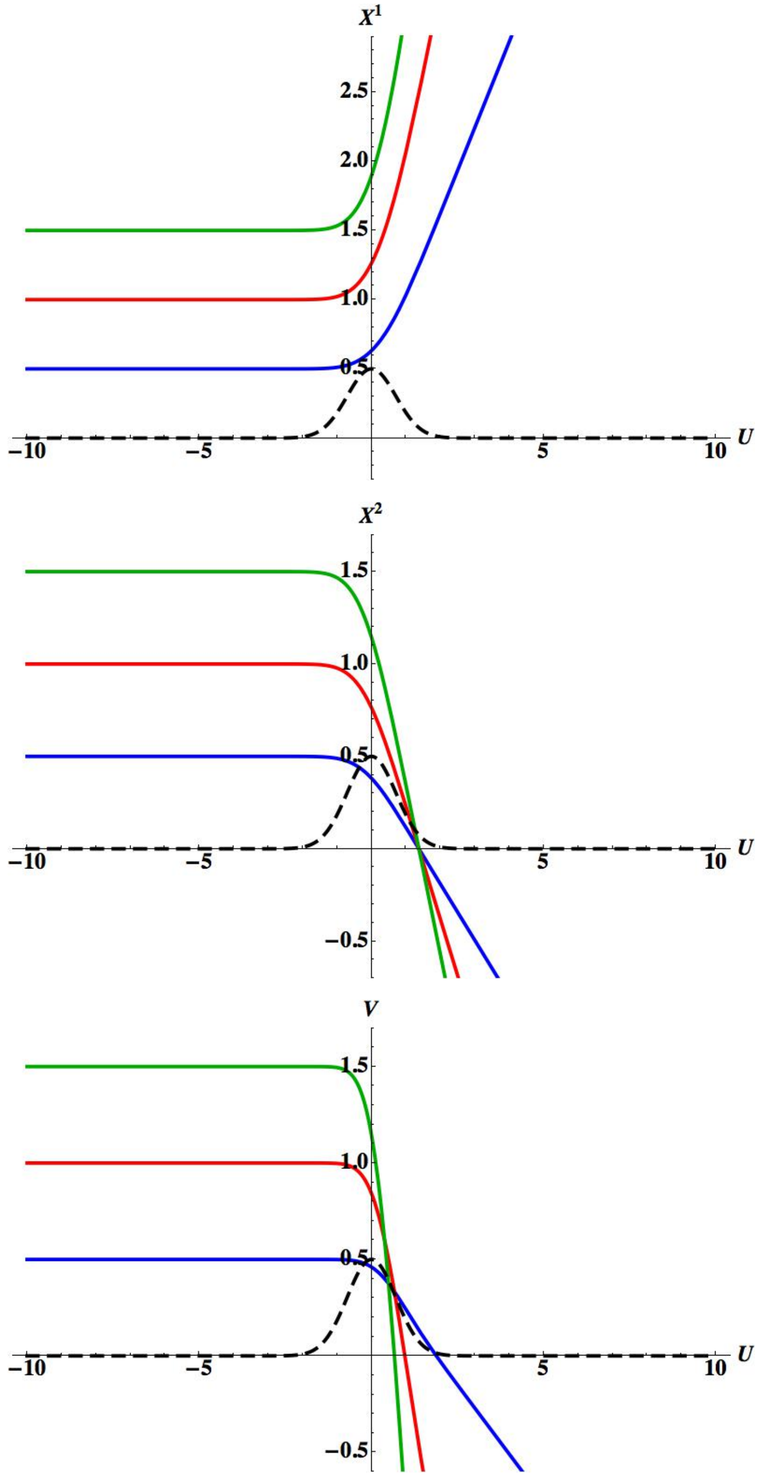
$$\boxed{K_{ij}(U)X^iX^j = \frac{e^{-U^2}}{\sqrt{\pi}}\Big((X^1)^2 - (X^2)^2\Big).} \qquad (7)$$



Bargmann pic $\sim$ anisotropic repulsive/attractiv oscillator with time-dependent frequency

$$\omega^2(U) = \frac{e^{-U^2}}{\sqrt{\pi}}$$

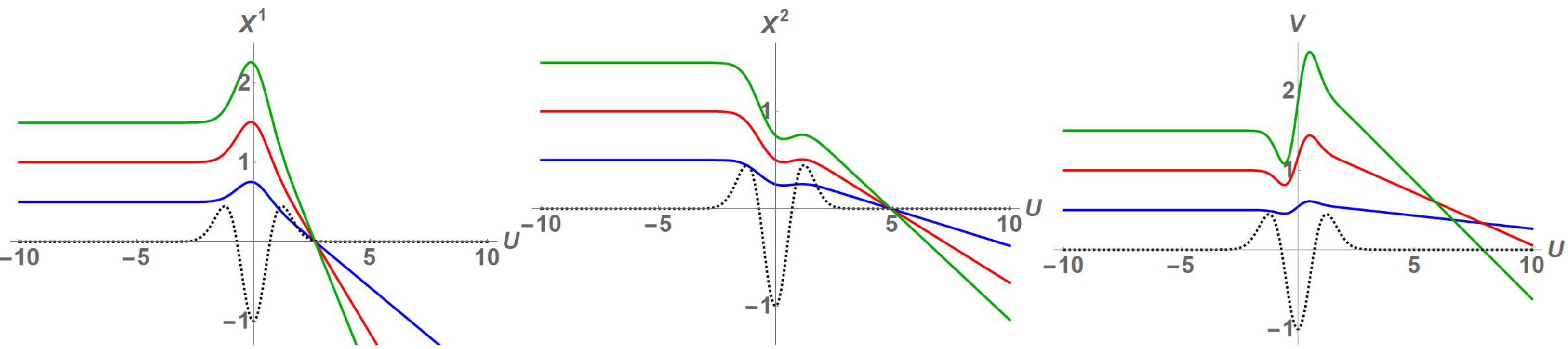
repulsive in X^1 **attractive** in X^2 $\rightsquigarrow$



Geodesics for Gaussian burst for blue/red/green positions in Beforezone.

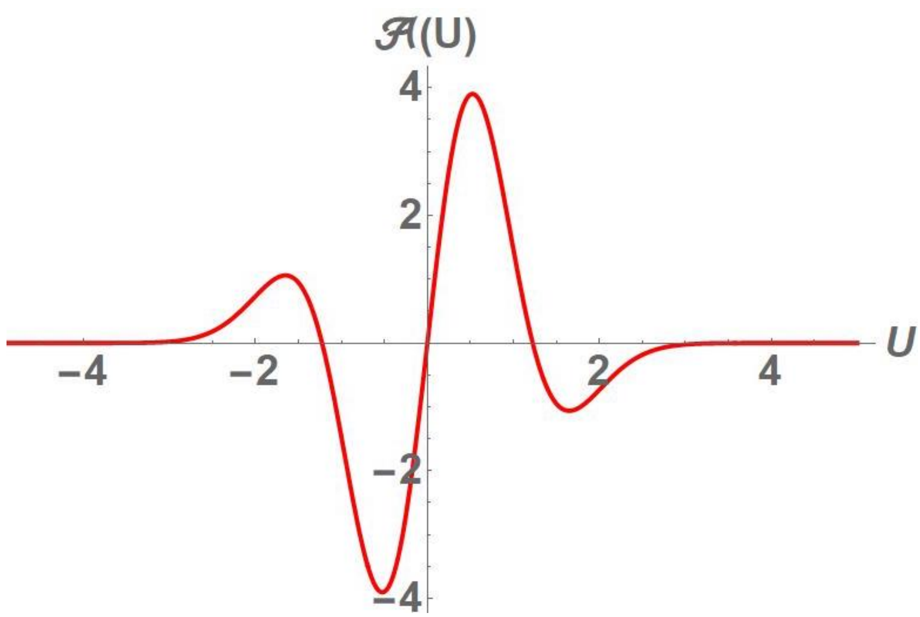
• Braginsky - Thorne system $\sim$ *2nd derivative* of Gaussian,

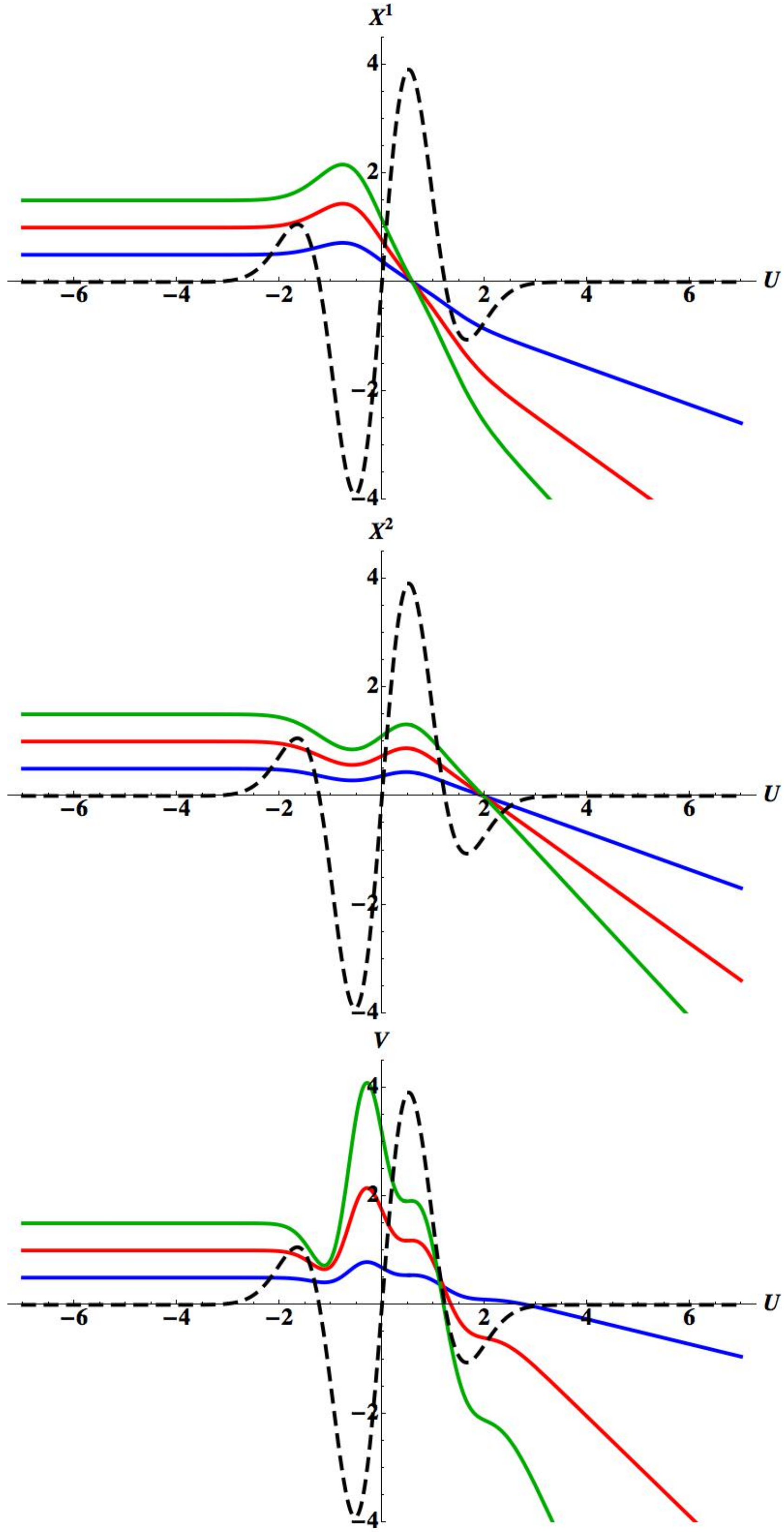
$$\mathcal{A}(U) = \frac{1}{2}\frac{d^2(e^{-U^2})}{dU^2}\,, \tag{8}$$



• "gravitational collapse" metric

$$\mathcal{A}(U) = \frac{1}{2}\frac{d^3(e^{-U^2})}{dU^3}\,. \tag{9}$$





Geodesics for particles initially at rest GW created by gravitational collapse, (9). *VM: YES* *DM : NO*

Isometries & Carroll

Souriau & Lévy-Leblond

Assumption: for large distances GW approximated with exact plane GW

H. Bondi, F. A. E. Pirani and I. Robinson, *"Gravitational waves in general relativity. 3. Exact plane waves,"* Proc. Roy. Soc. Lond. A **251** (1959) 519 :

plane GWs have 5-parameter group of isometry: 3 translations + 2 WHAT ?

Souriau *"Ondes et radiations gravitationnelles,"* Colloques Internationaux du CNRS No 220, pp. 243-256. Paris (1973): symmetry $\rightsquigarrow$ integration of geodesic eqns.

C. Duval, et al. *"Carroll symmetry of gravitational plane waves,"* Class. Quant. Grav. **34** (2017) 175003 [arXiv:1702.08284 [gr-qc]].

Duval et al *"Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,"* Class. Quant. Grav. **31** (2014) 085016

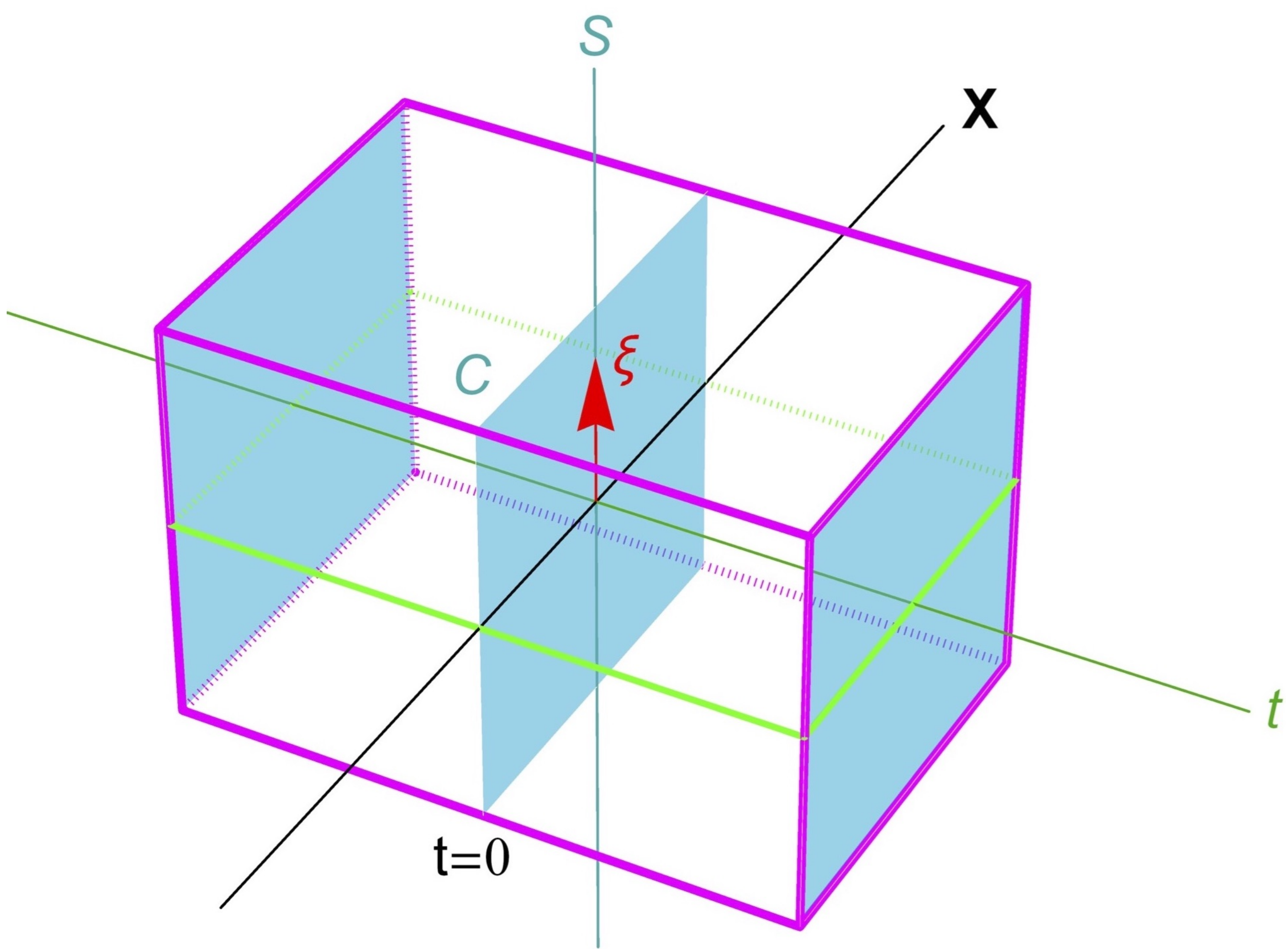
Carroll structure: $(C, \mathrm{g}, \xi, \nabla)$, where C $(d+1)$-dimensional manifold $\{s, \boldsymbol{x}\}$, g twice-symmetric covariant, positive, tensor field, whose kernel is generated by the nowhere vanishing, complete vector field ξ. ∇ is symmetric affine connection that parallel-transports both g and ξ.

Carroll group: diffeomorphisms of C that preserve "metric" g, vector field ξ, & connection ∇. Generated by vector fields X s.t.

$$L_X \mathrm{g} = 0, \quad L_X \xi = 0, \quad L_X \nabla = 0. \tag{10}$$

C. Duval, G. W. Gibbons, and P.A.H "Celestial Mechanics, Conformal Structures and Gravitational Waves," Phys. Rev. **D43**, 3907 (1991)

Restriction of Bargmann space $\{s, t, \boldsymbol{x}\}$ to $U = U_0 = \text{const}$ slice is Carroll manifold.



Flat

Carroll structure $C^{d+1} = \mathbb{R} \times \mathbb{R}^d = \{s, \boldsymbol{x}\}$

$$\mathrm{g} = \delta_{AB}\, dx^A dx^B, \quad \xi = \frac{\partial}{\partial s}, \quad \Gamma^k_{ij} = 0 \qquad (11)$$

More generally Baldwin-Jeffery-Rosen (BJR)

$$a_{ij}(u)\, dx^i dx^j + 2du\, dv\,. \qquad (12)$$

$a(u) \equiv (a_{ij}(u))$ positive 2×2 matrix.

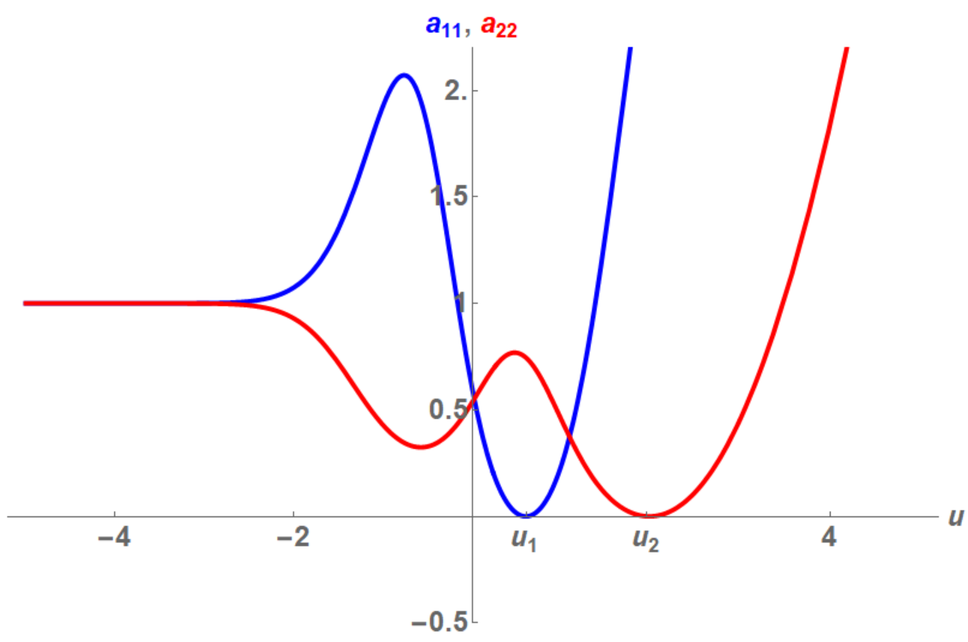
Isometries of plane GWs

J-M. Souriau 1973 Using BJR coordinates $(\boldsymbol{x}, u, v)$ metric takes form,

$$a_{ij}(u)\, dx^i dx^j + 2du\, dv \qquad (13)$$

$a(u) \equiv (a_{ij}(u))$ positive 2×2 matrix.

Brinkmann/Bargmann transcription : "potential" $K_{ij}X^iX^j$ traded for (linearly polarized) transverse metric $a_{ij}(u)$.



BJR coords singular zeros correspond to the u_i where where $\det(a) = 0$.

Souriau: in coordinate patch $u_1 < u < u_2$ isometries implemented on space-time $u \to u$

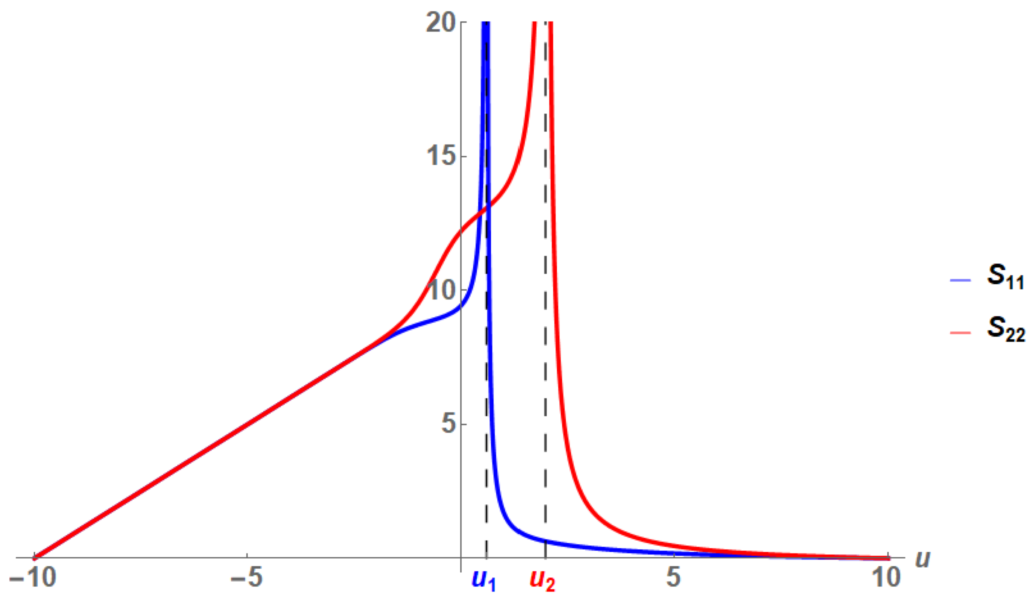
$$\begin{aligned} \boldsymbol{x} &\to \boldsymbol{x} + H(u)\,\mathbf{b} + \mathbf{c}, \\ v &\to v - \mathbf{b}\cdot\boldsymbol{x} - \tfrac{1}{2}\mathbf{b}\cdot H(u)\,\mathbf{b} + \nu, \end{aligned} \qquad (14)$$

Acts on $u = $ const slice. $H(u)$ is symmetric 2×2 Souriau matrix,

$$H(u) = \int_{u_0}^{u} a(t)^{-1} dt \qquad u_1 < u_0 < u_1 \qquad (15)$$

- $a = \mathrm{Id}$ (Minkowski) $\Rightarrow H(u) = u - u_0 \Rightarrow$ Galilei.

- 3rd derivative $\sim$ collapse profile (9)



Souriau matrix for collapse profile. In Beforezone, $H(u) \approx u\,\mathrm{Id}$*. In Afterzone* $H(u)$ *falls off rapidly.*

Brinkmann $\Leftrightarrow$ BJR transcription

Gibbons "Quantized Fields Propagating in Plane Wave Space-Times," Commun. Math. Phys. **45** (1975) 191.

1. Starting with Brinkmann profile, first solve Sturm-Liouville eqn with supplementary condition,

$$\boxed{\ddot{\mathbb{P}} = K\mathbb{P}, \qquad \mathbb{P}^\dagger\dot{\mathbb{P}} = \dot{\mathbb{P}}^\dagger\mathbb{P}} \tag{16}$$

for 2×2 matrix $\mathbb{P}$ where $\dot{\{\}} = d/dU$.

2. BJR profile: symmetric 2×2 matrix

$$\mathfrak{a}(U) = \mathbb{P}^\dagger\mathbb{P} \tag{17}$$

3. Brinkmann <=> BJR coords, $(\boldsymbol{X}, U, V)$, $(\boldsymbol{x}, u, v)$ related as

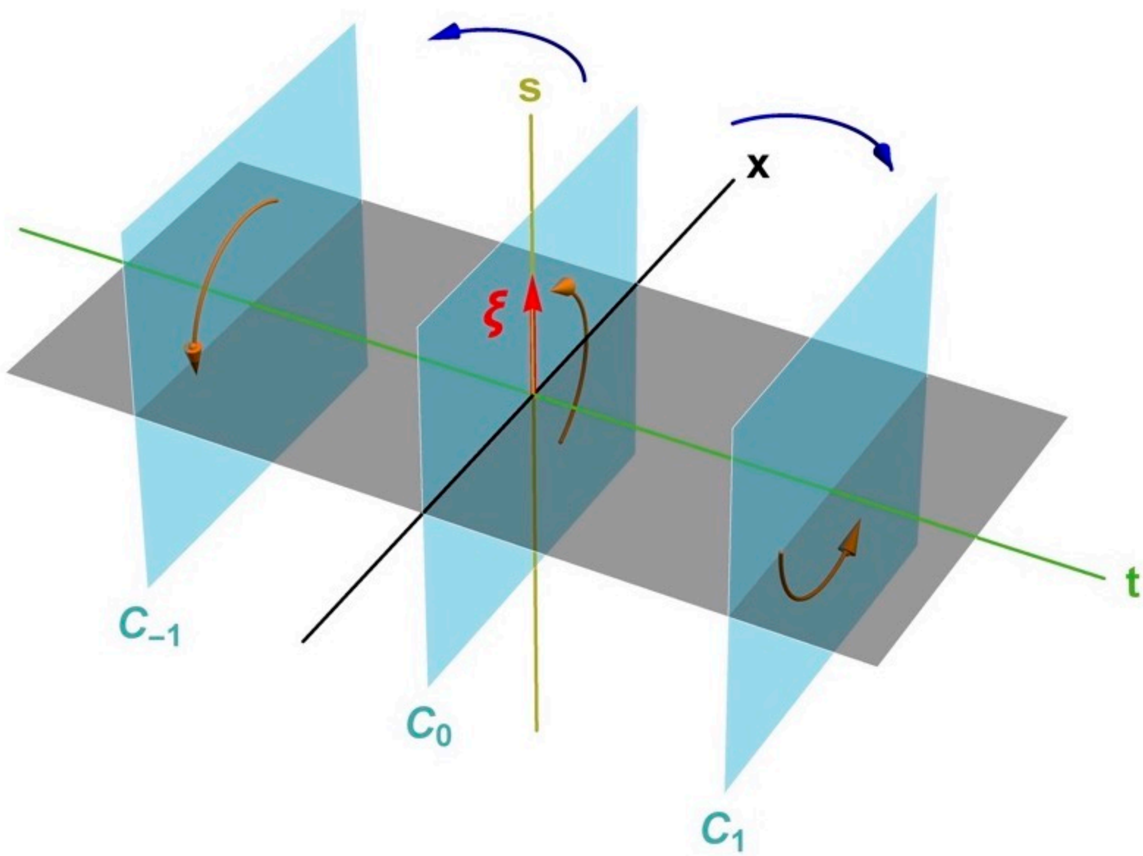
$$\boldsymbol{X} = \mathbb{P}(u)\boldsymbol{x},\ U = u,\ V = v - \frac{1}{4}\boldsymbol{x} \cdot \dot{\mathfrak{a}}(u)\boldsymbol{x}\,. \tag{18}$$

CARROLL SYMMETRY

Surprise: restriction to $C\Big|_{u=0} \Rightarrow H(u) = 0$ slice
$\rightsquigarrow$ boost implemented by

$$\boxed{\begin{cases} \boldsymbol{x}' &= \quad \boldsymbol{x} \\ v' &= v - \boldsymbol{b}\cdot\boldsymbol{x} \end{cases}} \qquad (19)$$

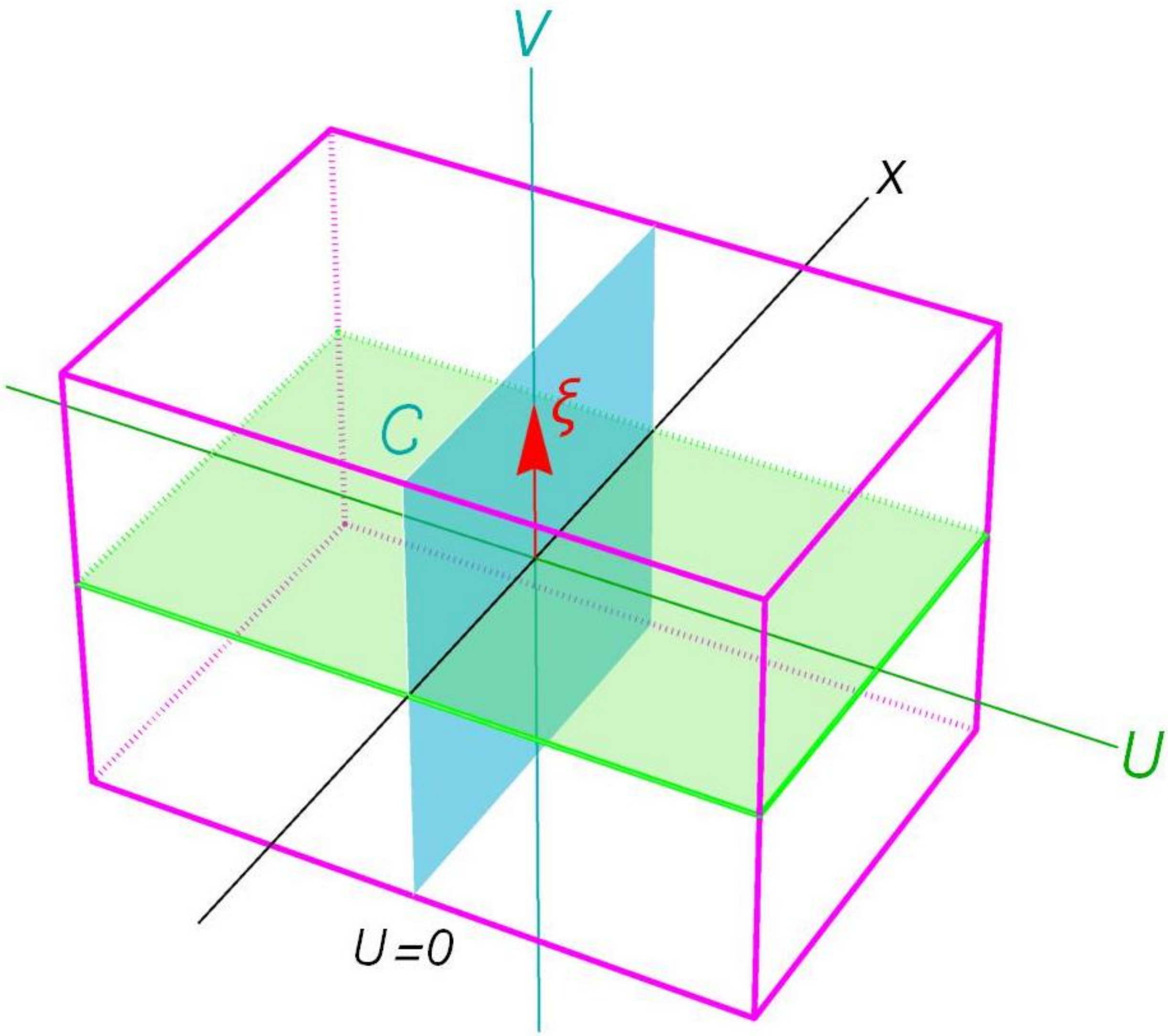
$\equiv$ Lévy-Leblond's **"Carroll" boost**



J.-M. Lévy-Leblond,
"Une nouvelle limite non-relativiste du group de Poincaré,"
Ann. Inst. H Poincaré **3** (1965) 1

NB: Recognized by Duval in . . . 2017 . . .

Unified framework Duval et al *"Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,"* Class. Quant. Grav. **31** (2014) 085016



Bargmann

Newton-Cartan $\sim$ **non-relativistic** $(U, \boldsymbol{X})$

Carroll $(\boldsymbol{X}, V)$

Geodesics from Carroll symmetry

Noether $\Rightarrow$ 5 isometries $\Rightarrow$ conserved quantities.
In BJR (from (14))

$$\mathbf{p} = a(u)\,\dot{\boldsymbol{x}}\,, \qquad \mathbf{k} = \boldsymbol{x}(u) - H(u)\,\mathbf{p}\,, \tag{20}$$

interpreted as conserved linear & boost-momentum, supplemented by $m = \dot{v} = 1$.

Extra const of motion (Jacobi invariant) $e = \frac{1}{2}\mathrm{g}_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu$ timelike/lightlike/ spacelike if e negative/zero/positive.

Conversely, geodesics determined by Noether quantities,

$$\boldsymbol{x}(u) = H(u)\,\mathbf{p} + \mathbf{k}, \tag{21a}$$

$$v(u) = -\tfrac{1}{2}\mathbf{p}\cdot \boxed{H(u)}\,\mathbf{p} + e\,u + d, \tag{21b}$$

Only quantity to calculate is Souriau matrix $\boxed{H(u)}$.

- In flat Minkowski $a = \mathbf{1} \Rightarrow H(u) = u\,\mathbf{1}$, yields free motion

$$\boldsymbol{x}(u) = u\,\mathbf{p} + \mathbf{k}, \tag{22a}$$

$$v(u) = \Big(-\tfrac{1}{2}|\mathbf{p}|^2 + e\Big)\,u + v_0. \tag{22b}$$

usual boosts / usual motions.

Momentum of particle at rest vanishes by (20), $\mathbf{p} = 0$ for $u \leq u_B$ because of initial condition $\dot{\boldsymbol{x}}(u) = 0$. $\mathbf{p}$ conserved $\Rightarrow$

$$\mathbf{p} = 0 \quad \text{for all } u \tag{23}$$

for any H i.e. for any metric a.

$$\boxed{\boldsymbol{x}(u) = \boldsymbol{x}_0, \quad v(u) = e\,u + v_0\,.} \tag{24}$$

In BJR particles initially at rest **remain at rest** *during and after passage of wave* !!

In Brinkmann coords both GWs and geodesics are global with no singularity. Solving SL eqns (16) [e.g. numerically] for P,

$$\boxed{\boldsymbol{X}(U) = P(u)\,\boldsymbol{x}^0, \quad \boldsymbol{x}^0 = \text{const}} \tag{25}$$

Complicated trajectory comes from $\mathbf{P(u)}$!!!

Displacement effect ??

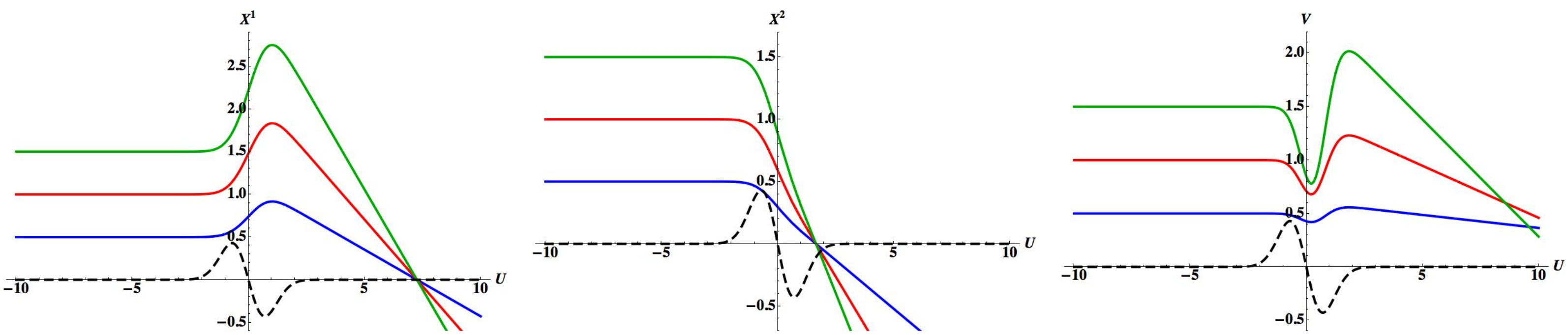
Zel'dovich-Polnarev: pure displacement for **flyby**

(give no proof)

Gibbons-Hawking: flyby profile proportional to *first derivative* of a Gaussian,

$$\mathcal{A}(U) = \frac{1}{2}\,\frac{d(e^{-U^2})}{dU}\,. \qquad (26)$$

Geodesics



Geodesics for flyby profile (26). velocity effect

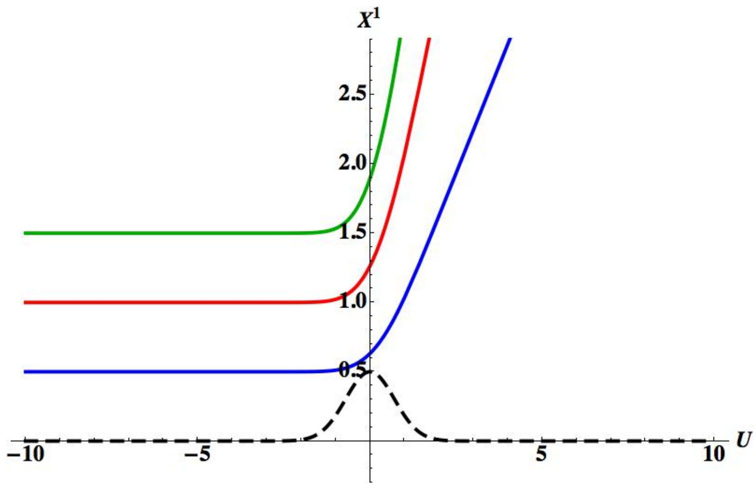
pure displacement ???

Miracle ! (numerical results) for specific choices of parameters:

toy example: **d=0** Gaussian burst

$$A_0\left(U\right)=k\left(\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2U^2}\right) \tag{27}$$

k and λ control height & width. Remember:



diverges in repulsive sector

1/2 DM effect : Strategy: eliminate unbounded motion in repulsive sector by putting $X^1 = 0$ (consistent with eqn. (6a))

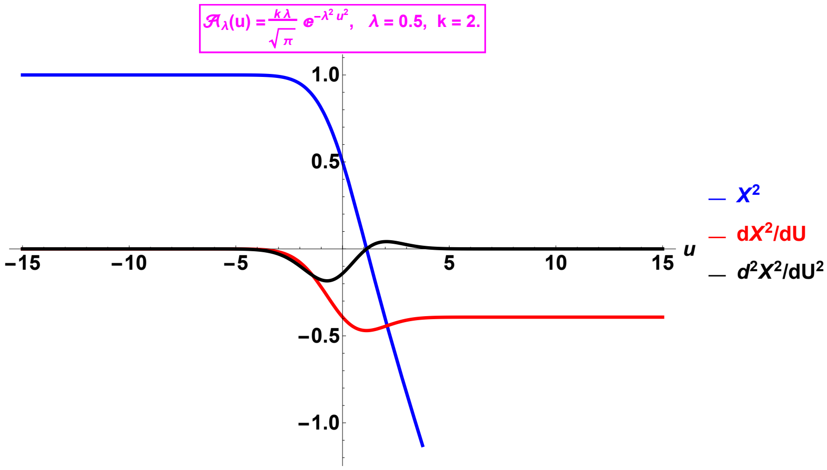
$$\frac{d^2X^1}{dU^2}-\frac{1}{2}\mathcal{A}X^1=0,$$

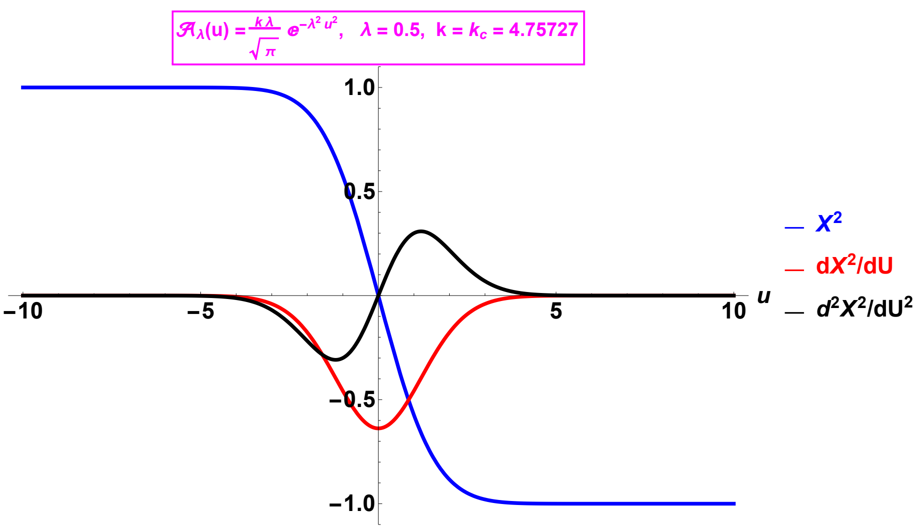
and "flatten" $X^2 \to$ const by fine-tuning. $\rightsquigarrow$ "jump solution"

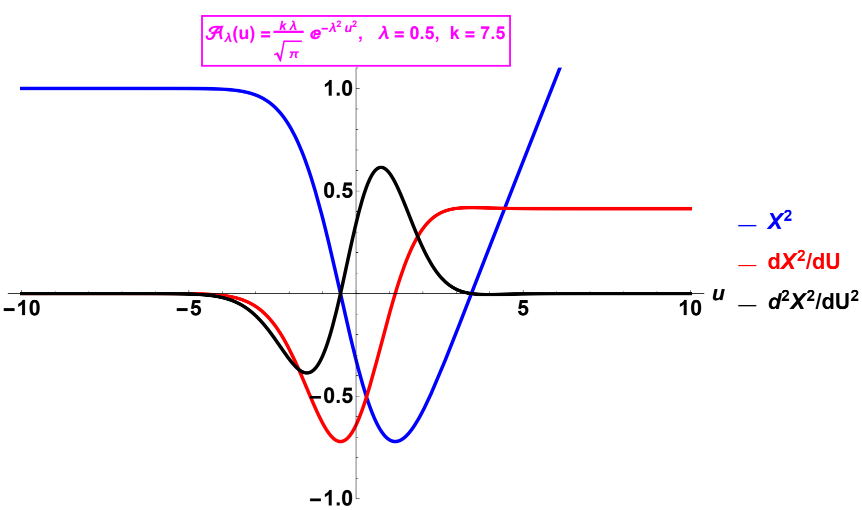
Trajectories $\sim$ "damped sin/cos". For large distance the motion (approx) along straight lines. Can they be made horizontal ?

Velocities & **forces** ($k \approx k_{crit} = 4.75727$, $X_1 \equiv 0$)

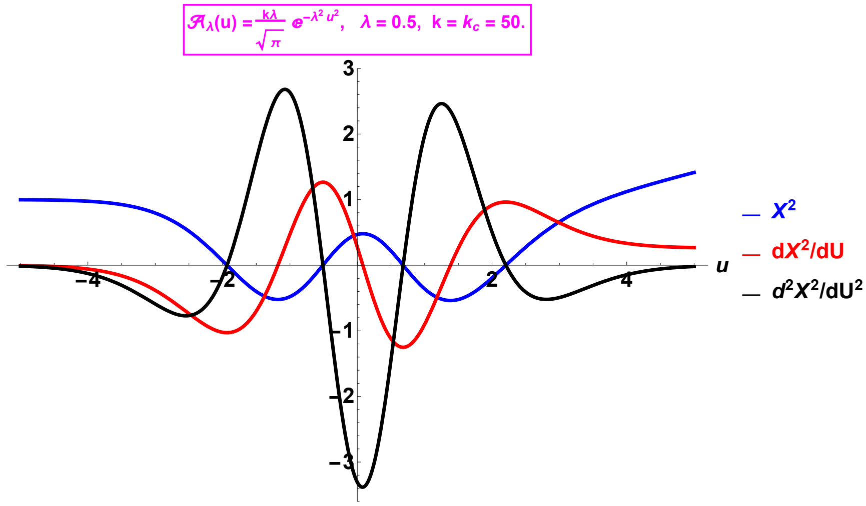
X^2 : trajectory, $\dfrac{dX^2}{dU}$: velocity, $\dfrac{\mathrm{d}^2\mathrm{X}^2}{\mathrm{dU}^2}$: force.

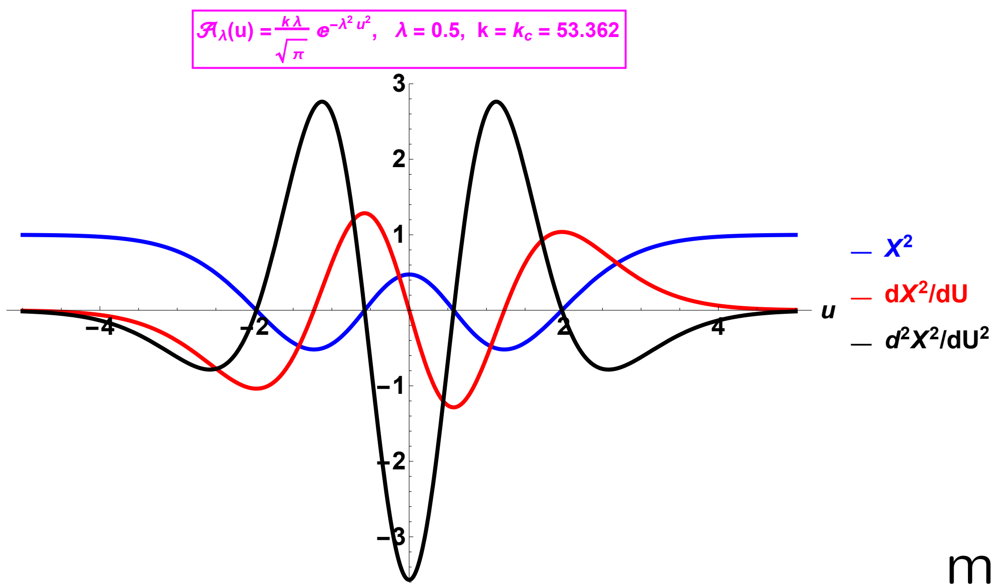






$(k \approx k_{crit} = 53.362,\ X_1 \equiv 0)$





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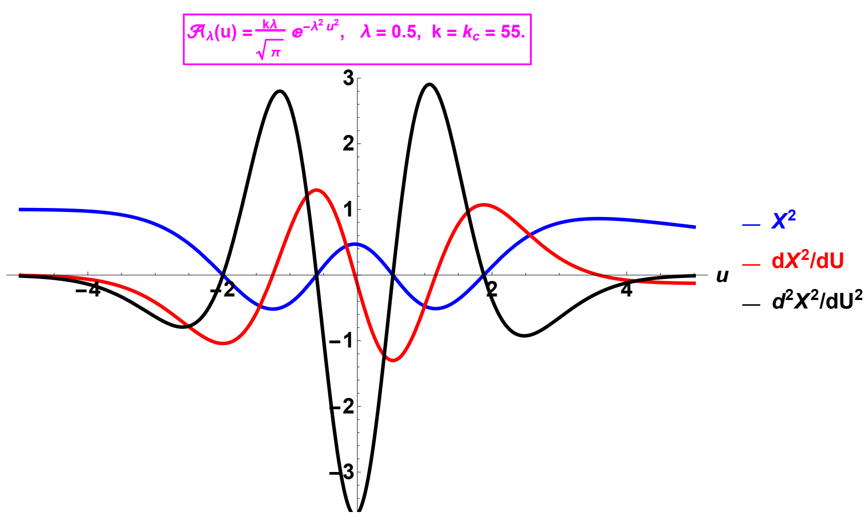
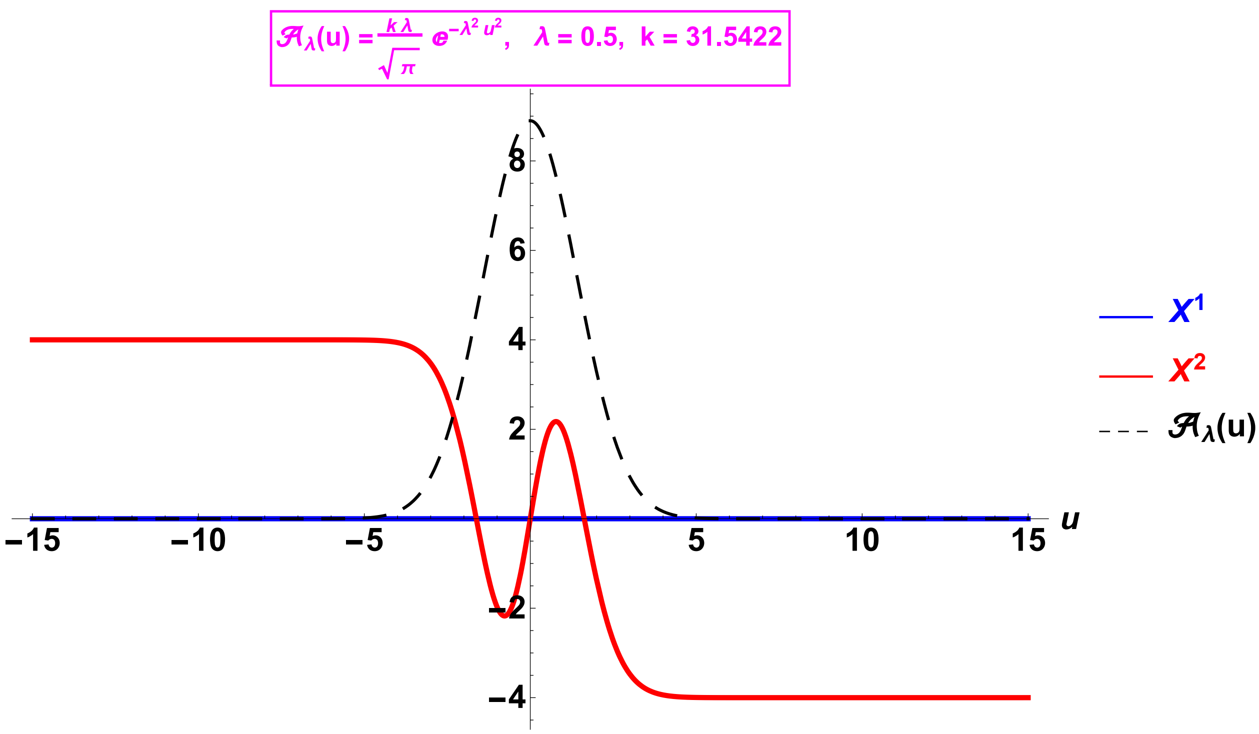


- 2-jumps $k = 30.6602\lambda$ [for $\lambda = 1$, $k \approx 10\pi$]

- 3-jumps $k = 63.0843\lambda$ [for $\lambda = 1$, $k \approx 20\pi$]



1/2 DM effect

Square profile

Toy model in $D = 1$ dim: Sturm-Liouville eqn (6a) - (6b)

$$\frac{d^2 Y}{dU^2} + \frac{1}{2}\mathcal{A}Y = 0 \tag{28}$$

for small λ limit of Gaussian

$$\mathcal{A}(U) = \begin{cases} 0, & U < -a \\ 2h^2, & -a < U < a \\ 0, & U > a \end{cases} \tag{29}$$

Localized in $(-a, a)$, height $2h^2$. Outside wave zone

$$\frac{d^2 Y}{dU^2} = 0 \tag{30}$$

Initial condition (before zone)

$$Y(U < -a) = Y_0. \tag{31}$$

General solution in Afterzone $U > a$:

$$Y(U > a) = y_1 + y_2 U \tag{32}$$

where $y_2 = 0$ means DM and $y_2 \neq 0$ means VM.

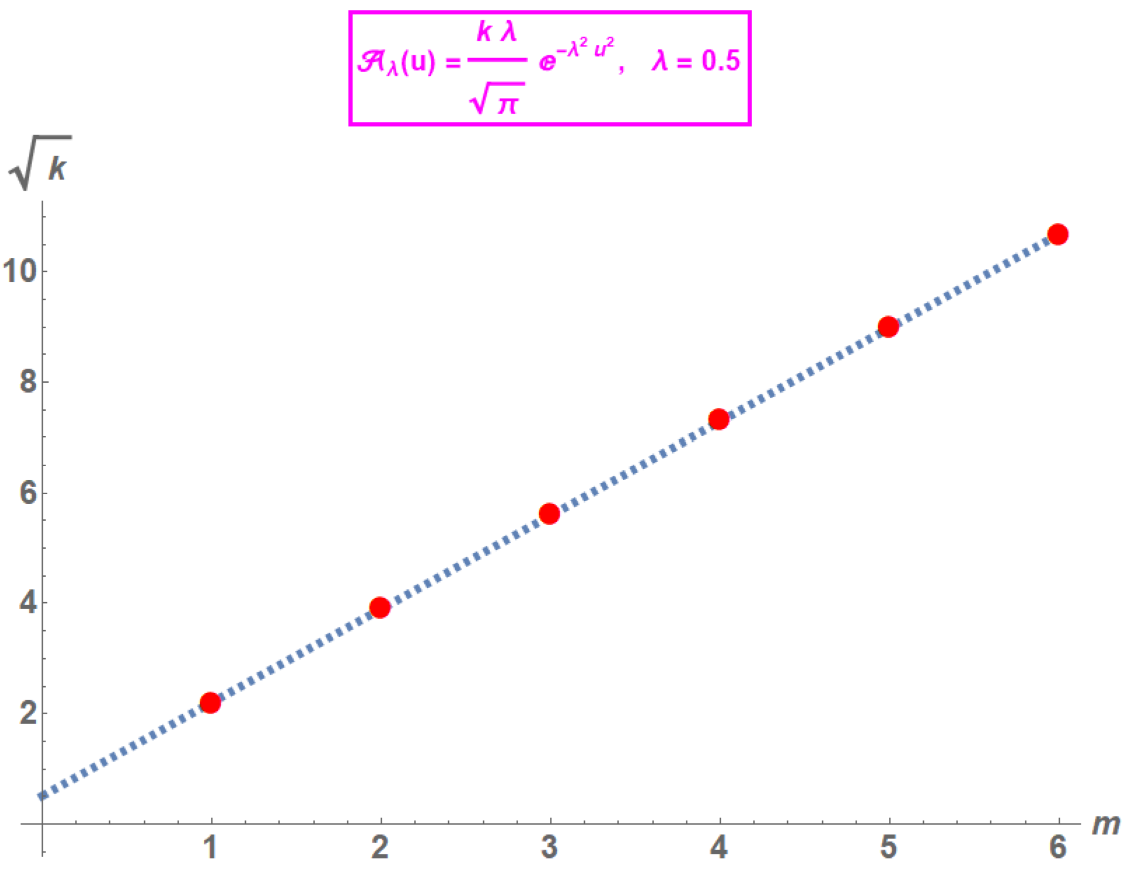
In wave zone $\frac{d^2 Y}{dU^2} + h^2 Y = 0$ oscillator $\Rightarrow$

$$Y(U) = Y_0 \cos h(U + a) \tag{33}$$

$$\left.\frac{dY}{dU}\right|_{U=a} \Rightarrow -Y_0 h \sin 2ha = 0 \Rightarrow$$

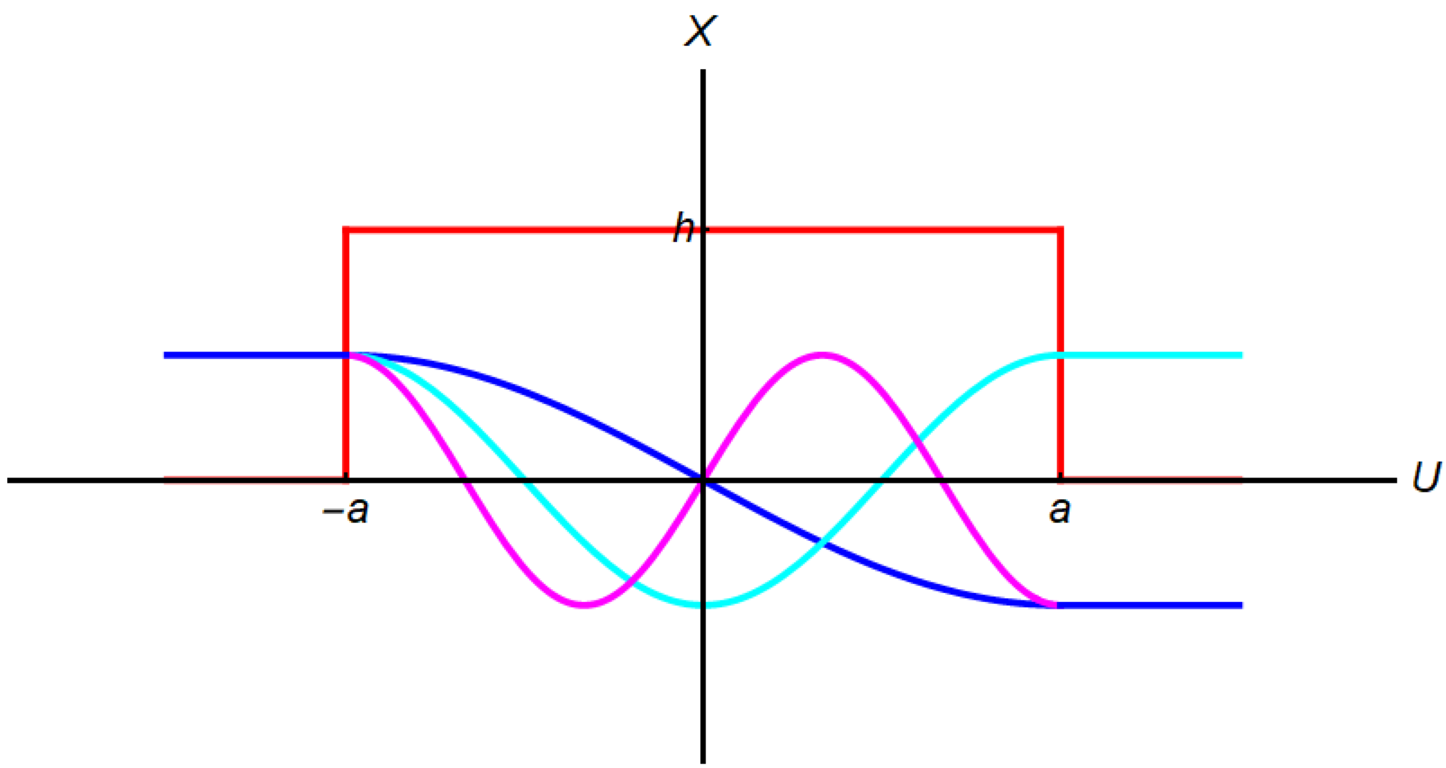
$$\boxed{2ah = \pi\, m, \quad m = 1, 2, \ldots} \tag{34}$$

Get DM if standing wave



k : height, m : number of 1/2 waves.

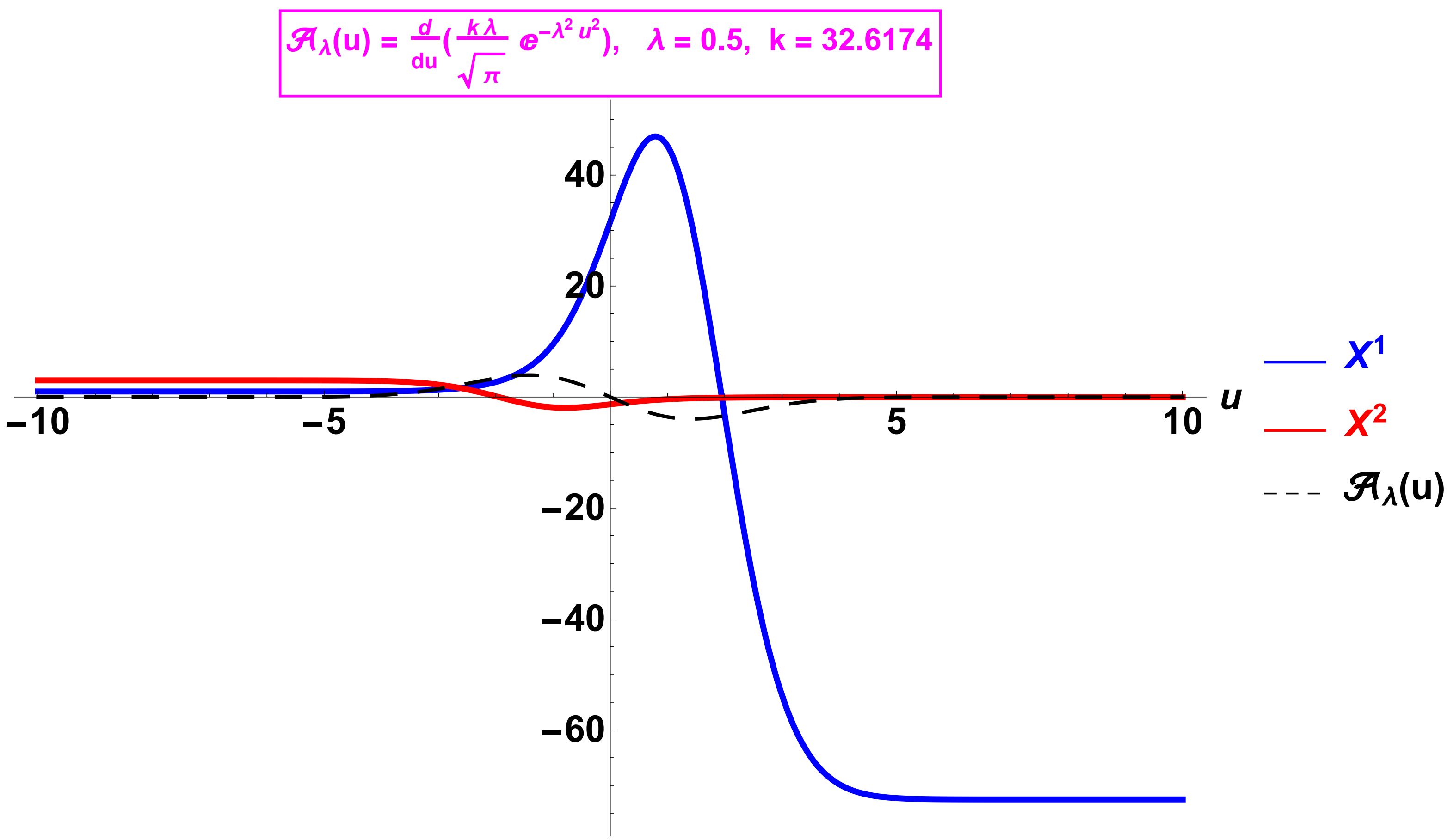
# of waves determined by # of enclosed zeros in $[-a, a]$.



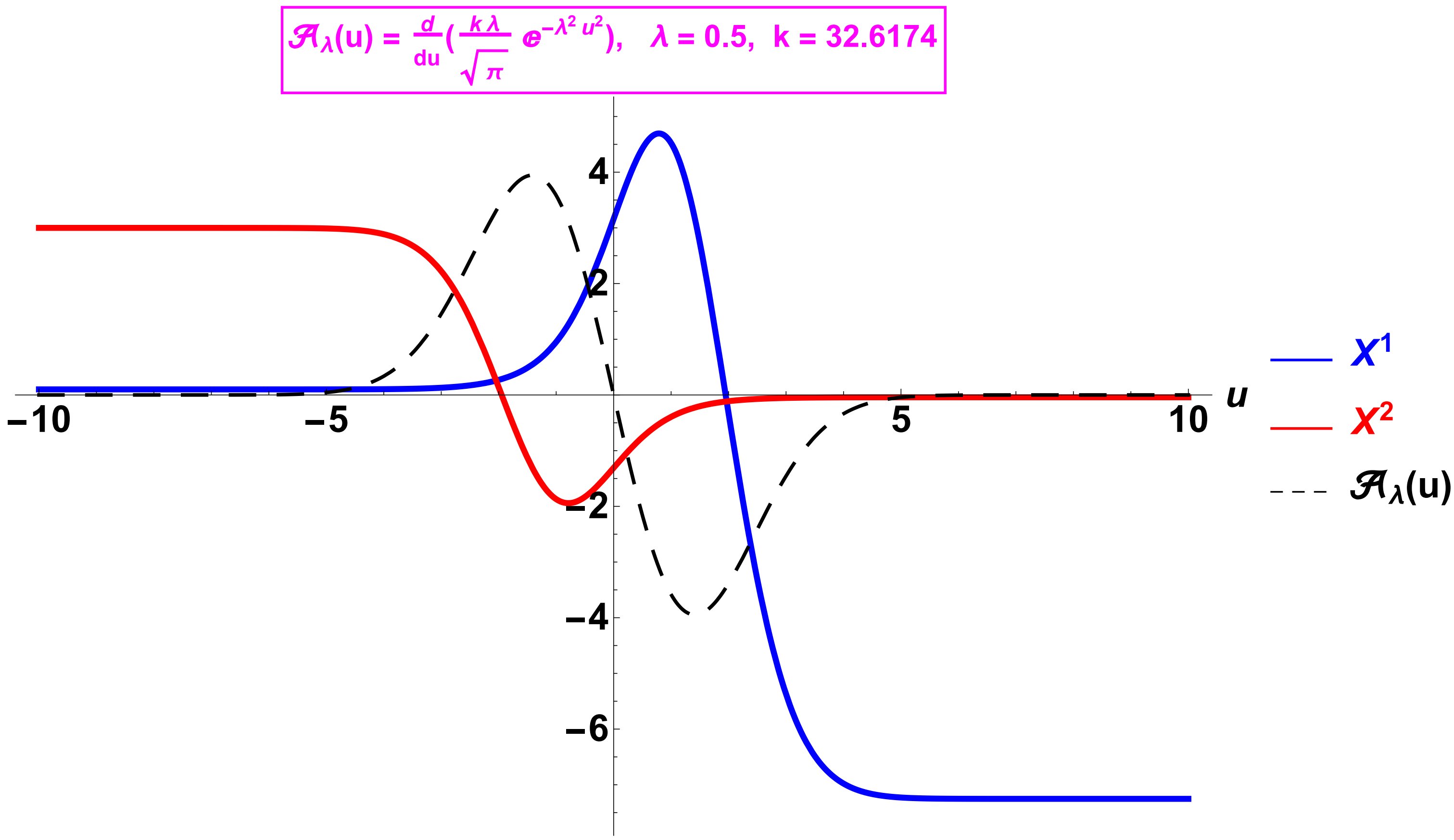
$m = 1$, $m = 2$, $m = 3$

d=1 flyby (Zeldovich - Polnarev):

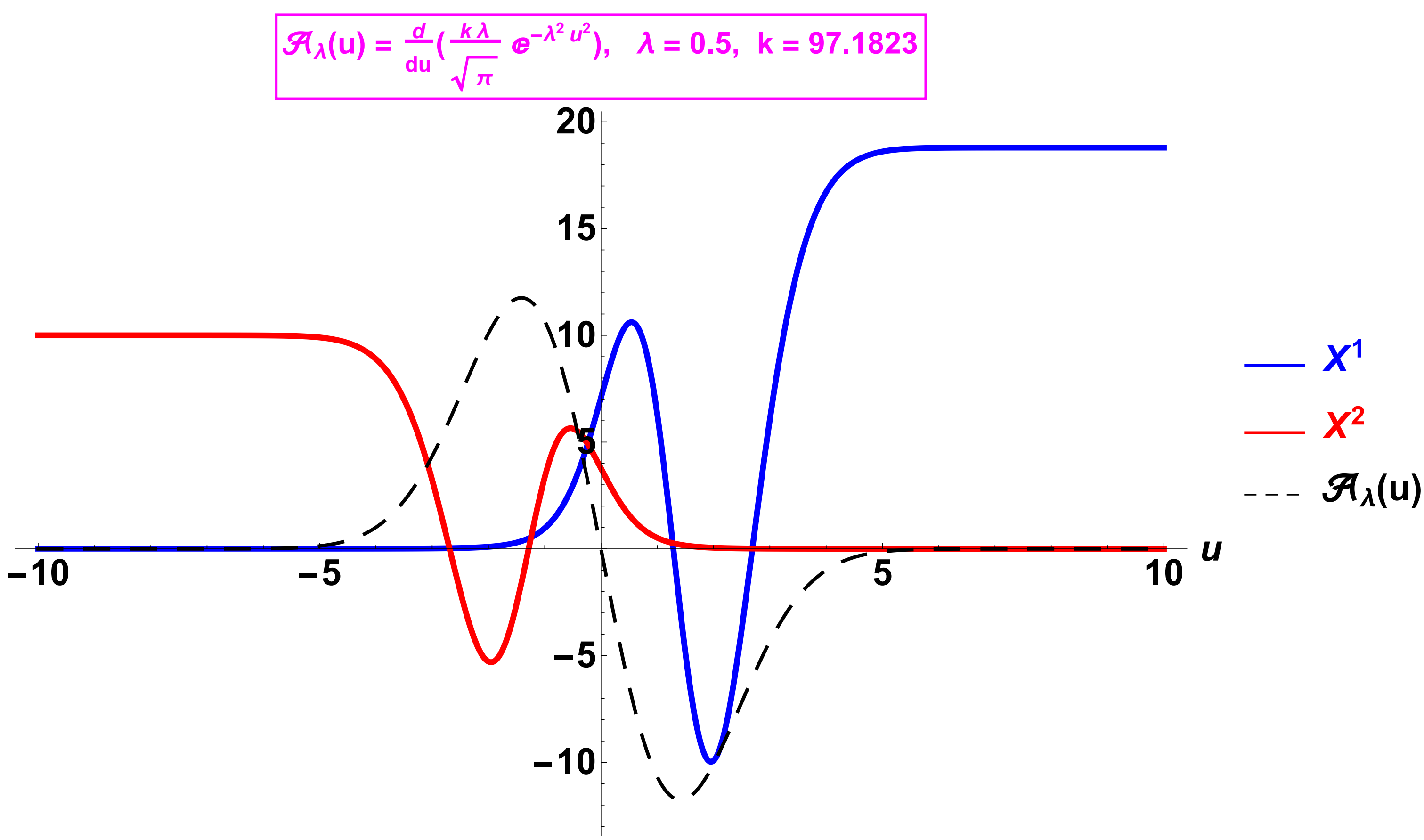
$$A_1(U) = \frac{d}{dU}\left(k\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2 U^2}\right) \tag{35}$$



DM effect in both components



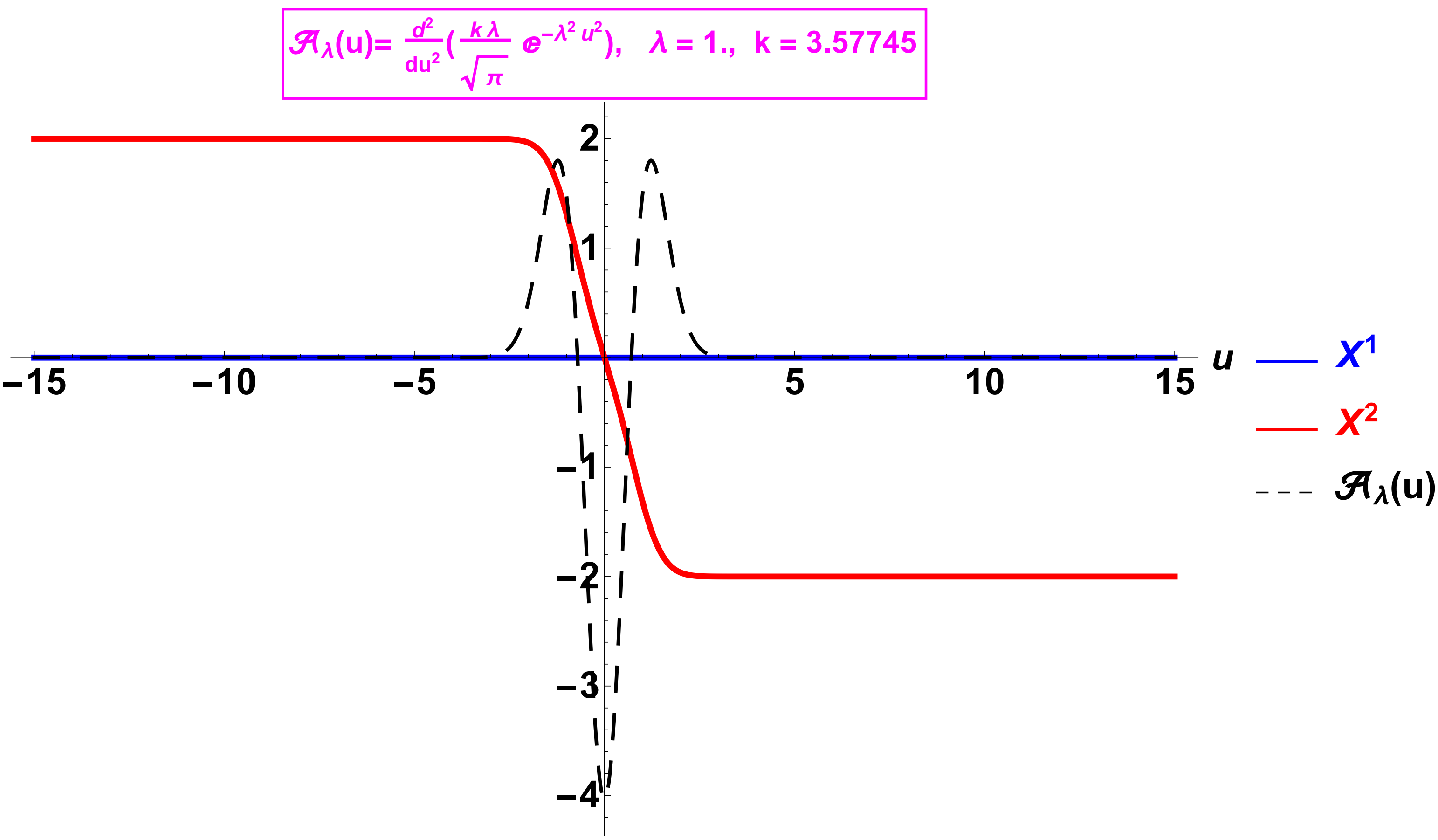
$\mathcal{A}_\lambda(u) = \frac{d}{du}(\frac{k\lambda}{\sqrt{\pi}} e^{-\lambda^2 u^2})$, $\lambda = 0.5$, $k = 32.6174$
X^1
X^2
$\mathcal{A}_\lambda(u)$
u



$\mathcal{A}_\lambda(u) = \frac{d}{du}(\frac{k\lambda}{\sqrt{\pi}} e^{-\lambda^2 u^2})$, $\lambda = 0.5$, $k = 97.1823$
X^1
X^2
$\mathcal{A}_\lambda(u)$
u

d=2 Braginsky-Thorne (1987)

$$\mathcal{A}(U) = \frac{1}{2}\frac{d^2(e^{-U^2})}{dU^2}, \qquad (36)$$

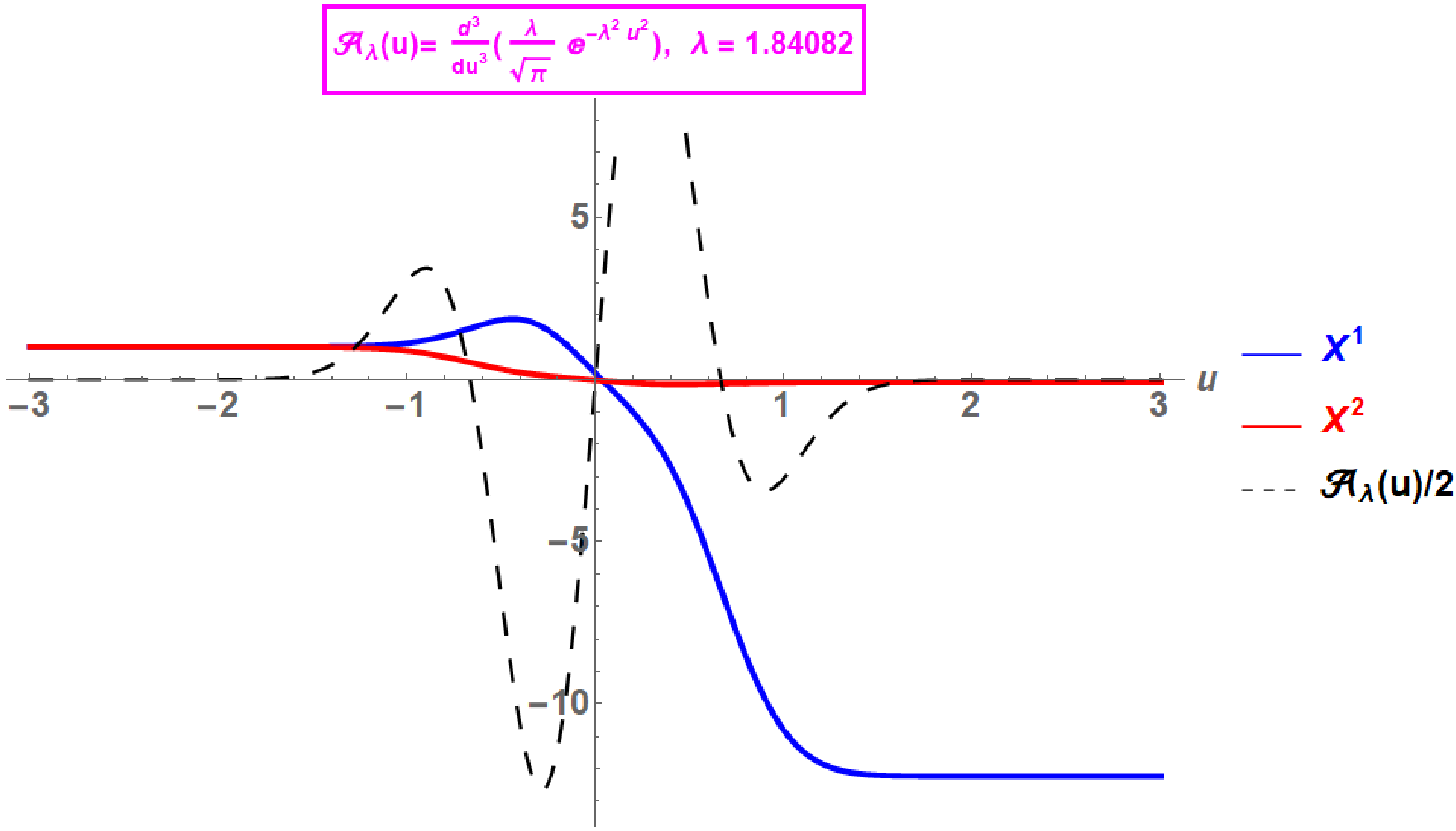


1/2 DM effect

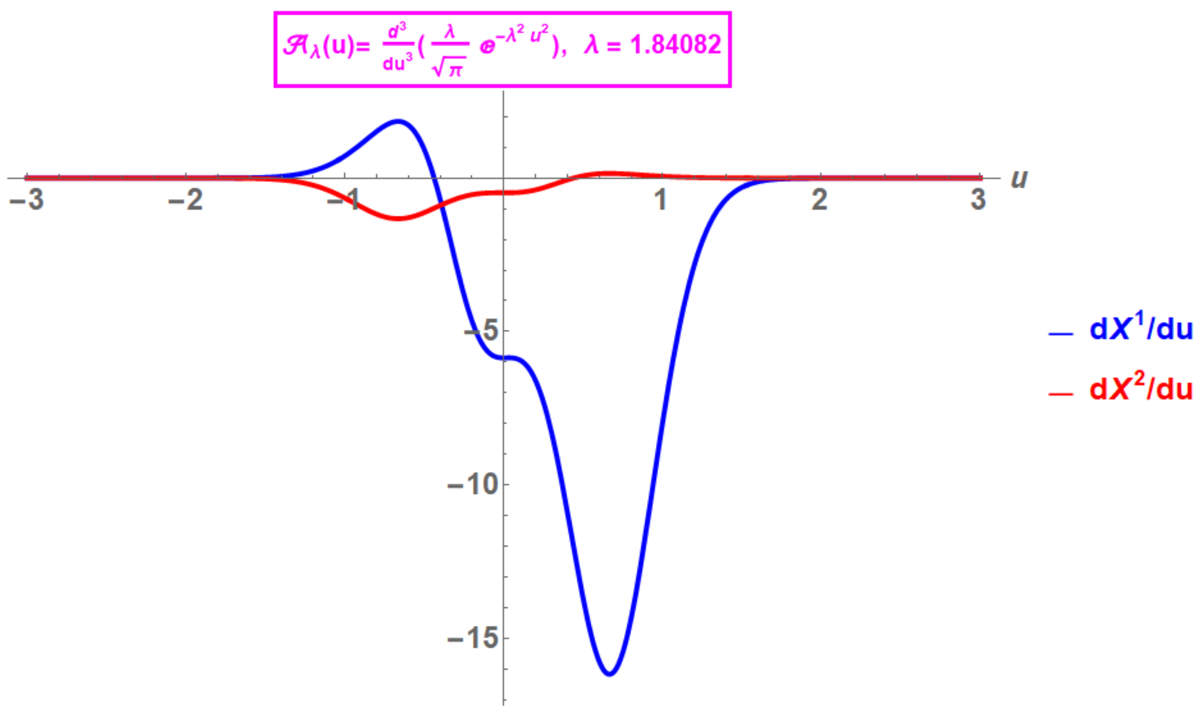
d=3 gravitational collapse

Gibbons-Hawking $\sim$

$$\mathcal{A}(U) = \frac{1}{2}\frac{d^3(e^{-U^2})}{dU^3}. \tag{37}$$



velocity:



DM effect in both components

Rule: $d = 0, 2, 4\ldots$ even # of derivations :

$$\mathcal{A}(-U) = \mathcal{A}(U) \Rightarrow$$

one component diverges exponentially $\Rightarrow$ should be killed.

$d = 1, 3\ldots$ odd # of derivations :

$$\mathcal{A}(-U) = -\mathcal{A}(U) \Rightarrow$$

attractive and repulsive components can combine. Velocity of one component $\equiv 0$ other can be fine-tuned to ≈ 0 exponentially.

CONCLUSION: DM possible for

exceptional values of parameters.

$\sim$ to integerer # of standing waves

in wave zone