The Nonabelian Plasma is Chaotic

Eigenstate Thermalization in SU(2) Gauge Theory

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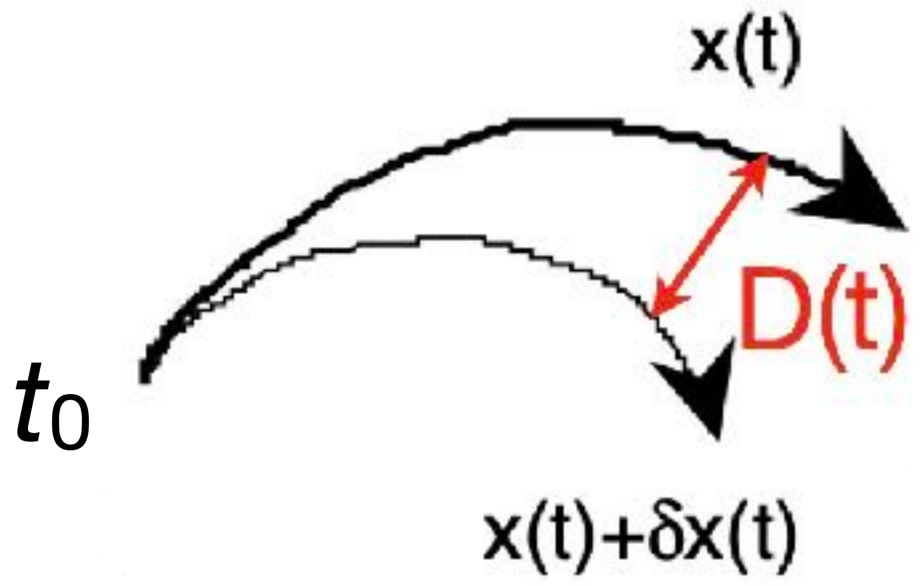
X. Yao, PRD 108, L031504 (2023)
BM, X. Yao, PRD 108, 094505 (2023)
L. Ebner, BM, A. Schäfer, C. Seidl, X. Yao, PRD 109, 014504 (2024)
arXiv:2401.15284

Prelude

Classical theory of chaos:
Lyapunov exponents and KS entropy

Lyapunov exponents - KS entropy

- **A constant growth rate of the observable entropy, i.e. the entropy measured after coarse graining, is a characteristic feature of *chaotic* dynamical systems.**
- Consider two evolutions of such a system starting from slightly different initial conditions $(\vec{x}(t_0), \vec{p}(t_0))$ and $(\vec{x}(t_0) + \delta\vec{x}(t_0), \vec{p}(t_0) + \delta\vec{p}(t_0))$. A dynamical system is *chaotic* if the distance in phase space between the two systems grows exponentially:



$$D(t) = \sqrt{|\delta\vec{x}(t)|^2 + |\delta\vec{p}(t)|^2} = D_0 e^{\lambda t}$$

- λ is called the (largest) *Lyapunov exponent*.
- More generally, one can construct a spectrum of modes around the original trajectory in phase space and obtain the associated spectrum of Lyapunov exponents λ_i. The rate of growth of the coarse grained entropy is known as the Kolmogorov-Sinai (KS) entropy h_{KS}. It is given by

$$dS/dt = h_{\mathrm{KS}} \equiv \sum_{\lambda_i > 0} \lambda_i.$$

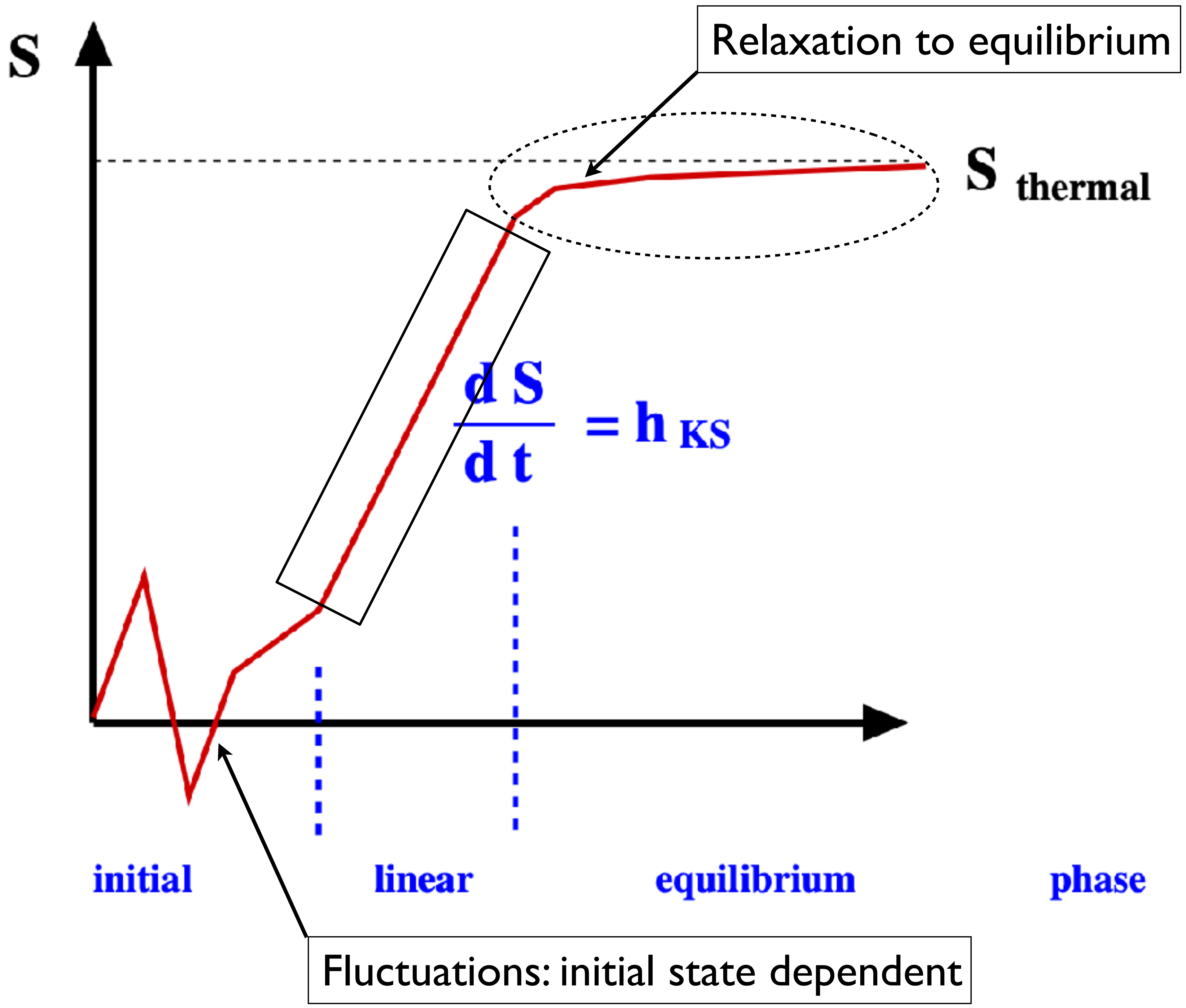
Thermalization of a chaotic system

Depending on the size of initial fluctuations, after some initial period, the measurable entropy of the system grows linearly with time:

$$dS/dt = h_{KS}.$$

After a time $\tau_{eq} = S_{eq}/h_{KS}$, the entropy of the system approaches the value of the entropy in thermal equilibrium, and further growth is impossible because the volume of accessible phase space at fixed total energy is finite.

This behavior can be calculated numerically in the classical limit of field theory.



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Deterministic Chaos in Non-Abelian Lattice Gauge Theory

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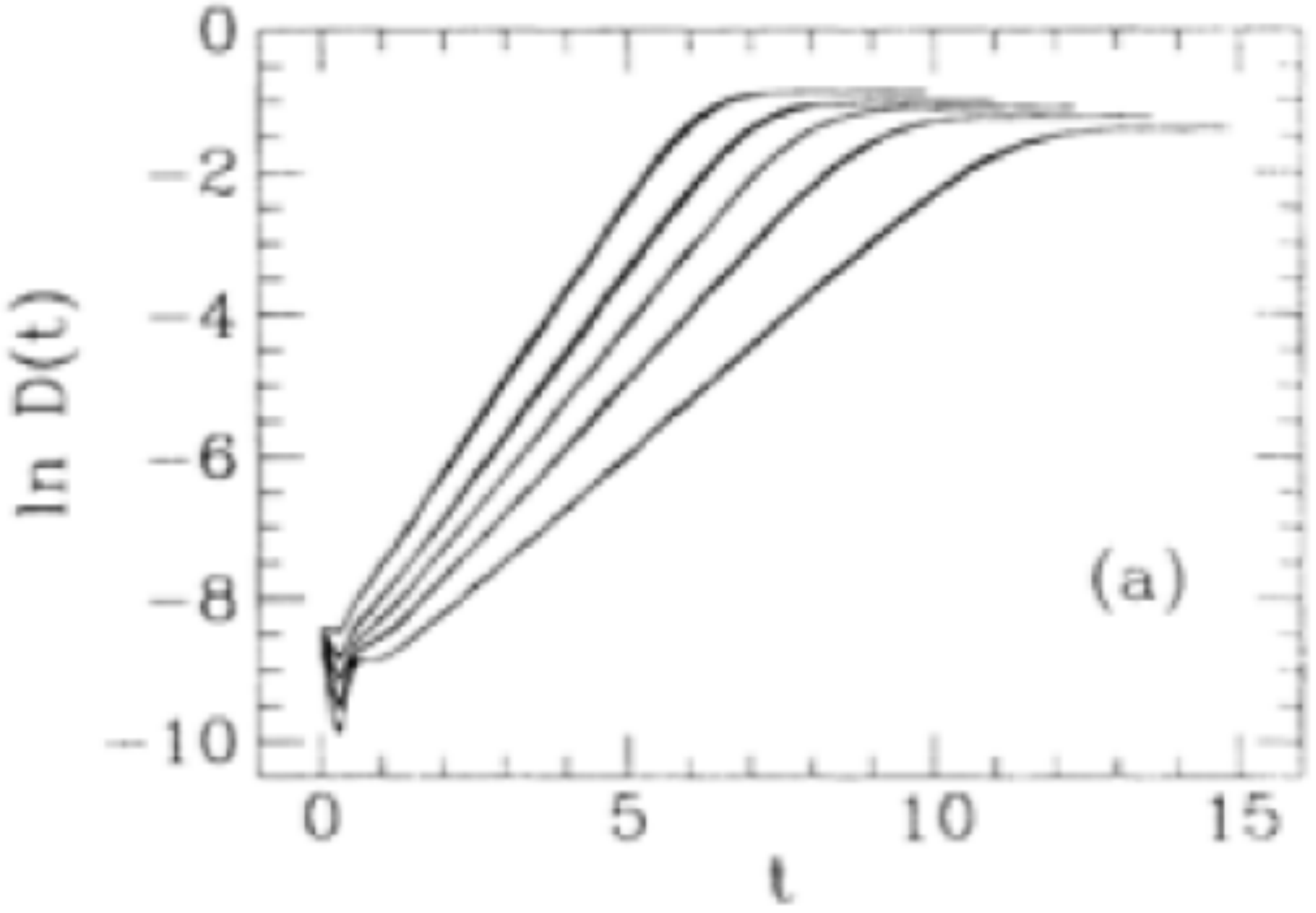
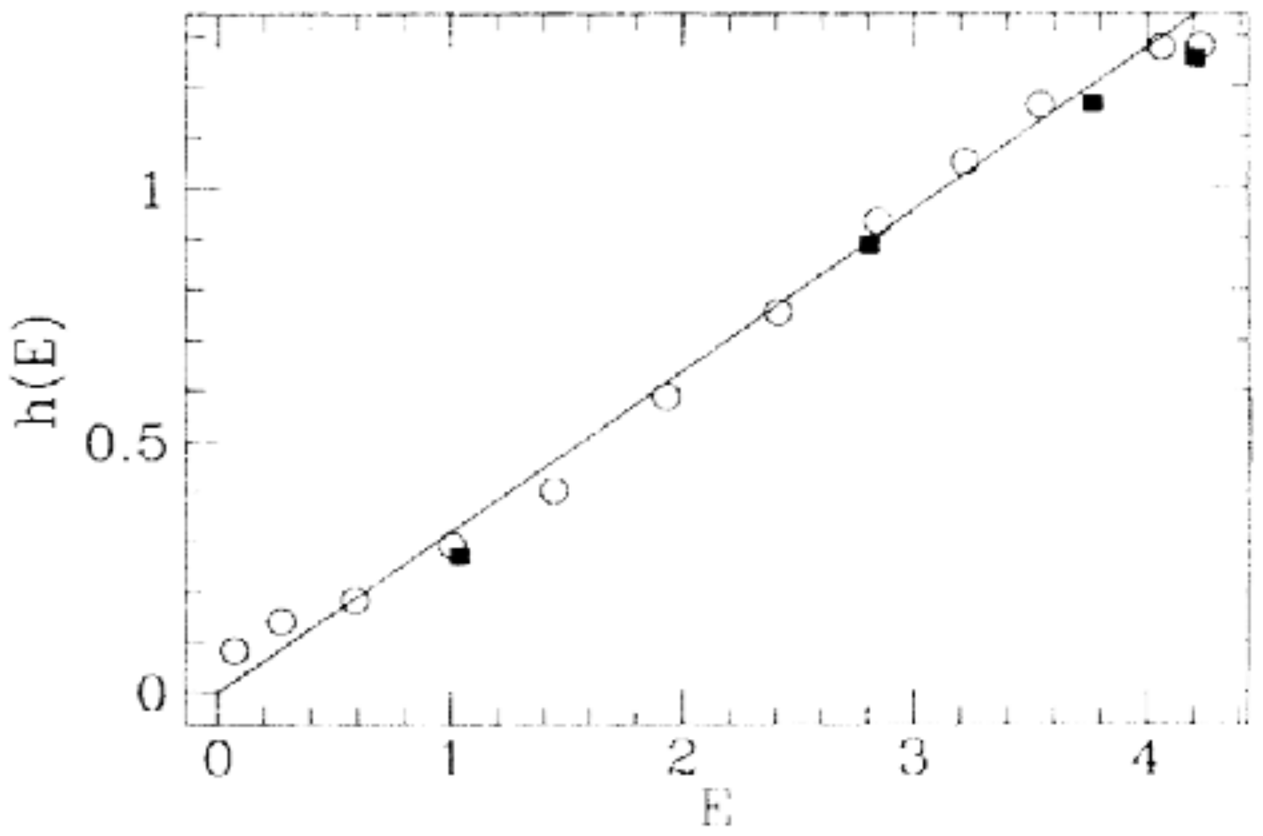
(Received 18 February 1992; revised manuscript received 10 April 1992)

We present numerical evidence that the real-time Hamiltonian dynamics of SU(2) gauge theory on a spatial lattice exhibits deterministic chaos in the semiclassical limit. We determine the largest Lyapunov exponent of the gauge field as a function of energy density, and derive a nonperturbative expression for the thermalization time.



$$h_{\mathrm{SU(2)}}(E)\,a \approx 0.17 g^2 E\,a$$



Physics Letters B 298 (1993) 257–262
North-Holland

Lyapunov exponent of classical SU(3) gauge theory

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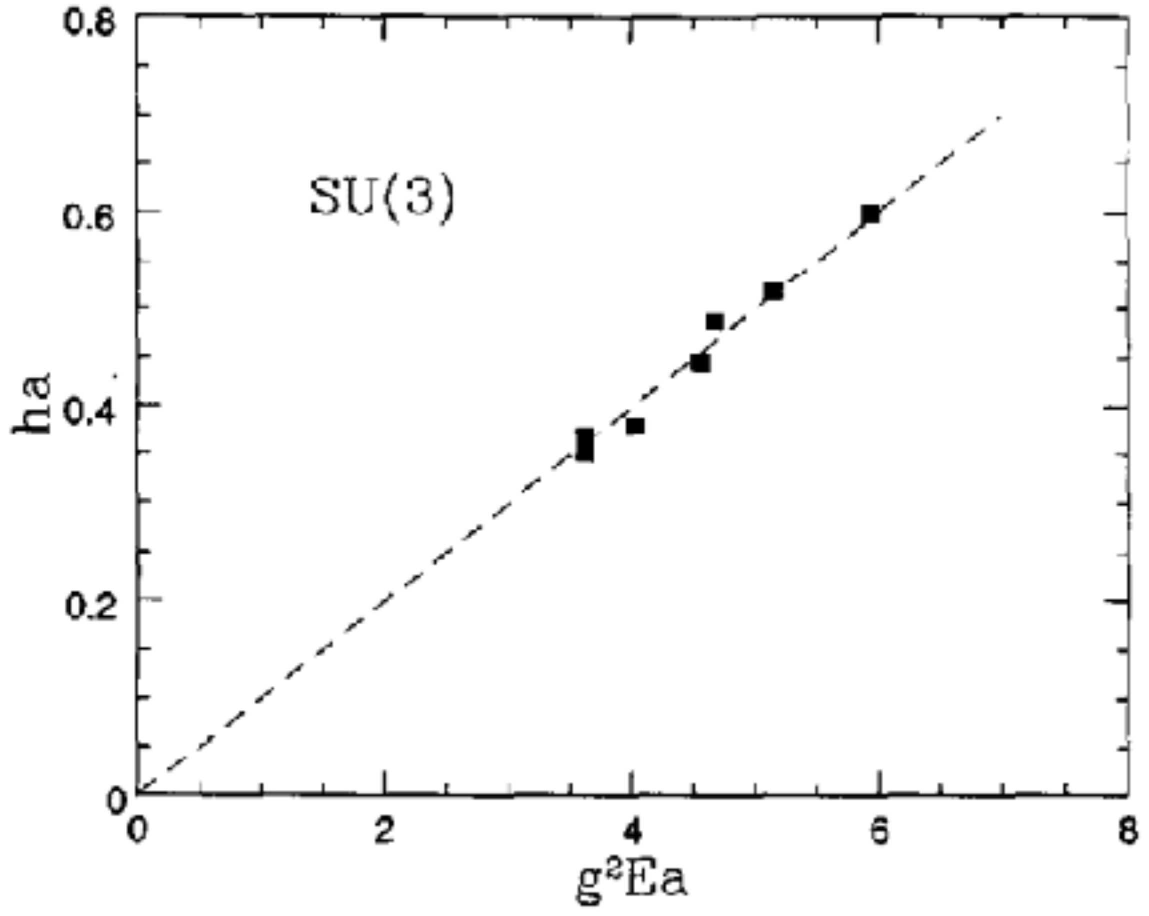
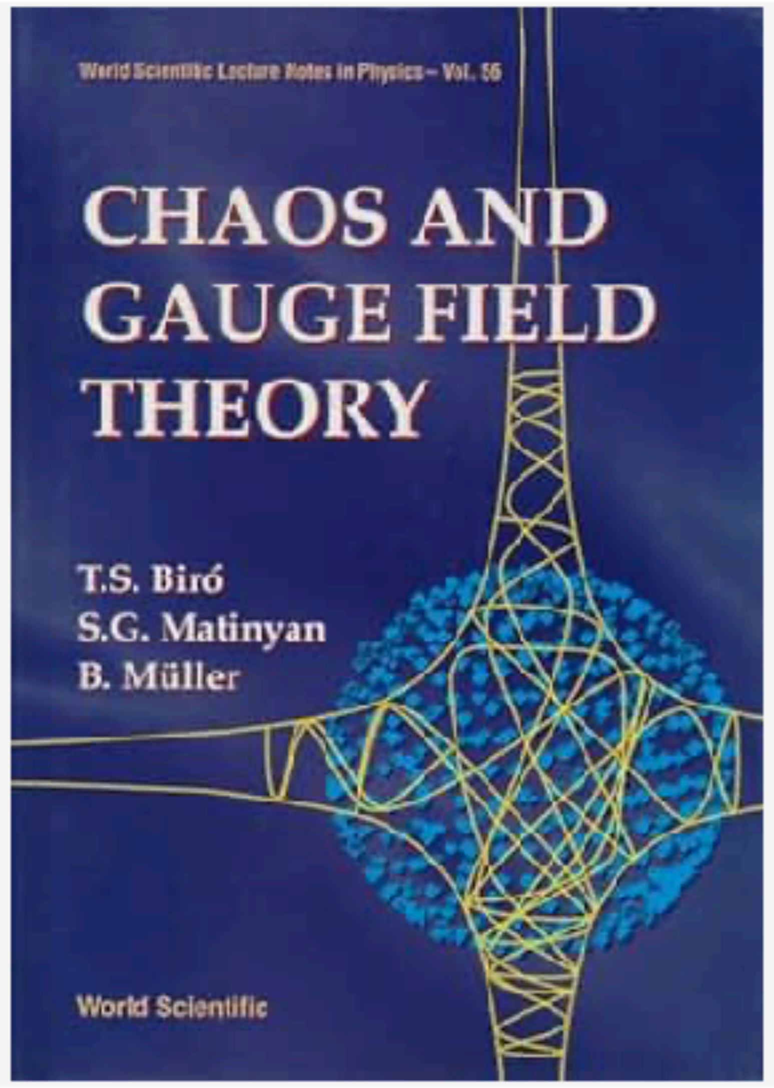
Received 28 September 1992

The classical SU(3) gauge theory is shown to be deterministic chaotic. Its largest Lyapunov exponent is determined, from which a short time scale of thermalization of a pure gluon system is estimated. The connection to gluon damping rate is discussed.

$$h_{\mathrm{SU(3)}}(E)\,a \approx 0.10 g^2 E\,a$$



$$h_{\mathrm{SU(N)}} \approx 0.17 g^2 N_c T \approx 2\gamma_{\mathrm{pl}}(\omega = 0)$$



CHAOS AND GAUGE FIELD THEORY (World Scientific Lecture Notes in Physics)

by Tamas S Biro, Sergei Matinyan, et al. | Mar 1, 1995

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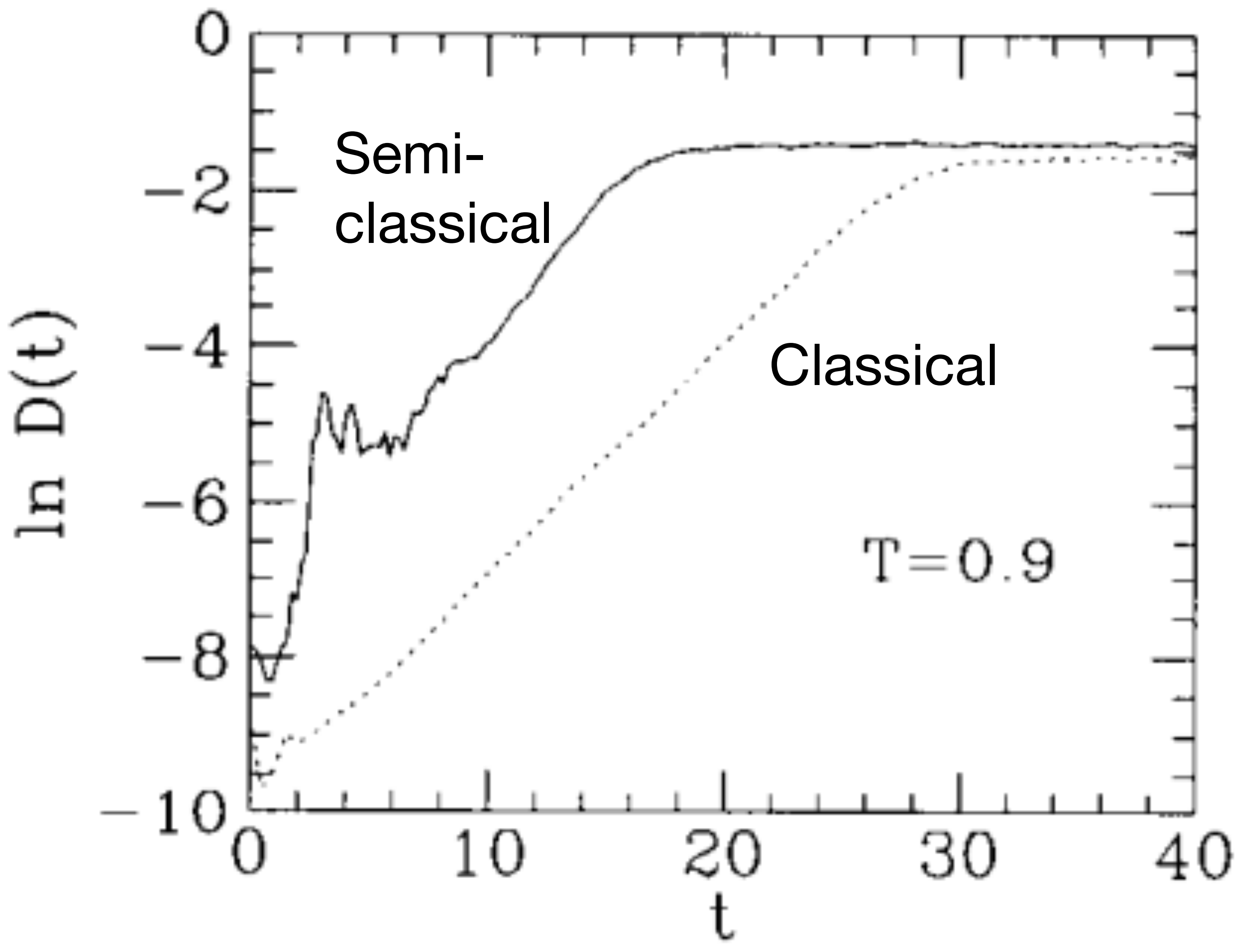

Nuclear Physics A568 (1994) 727–744
North-Holland

NUCLEAR PHYSICS A

Variational approach to real-time evolution of Yang–Mills gauge fields on the lattice

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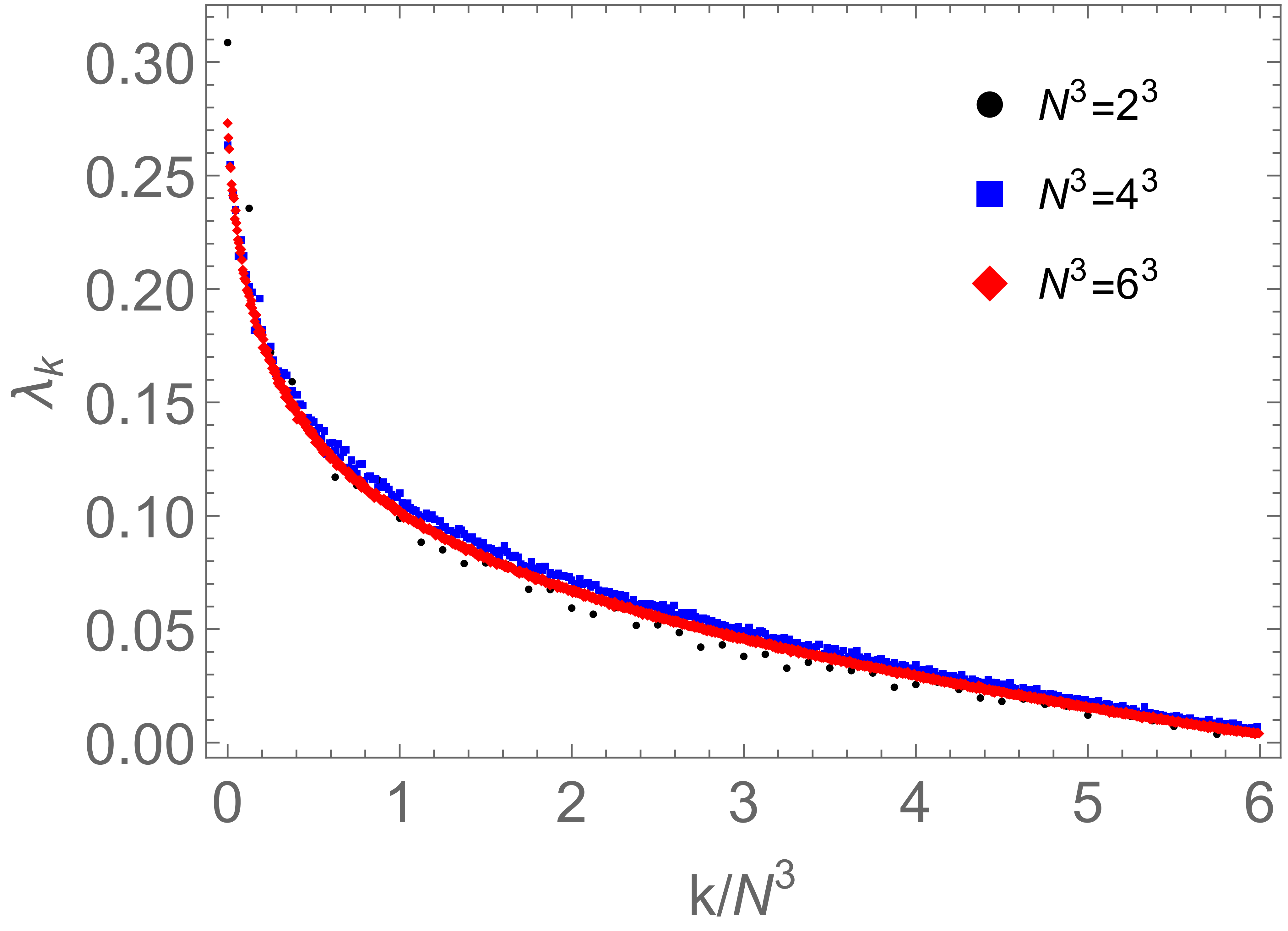
Received 26 May 1993

Abstract

Applying a variational method to a gaussian wave ansatz, we have derived a set of semi-classical evolution equations for SU(2) lattice gauge fields, which take the classical form in the limit of a vanishing width of the gaussian wave packet. These equations are used to study the quantum effects on the classical evolutions of the lattice gauge fields.

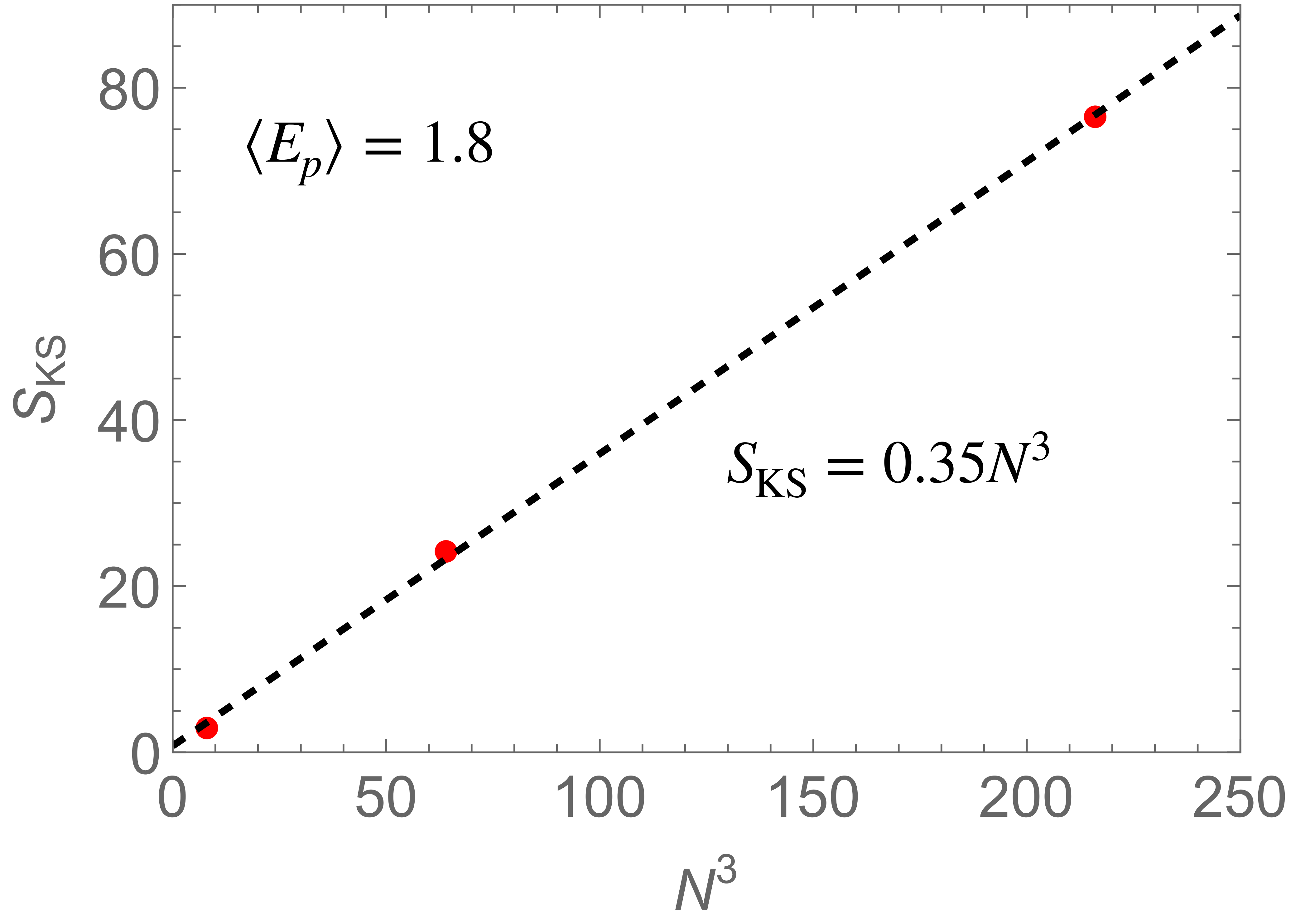
Lyapunov Spectrum SU(2)

Number of unstable modes with positive Lyapunov exponents = number of dynamical modes of the lattice gauge theory



Sum of all positive Lyapunov exponents exhibits volume growth: S_{KS} is extensive

[J. Bolte, BM, A. Schäfer, PRD 61 (2000) 054506]



How chaotic is QCD?

- Maldacena, Shenker, and Stanford [JHEP 08 (2016) 106] argued that there is an upper bound on Lyapunov exponents: $\lambda \leq 2\pi T$, where *T* is the temperature reached after equilibration.
- Our numerical simulations for the SU(3) gauge theory found [PRD 52 (1995) 1260]

$$\lambda_{\rm max} = 0.53 g^2 T$$

Which saturates the MSS bound when $\alpha_s = g^2/4\pi \approx 1$.

- Thermalization in QCD at realistic coupling may thus be about three times slower than at infinitely strong coupling as realized in BH formation or AdS/CFT.
- But this still gives a rather short thermalization time around 1 fm/*c*.
- *Next challenge:* Compute entropy growth in the quantum lattice gauge theory.

Chaos and quantization

Full chaos implies small perturbations around classical solutions generally grow exponentially and that their eigenvalue spectrum contains modes with imaginary frequencies.

At the quantum level, these modes correspond to tachyons which invalidate canonical quantization.

This problem is often avoided by *dynamical mass generation* [S.G. Matinyan & BM, PRL 78 (1997) 2515]. In simple cases, analytical solutions can be found involving spontaneous symmetry breaking (e.g. abelian Higgs model).

In the case of nonabelian gauge theories the mechanism is confinement, and no analytical approach is known that avoids the appearance not tachyonic modes around any classical field configuration.

Famous example: instability of a uniform chromomagnetic field (Savvidy 1977). Attempts to construct a nonperturbative ground state ("Copenhagen vacuum") ultimately failed.

Euclidean lattice gauge theory provides a successful approach for the ground state and thermal equilibrium. Dynamics of highly excited state requires a different approach.

Quantum Chaos

SU(2) Lattice Gauge Theory

Thermalization

Current description of rapid thermalization uses either semiclassical approximations (kinetic theory) or holographic techniques: One method neglects potentially important quantum effects, the other method describes a quantum field theory that differs from QCD. Can we do better?

How does apparent thermalization happen in a closed quantum system, when energy is conserved?

Time evolution of local operator expectation value in terms of energy eigenstates is:

$$\langle O\rangle(t) = \mathrm{Tr}[O\rho(t)] = \sum_{n,m}\langle n|O|m\rangle\langle m|\rho(0)|n\rangle e^{i(E_n-E_m)t}$$

↓ After some time?

$$\langle O\rangle_{\mathrm{mc}}(E) \qquad E = \mathrm{Tr}(H\rho)$$

Microcanonical ensemble average

Eigenstate Thermalization Hypothesis (ETH)

For most non-integrable systems, matrix elements of “typical” local operators for “typical” energy eigenstates can be represented as

$$\langle n|O|m\rangle = \langle O\rangle_{\rm mc}(E)\delta_{nm} + e^{-S(E)/2} f(E,\omega) R_{nm}$$

$$E = (E_n + E_m)/2$$

$$\omega = E_n - E_m$$

Diagonal part close to microcanonical ensemble average

Correction suppressed exponentially by system size

Gaussian (?) random matrix

Spectral function decays with ω

Deutsch, PRA 43, 2046 (1991)

Srednicki, PRE 50, 888 (1994)

L. D’Alessio, Y. Kafri, A. Polkovnikov, M. Rigol, Adv. Phys. 65 (2016) 239 [1509.06411]

From ETH to Thermalization

For large systems and initial states with small energy spread, ETH leads to

(1) Long time average $\overline{O} \approx$ thermal expectation value $\langle O \rangle_T$ —> ergodic

(2) Fluctuations of $\langle O \rangle(t)$ around $\overline{O}$ are exponentially small in system size

(3) Quantum fluctuations $\approx$ thermal fluctuations

(4) Real-time autocorrelation function

$$\langle n|O(t)O(0)|n\rangle - \langle n|O(t)|n\rangle\langle n|O(0)|n\rangle \approx \int d\omega e^{-i\omega t} e^{\beta\omega/2} |f(E,\omega)|^2$$

where $f(E,\omega)$ is related to the spectral function (depending on operator O)

The system, when observed through O, behaves like a system in thermal equilibrium.

What must be demonstrated?

Fundamental ETH relation: $\langle n|O|m\rangle = \langle O\rangle_{\rm mc}(E)\delta_{nm} + e^{-S(E)/2} f(E,\omega) R_{nm}$

- Discretize the continuum theory on a spatial lattice, choose boundary conditions
- Show that the diagonal part is exponentially close to the microcanonical average
- Show that the off-diagonal part is a (Gaussian) random matrix
- Show that the spectral function decays for large ω
- Consider “physical”, i.e. gauge invariant, multiplicatively renormalizable operators
- Operators could be local or sufficiently smeared
- Demonstrate RG behavior for several $g(a)$ when $a \to 0$, to establish the continuum limit
- Demonstrate ETH for several system sizes for fixed $g(a)$, to establish the infinite volume limit

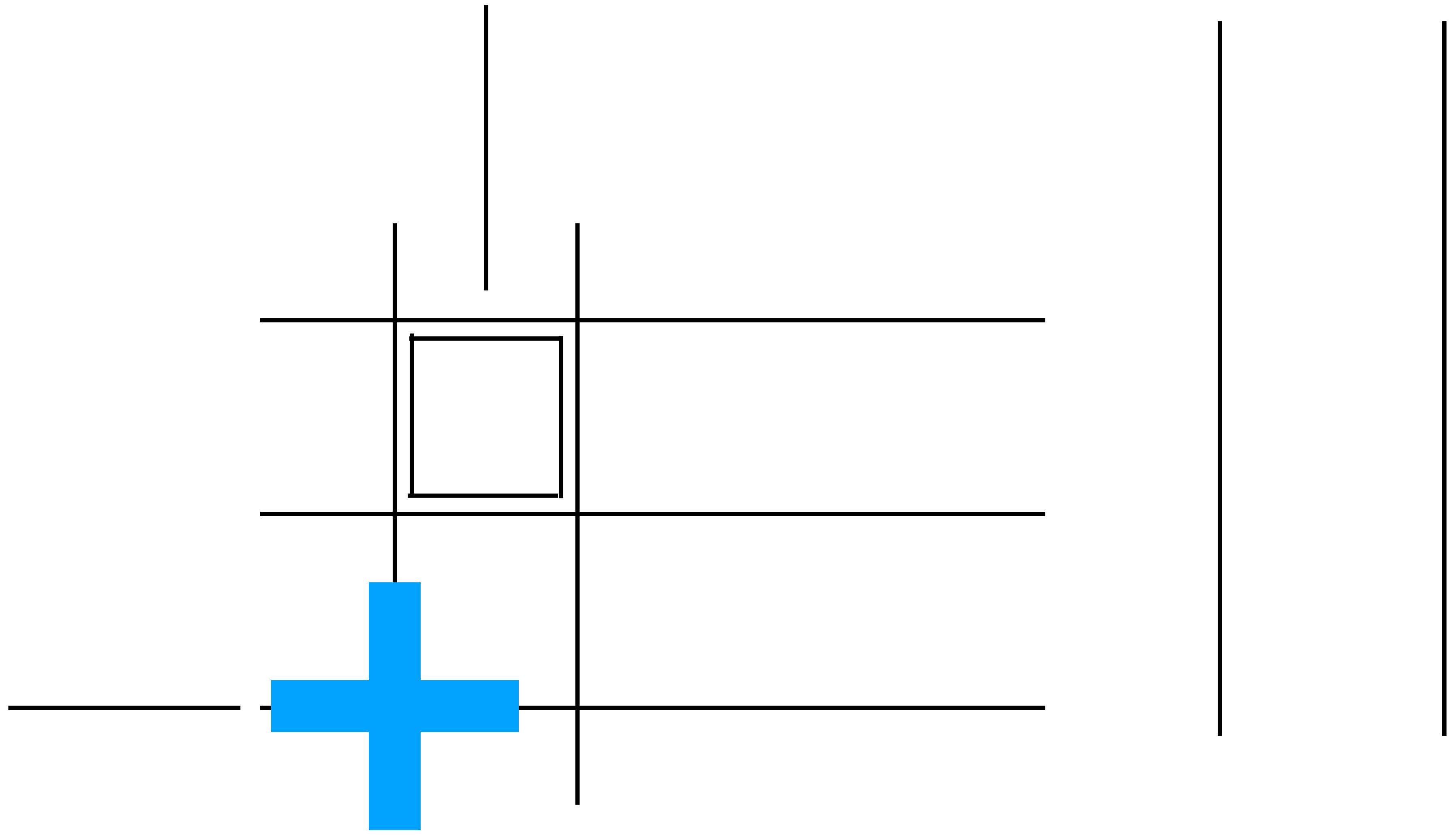
(2+1)-D SU(2) Lattice Gauge Theory

Kogut-Susskind Hamiltonian: $$H = \frac{g^2}{2}\sum_{\text{links}}(E_i^a)^2 - \frac{2}{a^2 g^2}\sum_{\text{plaquettes}}\Box(\boldsymbol{n})$$

$$\Box(\boldsymbol{n}) = \mathrm{Tr}[U^\dagger(\boldsymbol{n},\hat{y})U^\dagger(\boldsymbol{n}+\hat{y},\hat{x})U(\boldsymbol{n}+\hat{x},\hat{y})U(\boldsymbol{n},\hat{x})]$$

$$[E_i^a, U(\boldsymbol{n},\hat{j})] = -\delta_{ij}T^a U(\boldsymbol{n},\hat{j})$$

$$[E_i^a, E_i^b] = i f^{abc} E_i^c$$



Gauss's law: Every vertex transforms as a singlet for a state to be physical

Electric basis on links: $|j\, m_L\, m_R\rangle$

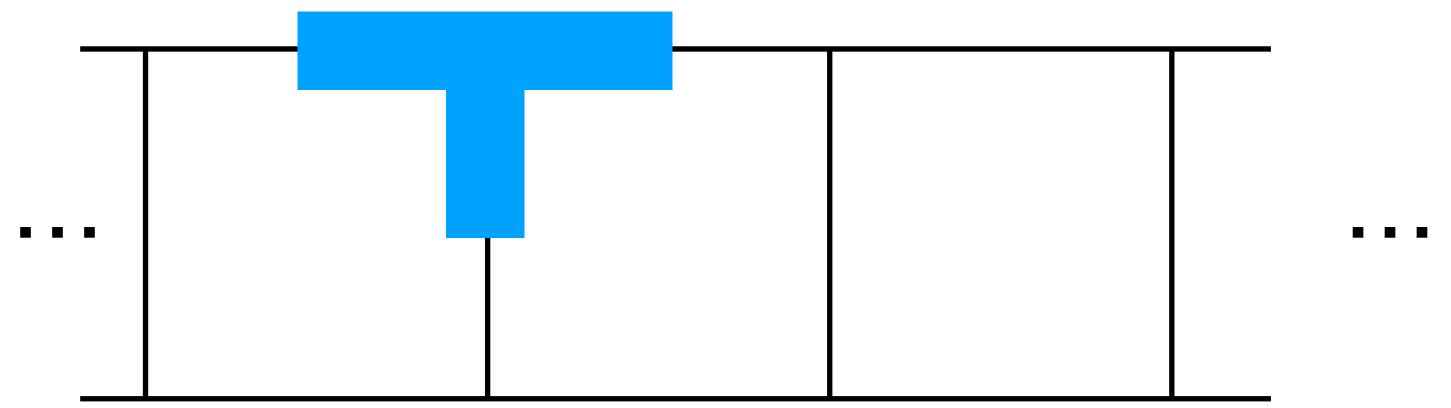
$$E^2|j\, m_L\, m_R\rangle = j(j+1)|j\, m_L\, m_R\rangle$$

S.A. Chin et al.. PRD 31 (1985) 3201

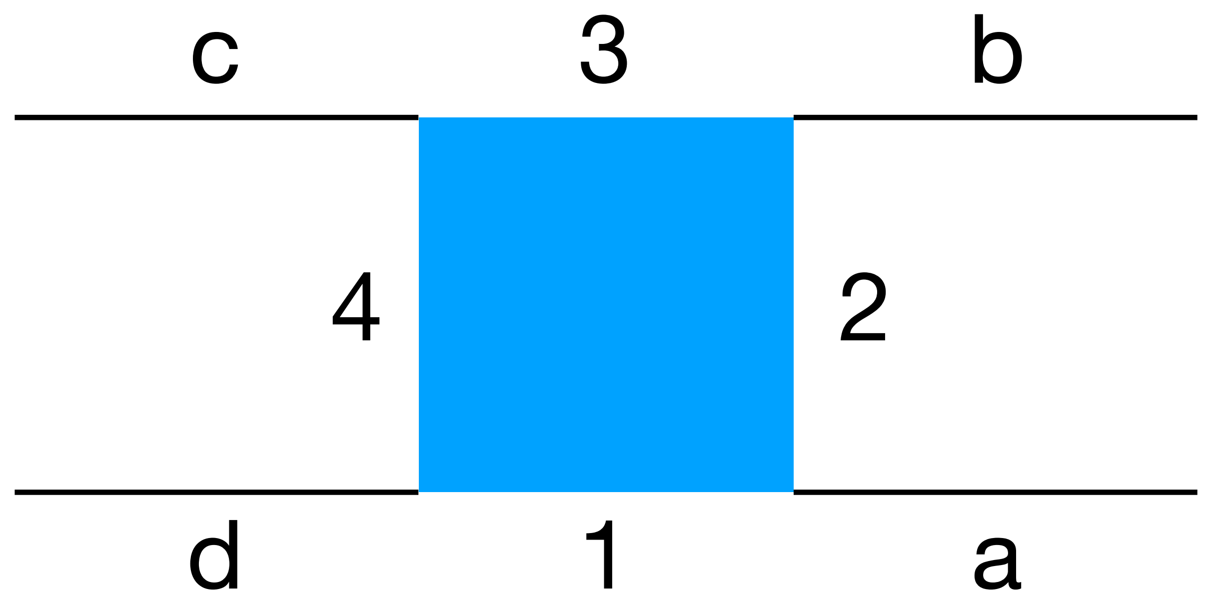
Byrnes, Yamamoto, quant-ph/0510027

(2+1)-D SU(2) on Periodic Plaquette Chain

Each vertex has three links: singlet is uniquely defined by the j values on the three links



Matrix elements between physical states (singlets) expressed in 6j symbols



j: initial

J: final

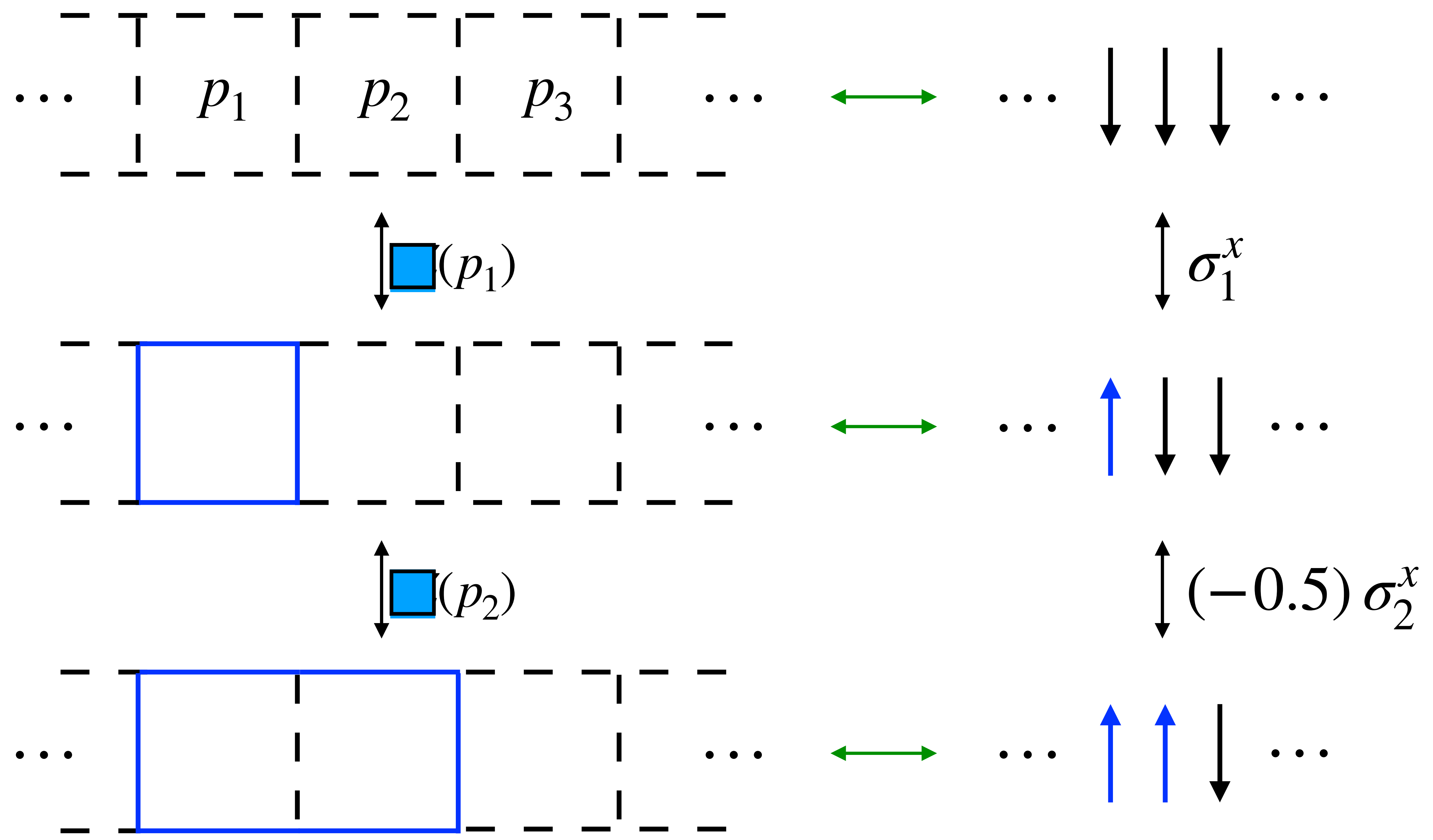
$$\langle J_1 J_2 J_3 J_4 | \Box | j_1 j_2 j_3 j_4 \rangle = \prod_{\alpha=a,b,c,d} (-1)^{j_\alpha} \prod_{\alpha=1,2,3,4} \left[(-1)^{j_\alpha + J_\alpha} \sqrt{(2j_\alpha+1)(2J_\alpha+1)} \right]$$

$$\begin{Bmatrix} j_a & j_1 & j_2 \\ 1/2 & J_2 & J_1 \end{Bmatrix} \begin{Bmatrix} j_b & j_2 & j_3 \\ 1/2 & J_3 & J_2 \end{Bmatrix} \begin{Bmatrix} j_c & j_3 & j_4 \\ 1/2 & J_4 & J_3 \end{Bmatrix} \begin{Bmatrix} j_d & j_4 & j_1 \\ 1/2 & J_1 & J_4 \end{Bmatrix}$$

Klco, Stryker, Savage, 1908.06935

Reduced Hilbert space with $j_{\max} = 1/2$

SU(2) with $j_{\max} = 1/2$
can be mapped onto spin chain
[X. Yao, PRD 108 (2023) L031504]



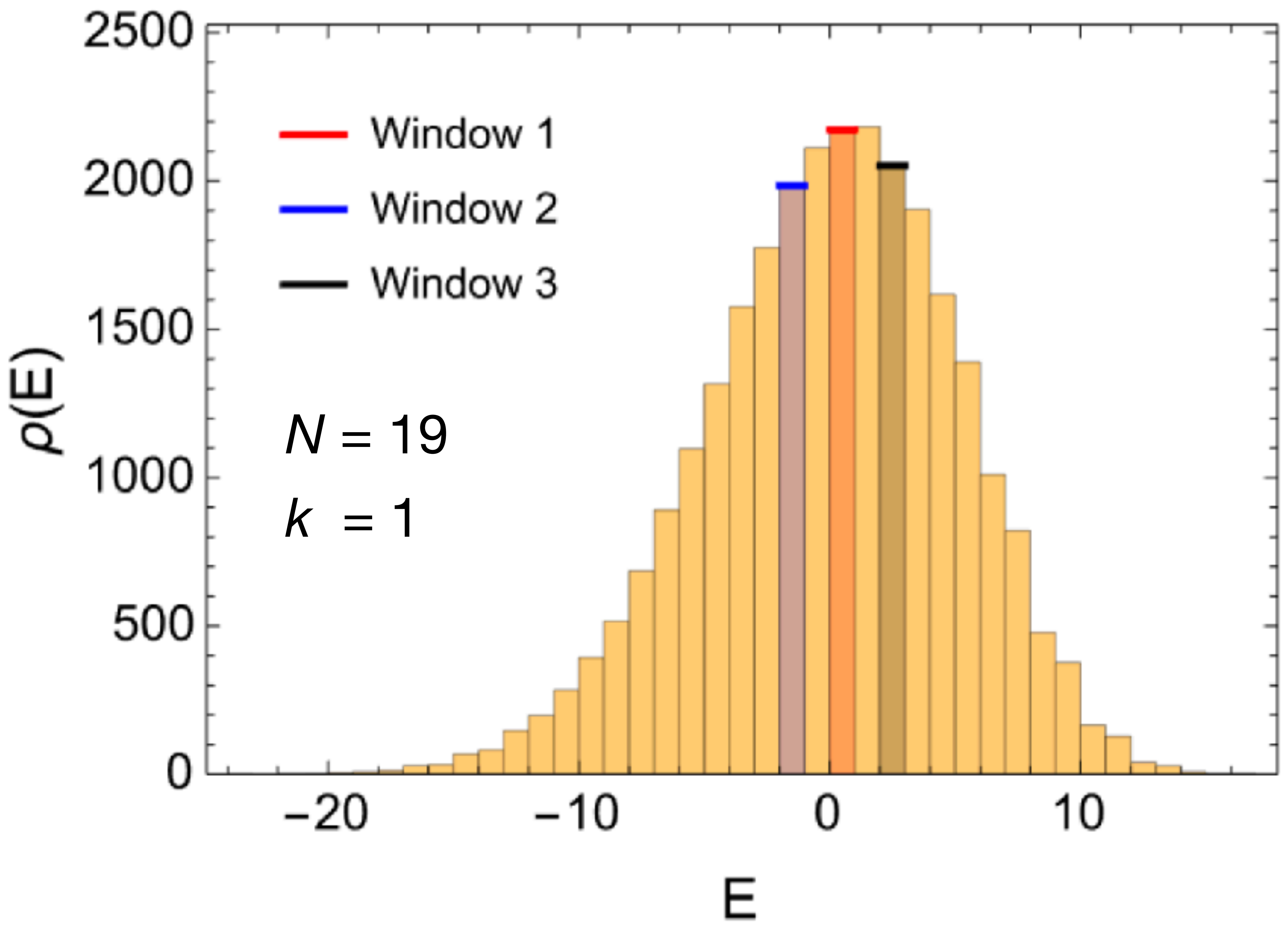
Project onto momentum eigenstates
$-N/2 \leq k \leq N/2$ (N plaquettes)

$$aH = J\sum_{i=0}^{N-1}\sigma_i^z\sigma_{i+1}^z + h_z\sum_{i=0}^{N-1}\sigma_i^z + h_x\sum_{i=0}^{N-1}(-0.5)^{(\sigma_{i-1}^z+\sigma_{i+1}^z)/2+1}\sigma_i^x$$

$$J = -3ag^2/16,\ h_z = 3ag^2/8,\ h_x = -2/(ag^2)$$

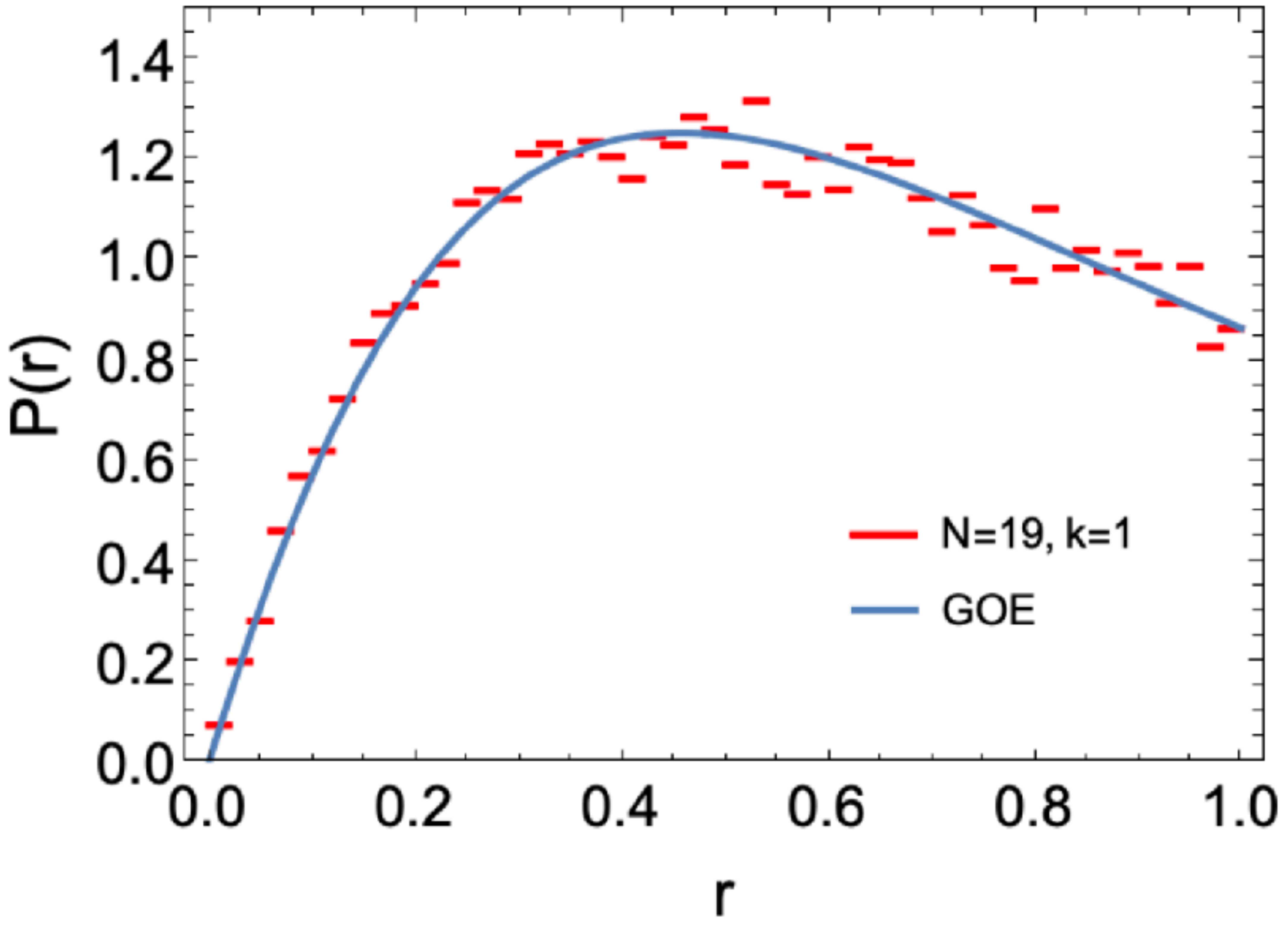
Plaquette Chain with $j_{\max} = 1/2$: Spectrum

Look at matrix elements in 3 energy windows around peak with ca. 2000 eigenstates each



Restricted gap ratio distribution

$$0 < r_\alpha = \frac{\min[\delta_\alpha, \delta_{\alpha-1}]}{\max[\delta_\alpha, \delta_{\alpha-1}]} \leq 1$$

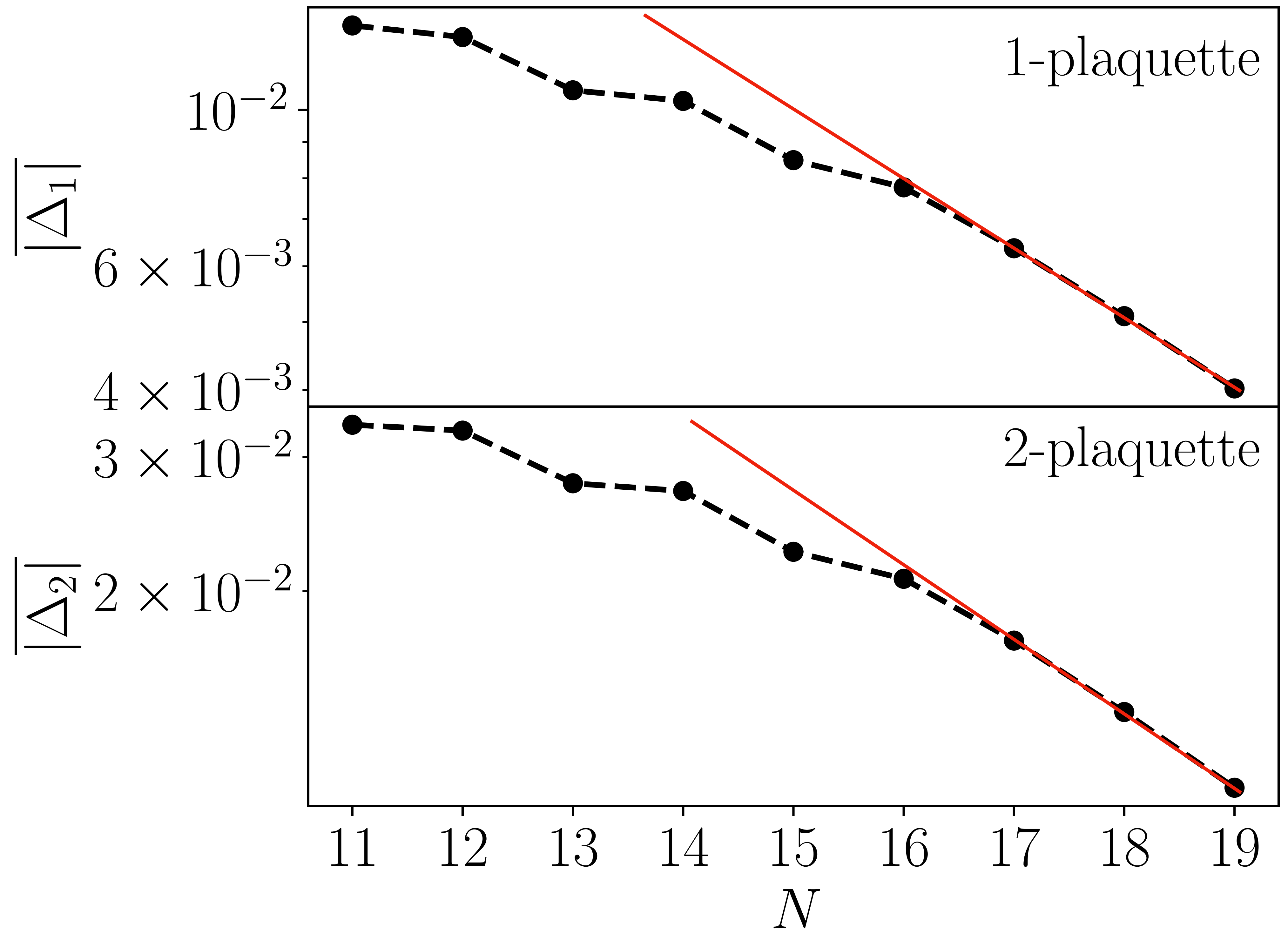


Plaquette Chain with $j_{\max} = 1/2$: Diagonal Part

Consider 1-plaquette and 2-plaquette operators with $ag^2 = 1.2$

Proxy for microcanonical ensemble:

$$\Delta_i(n) = \langle n|O_i|n\rangle - \frac{1}{21}\sum_{m=n-10}^{n+10}\langle m|O_i|m\rangle$$

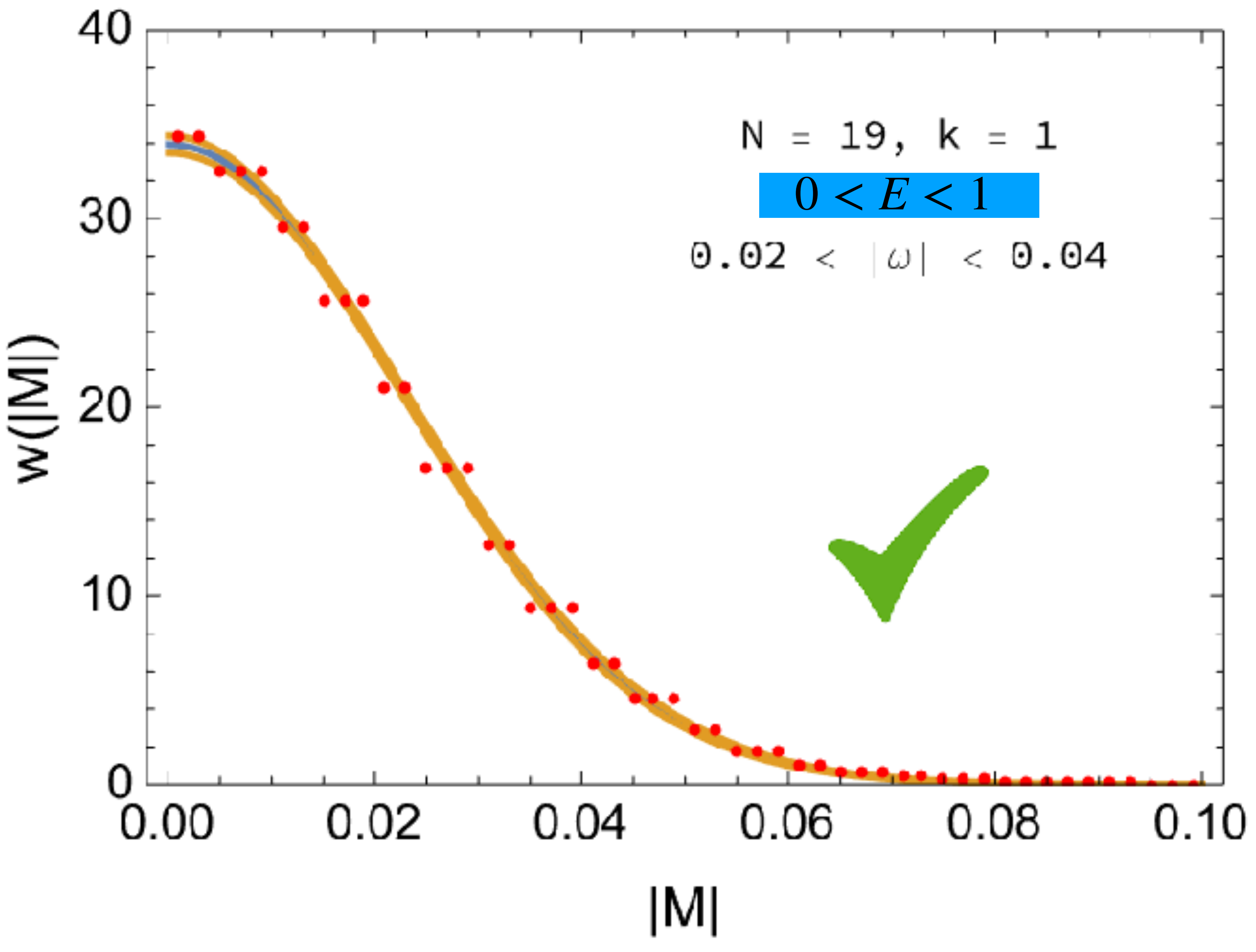


Exponential decay with system size for $N \geq 16$ ✔

[X. Yao, PRD 108 (2023) L031504]

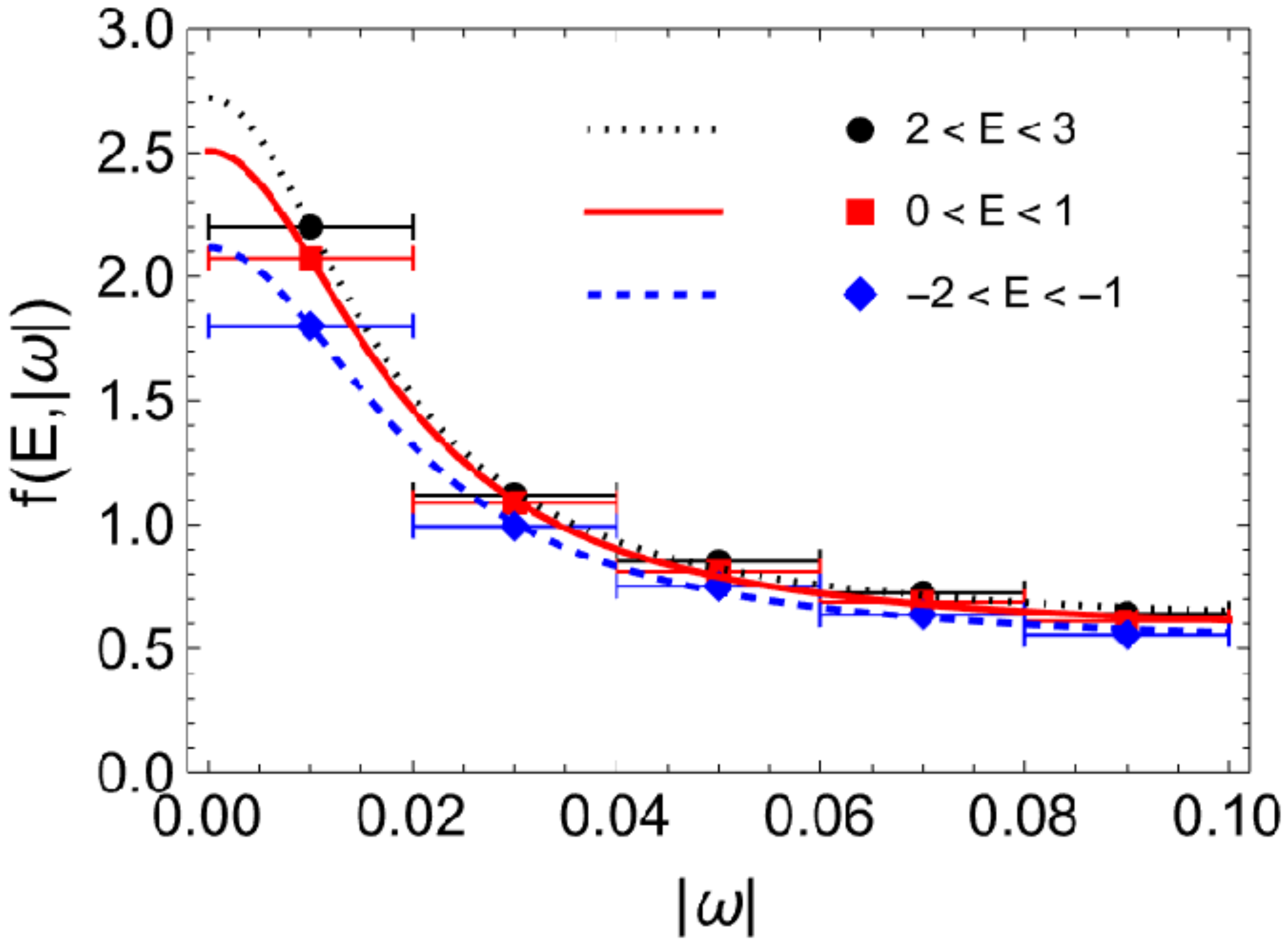
Chain with $j_{\max} = 1/2$: Off-Diagonal Part

Consider off-diagonal part $M_{mn} \equiv \langle m | H_{\rm el} | n \rangle$



Well described by Gaussian

$$\sigma^2 = {\rm Tr}[\boldsymbol{M}^2] = \frac{f_{\rm el}(E,\omega)^2}{\rho(E)}$$

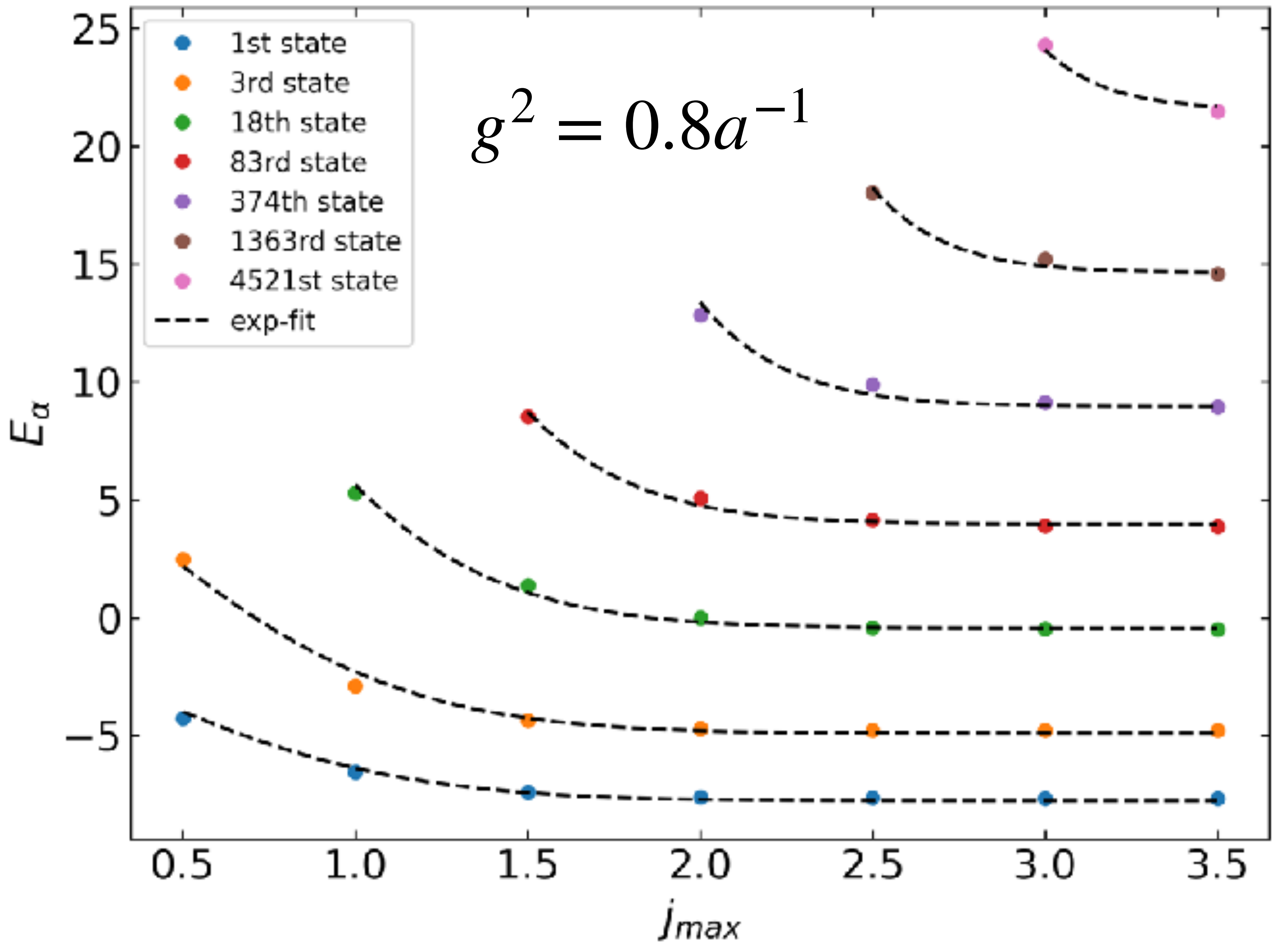


Spectral function at small $|\omega|$ is well described by a diffusive transport peak

$$f(E,\omega) = \frac{a}{\omega^2 + b^2} + c$$

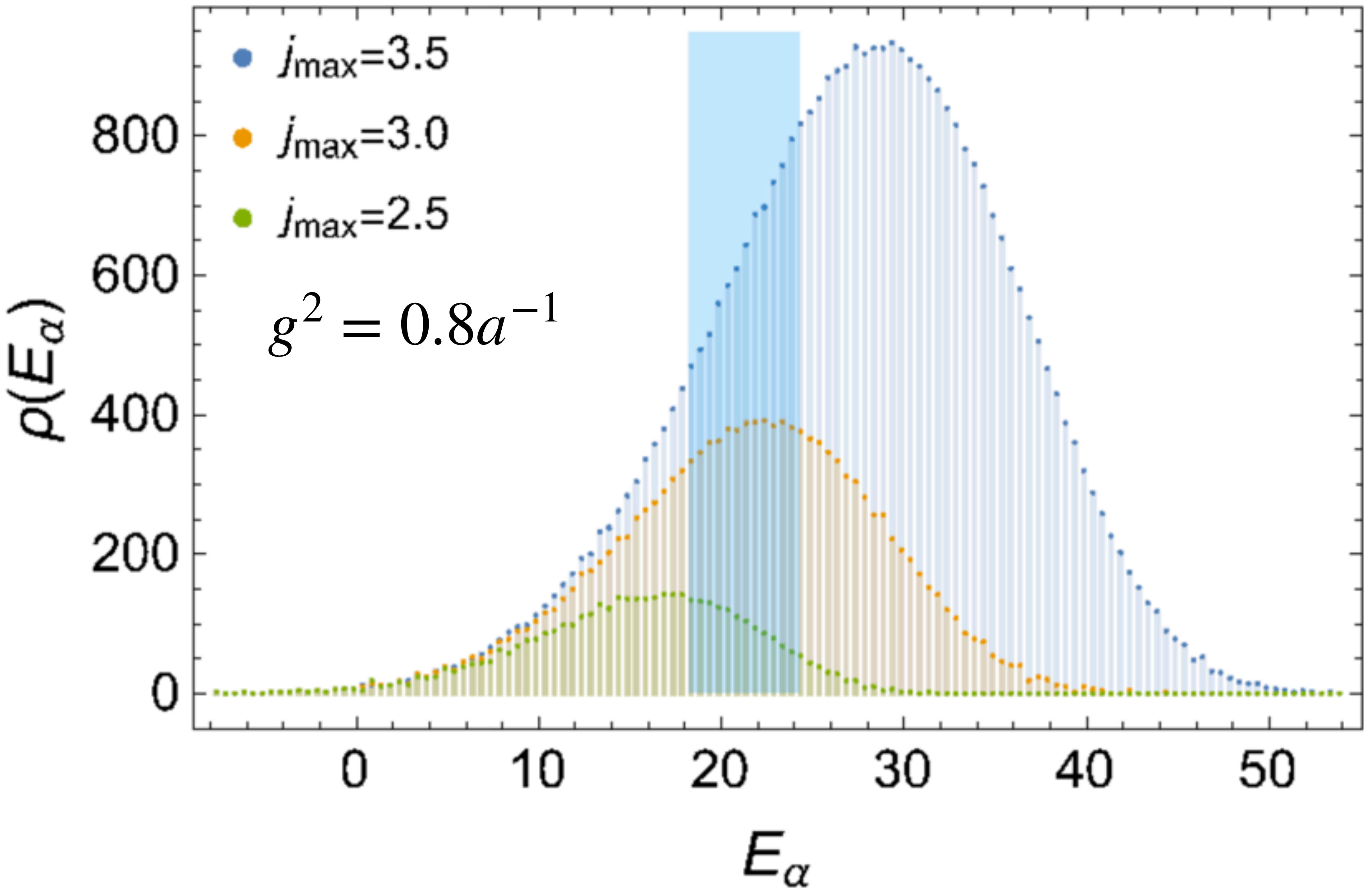
$j_{\max}$ Cutoff Dependence and Convergence

Energy eigenvalues on N = 3 chain vs. $j_{\max}$



Take $j_{\max} = 3.5$, only use states within 5% error from asymptotic eigenenergy values

Energy level spectrum for different $j_{\max}$

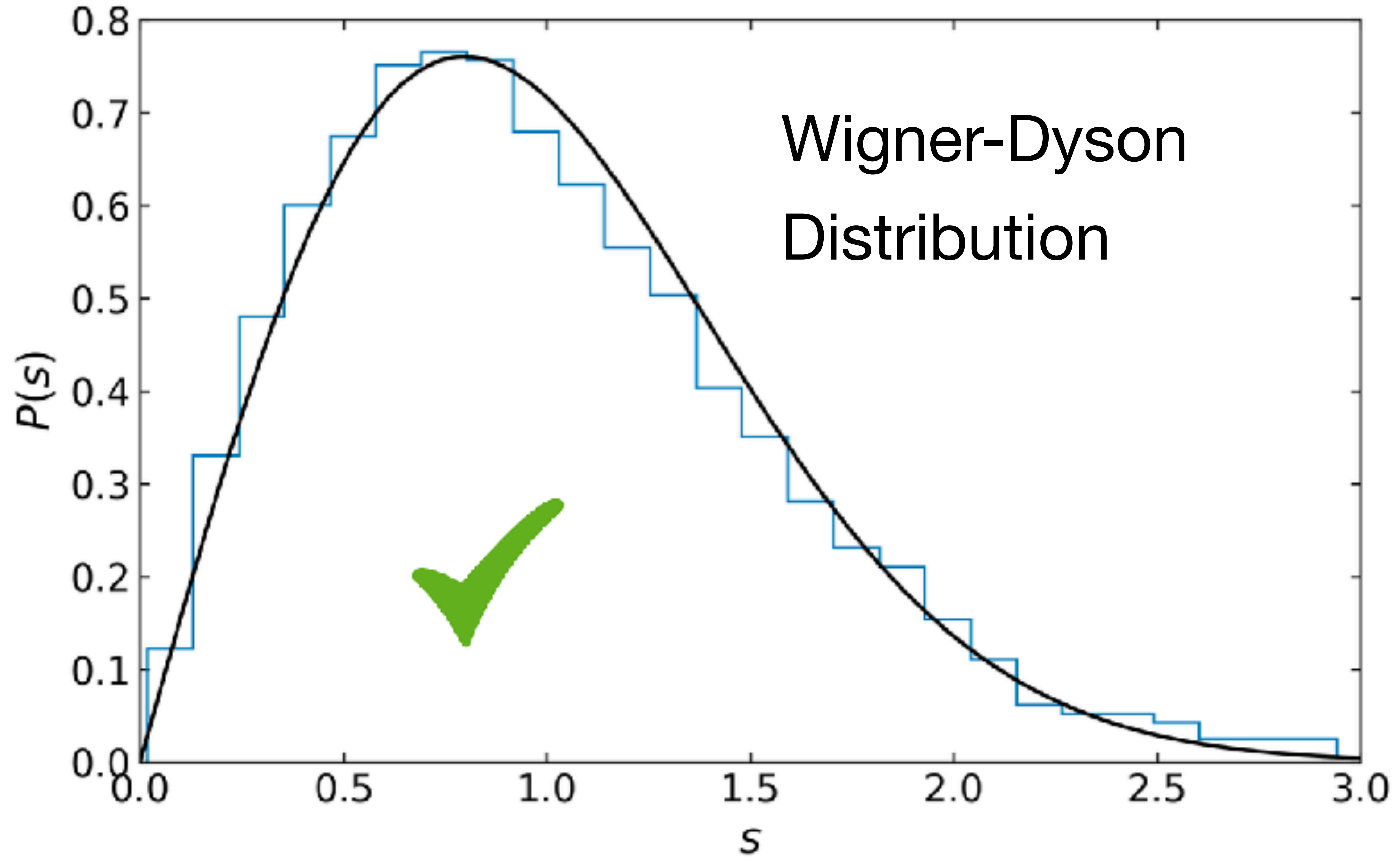


Select converged region 18 < E < 24

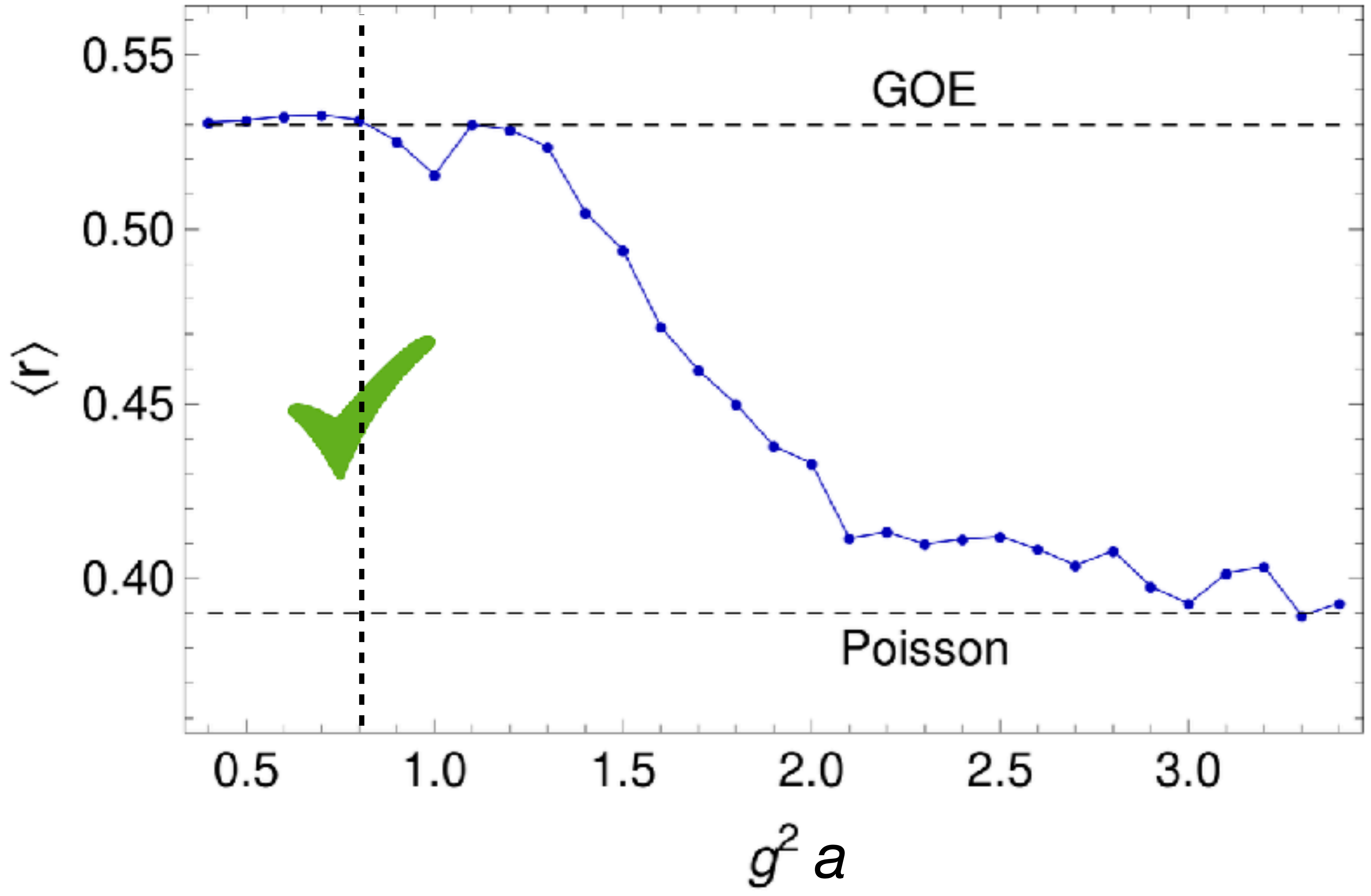
Ebner, BM, Schäfer, Seidl, Yao, PRD 109 (2024) 014504

N = 3 Chain with $j_{\max} = 7/2$: Spectrum

Nearest-neighbor level statistics exhibits GOE characteristics at $g^2a = 0.8$

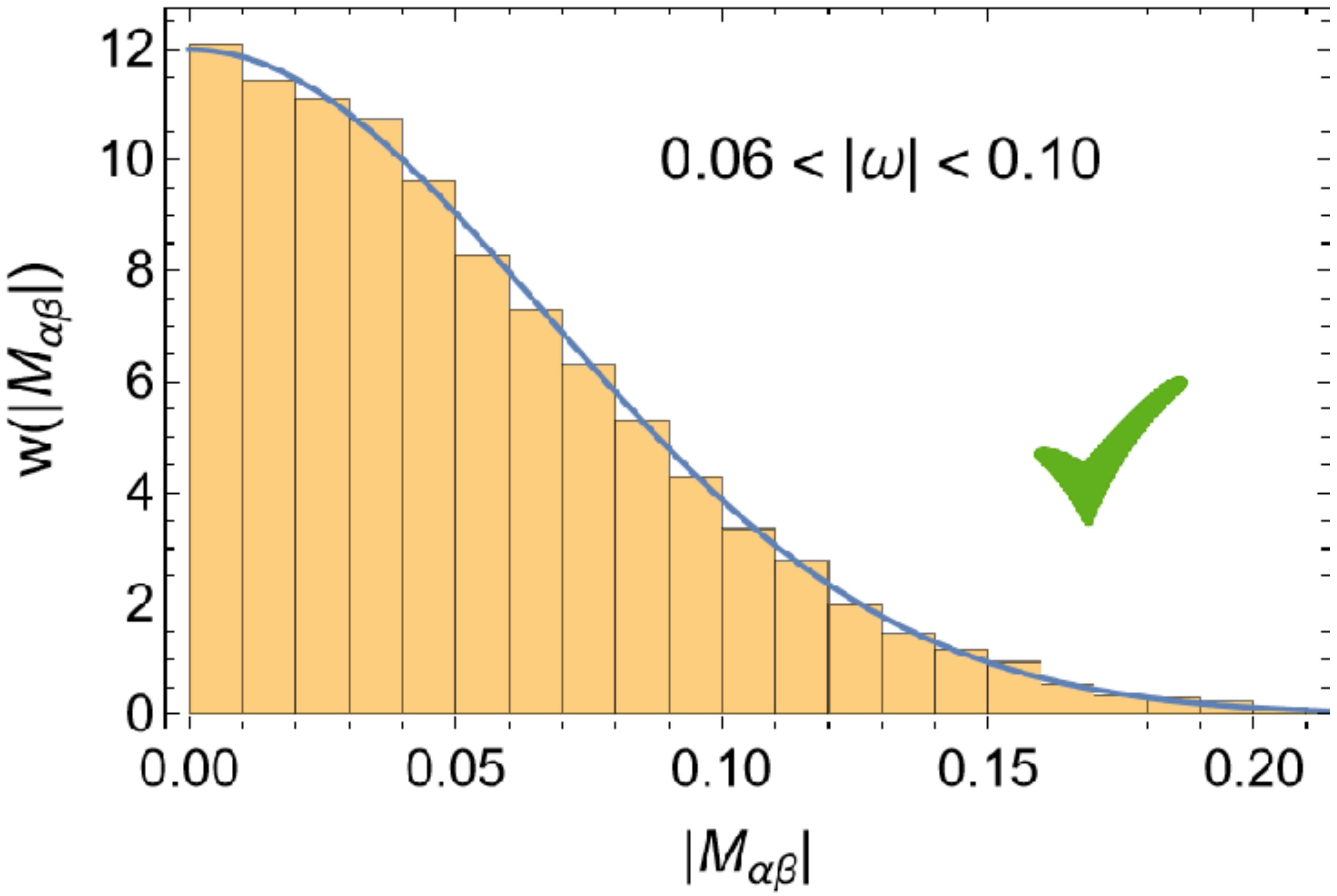


Mean restricted gap ratio shows GOE behavior at weak coupling and Poisson at strong coupling

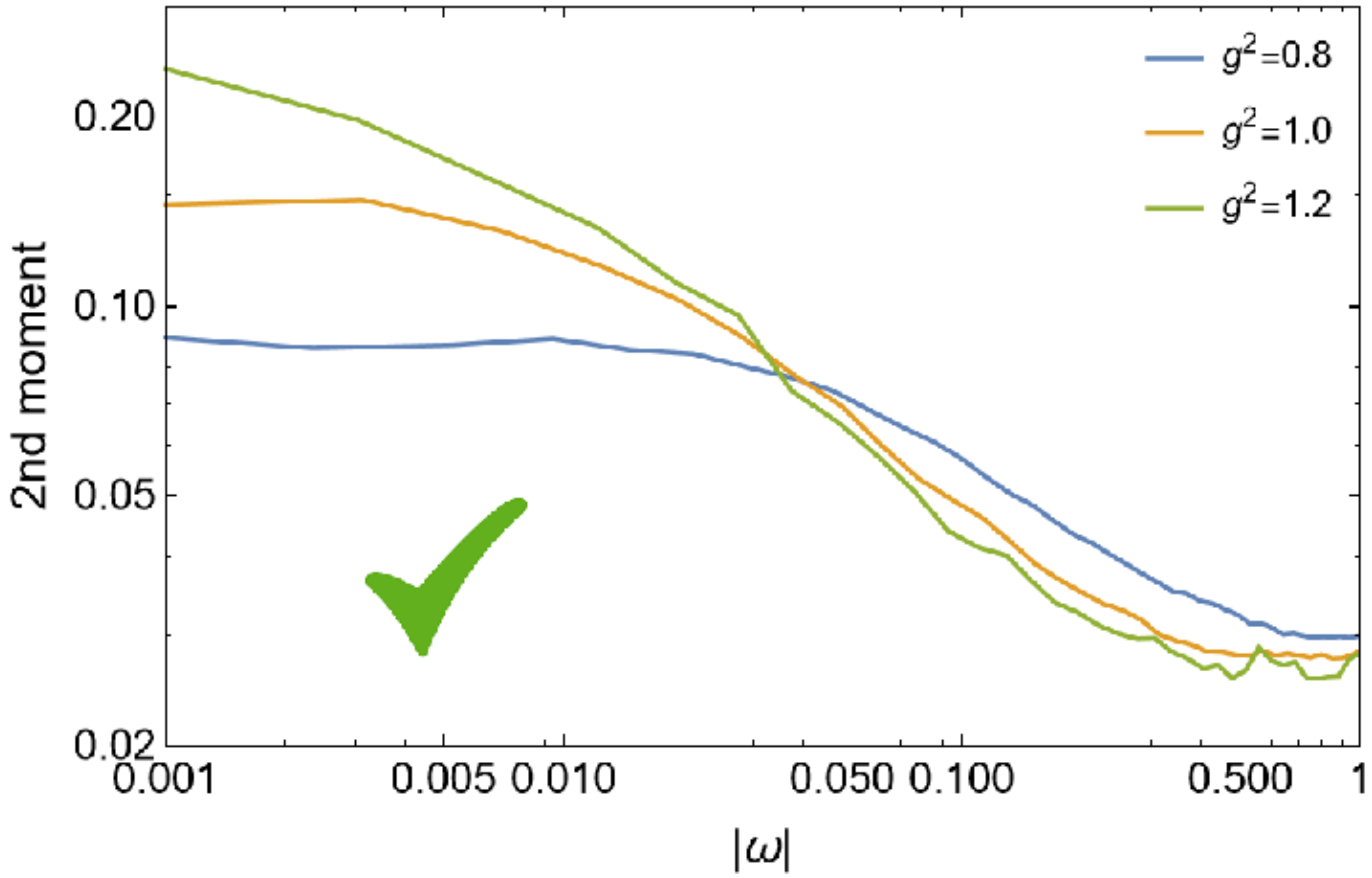


N = 3 Chain with $j_{\max} = 7/2$: Off-Diagonal Part

Off-diagonal elements of $H_{\rm el}$ are Gaussian distributed



Spectral function at small |ω| exhibits a diffusive transport peak with plateau



Plateau disappears when system is non-chaotic

Test GOE Behavior: $N = 3, j_{\max} = 7/2$

Construct band matrix by dropping deciphered matrix elements at time *T*

$$O^T_{mn} = \begin{cases} \langle m|O|n\rangle, & |E_m - E_n| \leq \frac{2\pi}{T} \\ 0, & |E_m - E_n| > \frac{2\pi}{T} \end{cases}$$

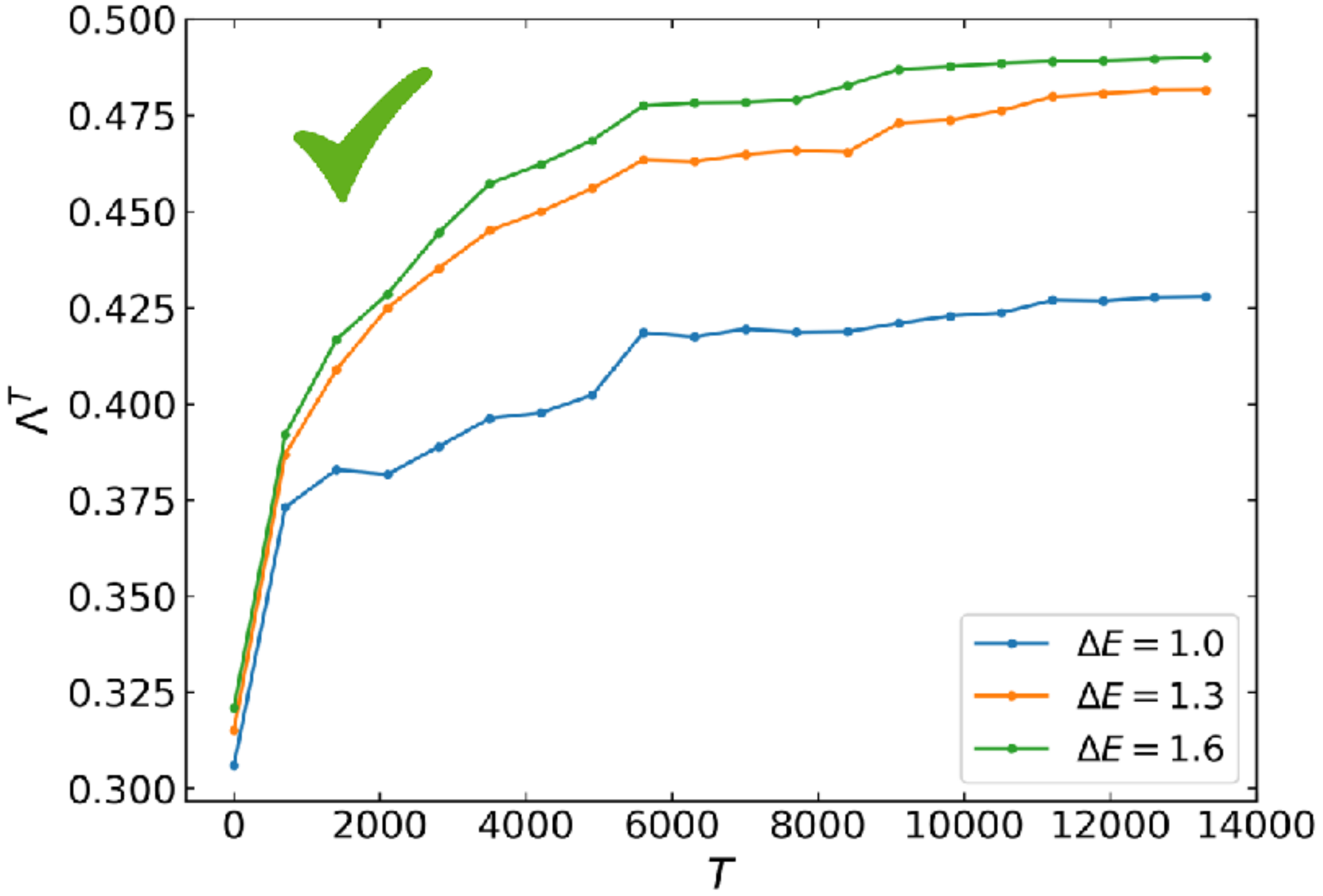
GOE measure Λ^T

$$O^T_c = O^T - \mathrm{Tr}[O^T]/d$$

$$\Lambda^T = \frac{\left(\mathrm{Tr}[(O^T_c)^2]\right)^2}{d\ \left(\mathrm{Tr}[(O^T_c)^4]\right)}$$

For Gaussian Orthogonal Ensemble (GOE):

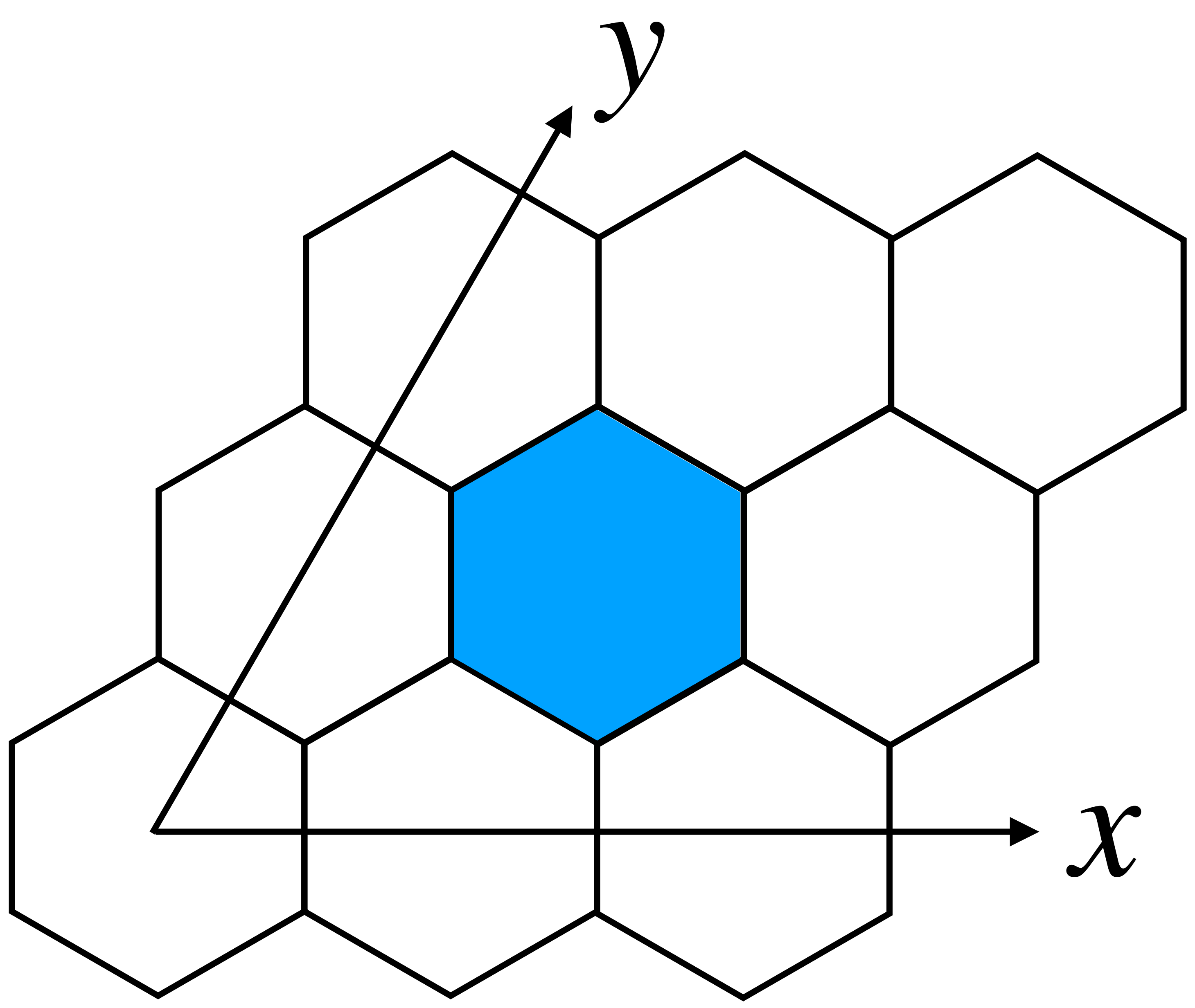
$$\Lambda^T = 0.5$$



(2+1)-D SU(2) on Honeycomb Lattice

On square lattice each vertex has four links and singlet is not unique

Solution: use honeycomb lattice



$$H_{\rm el} = \frac{g^2}{2}\frac{3\sqrt{3}}{2}\sum_{\boldsymbol{n}}\sum_{i=1}^{3} E_i^2(\boldsymbol{n})$$

$$H_{\rm mag} = -\frac{4\sqrt{3}}{9a^2g^2}\sum_{\boldsymbol{n}} \text{⬢}(\boldsymbol{n})$$

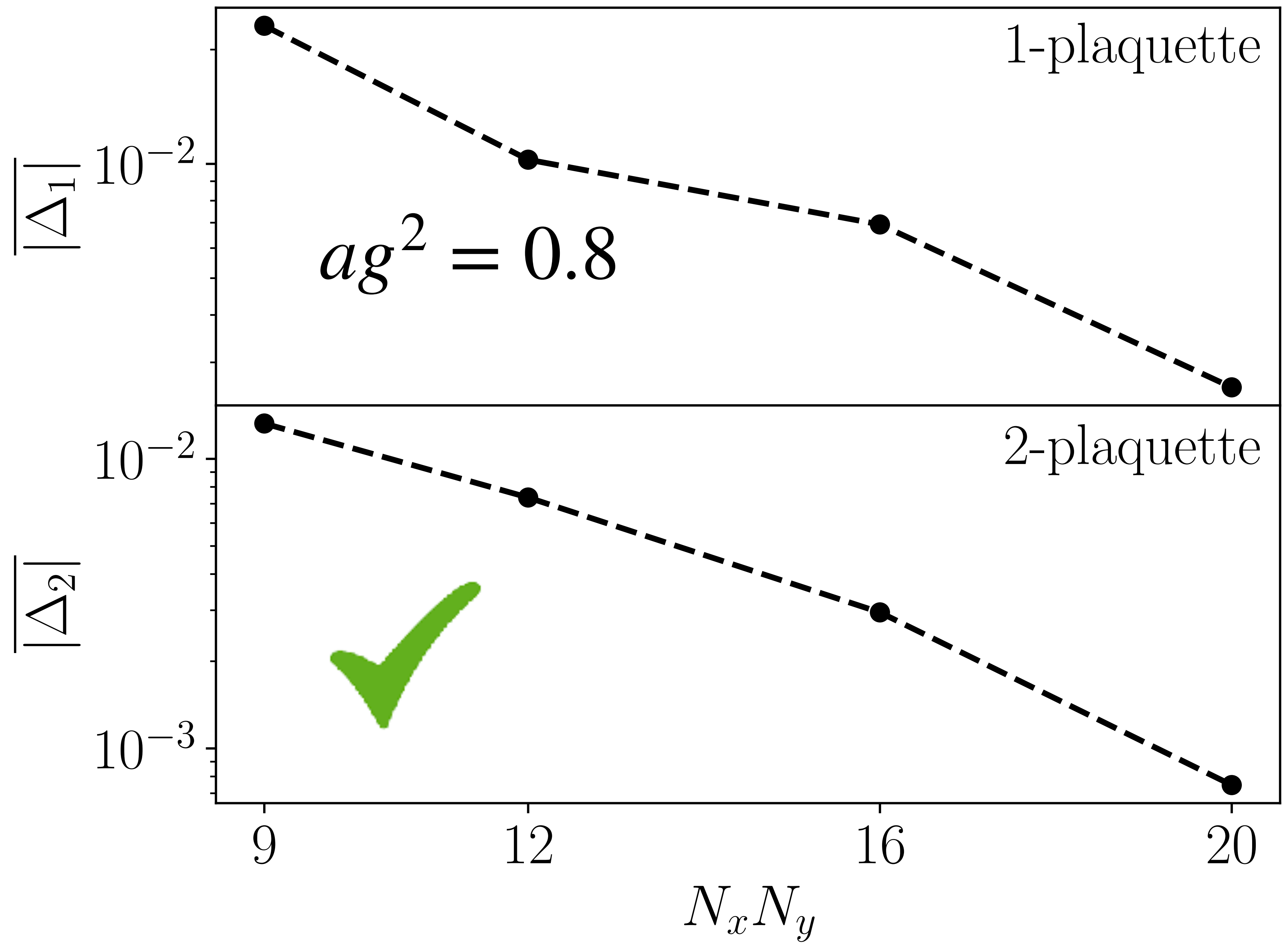
$\langle J_i | \text{⬢} | j_i \rangle$ between physical states

$= \text{product of six } 6j \text{ symbols}$

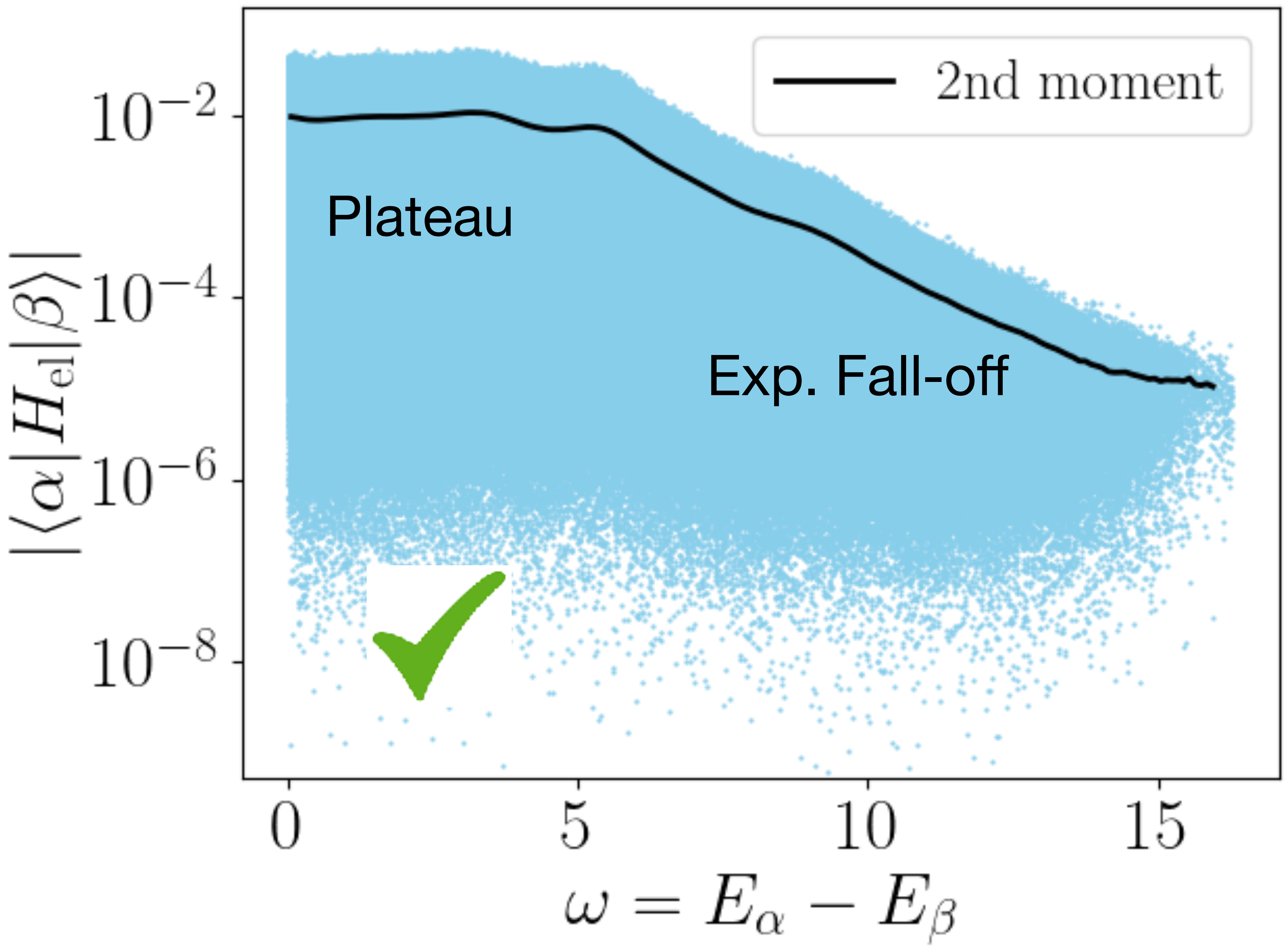
BM, X. Yao, PRD 108 (2023) 094505

ETH Tests for Honeycomb Lattice with $j_{\max} = 1/2$

Diagonal matrix element test for local operators (1 and 2 plaquettes)



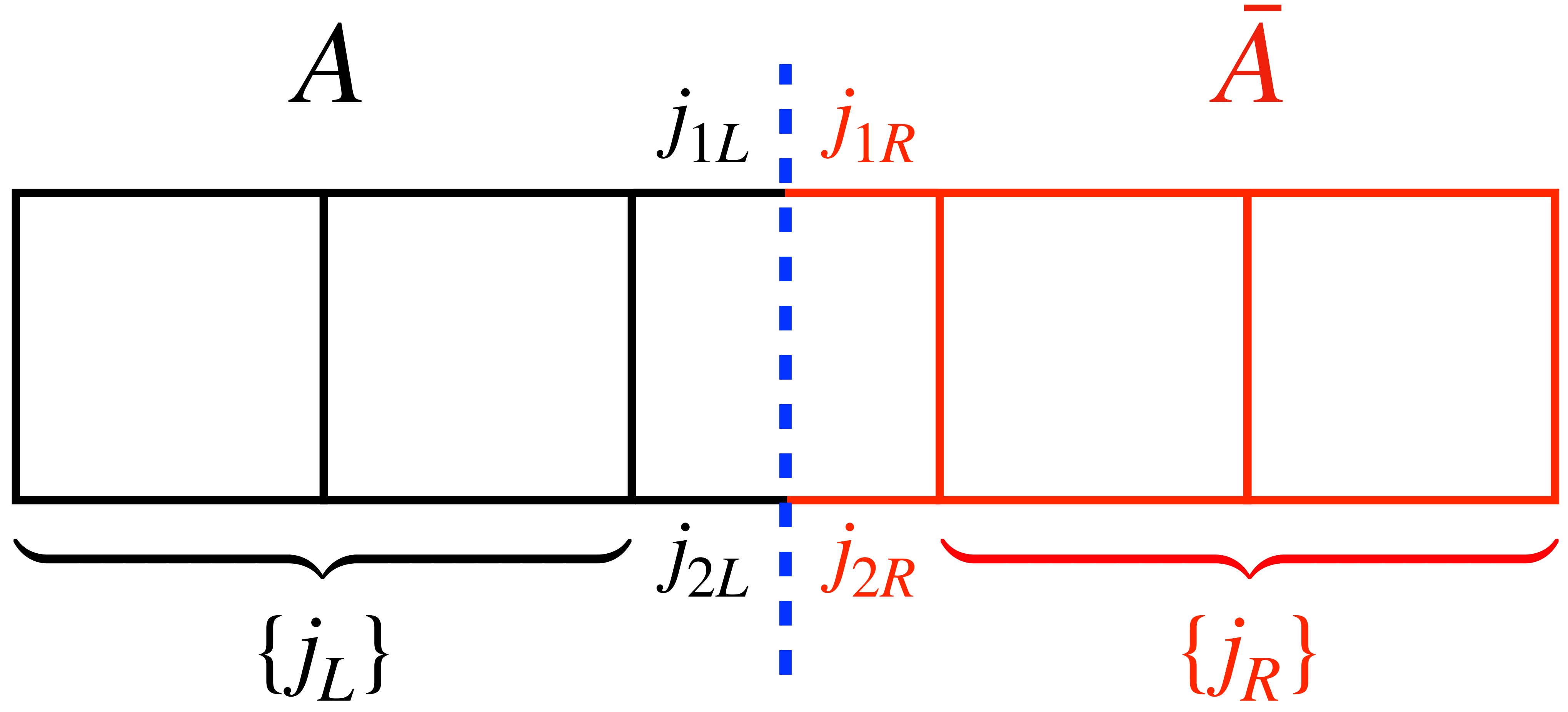
Off-diagonal matrix elements of $H_{\rm el}$



Entanglement entropy

Entanglement entropy of subsystems first grows with size and then declines when the subsystem exceeds half the size of the full system [D.N. Page, PRL 71 (1993) 3743].

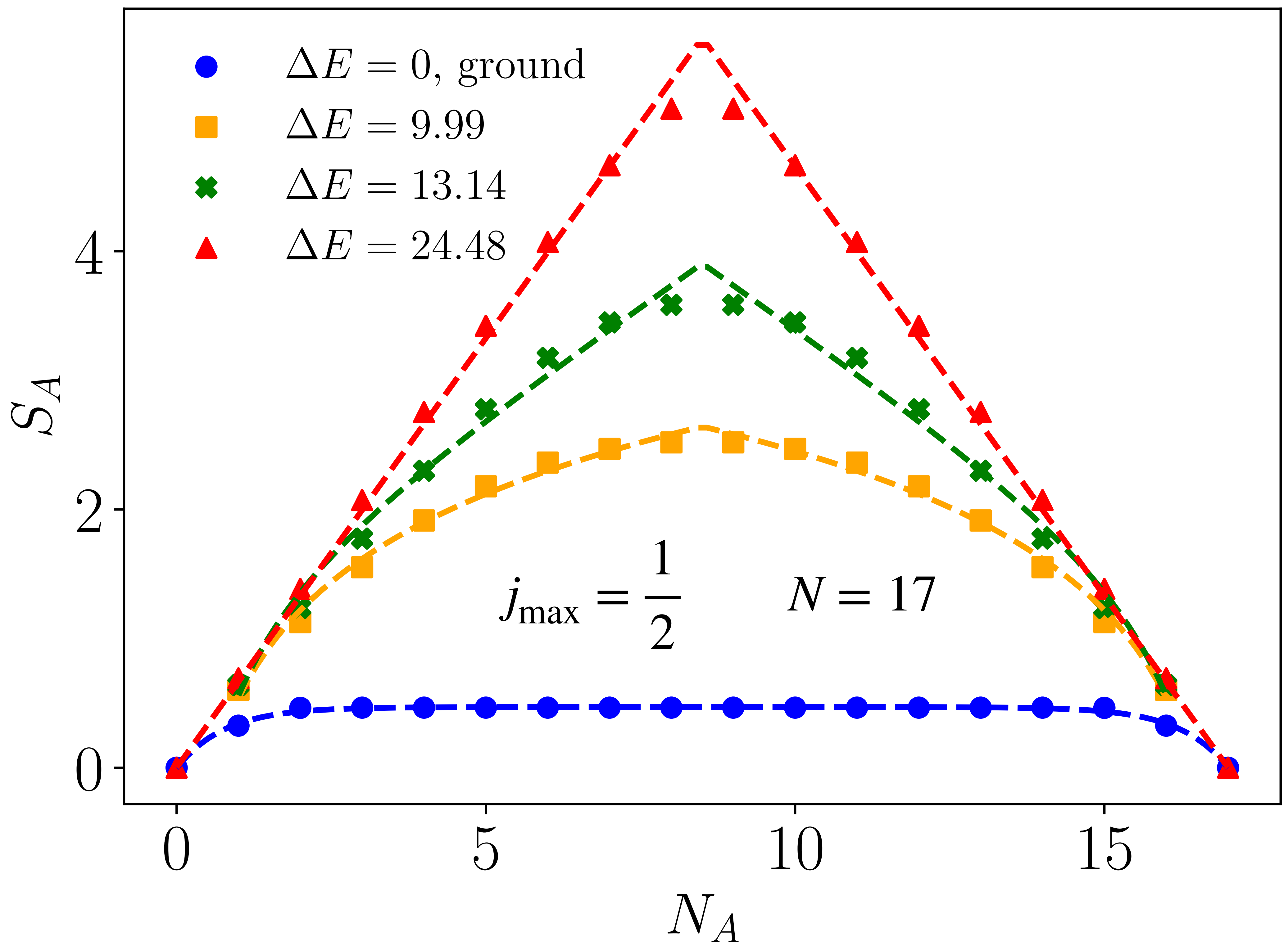
$$S_A = \mathrm{Tr}_A(\rho_A \ln \rho_A) \qquad \text{with} \qquad \rho_A = \mathrm{Tr}_{\bar{A}} |\psi\rangle \langle\psi|$$

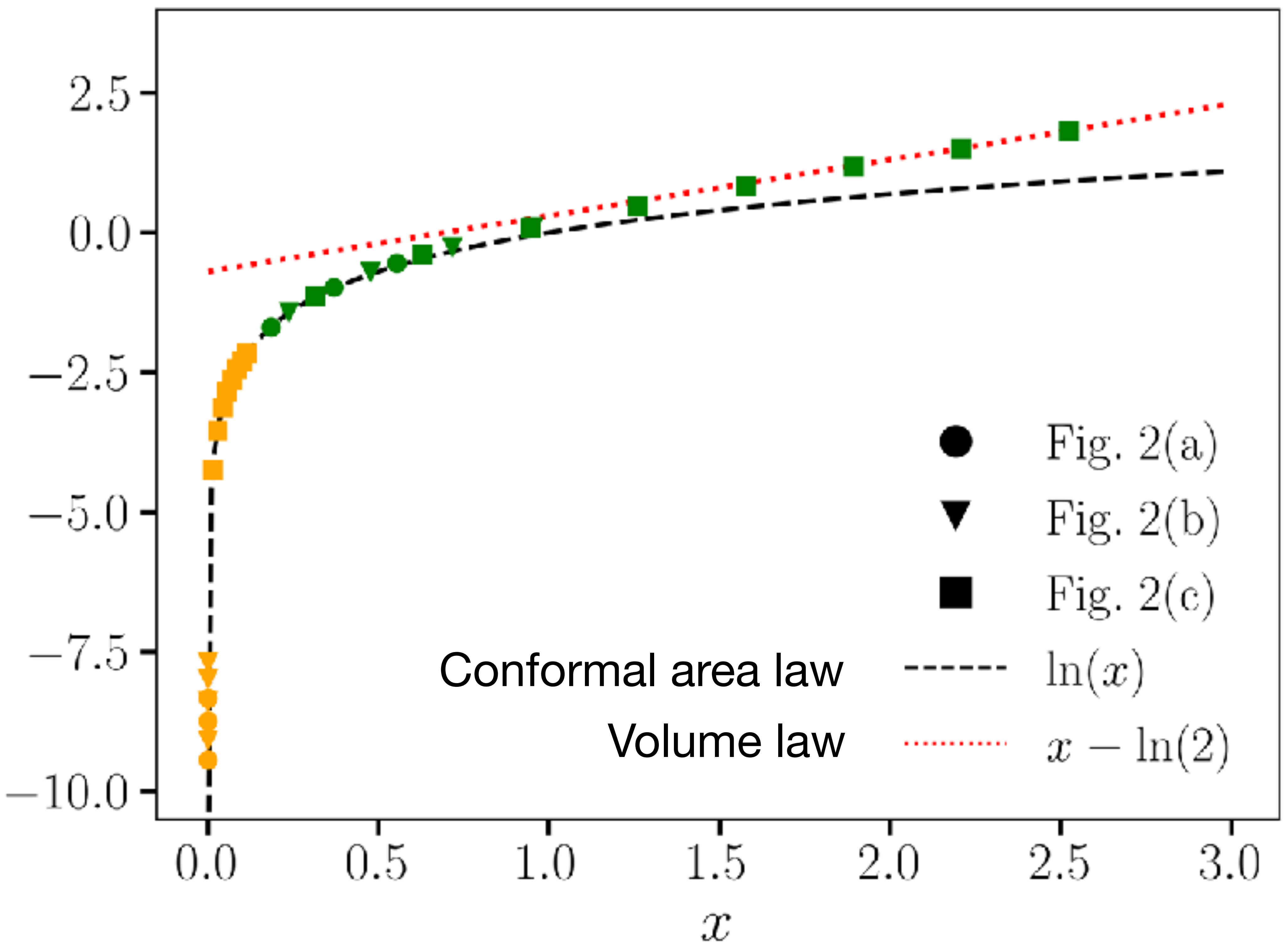


Ebner, BM, Schäfer, Seidl, Yao, 2401.15184

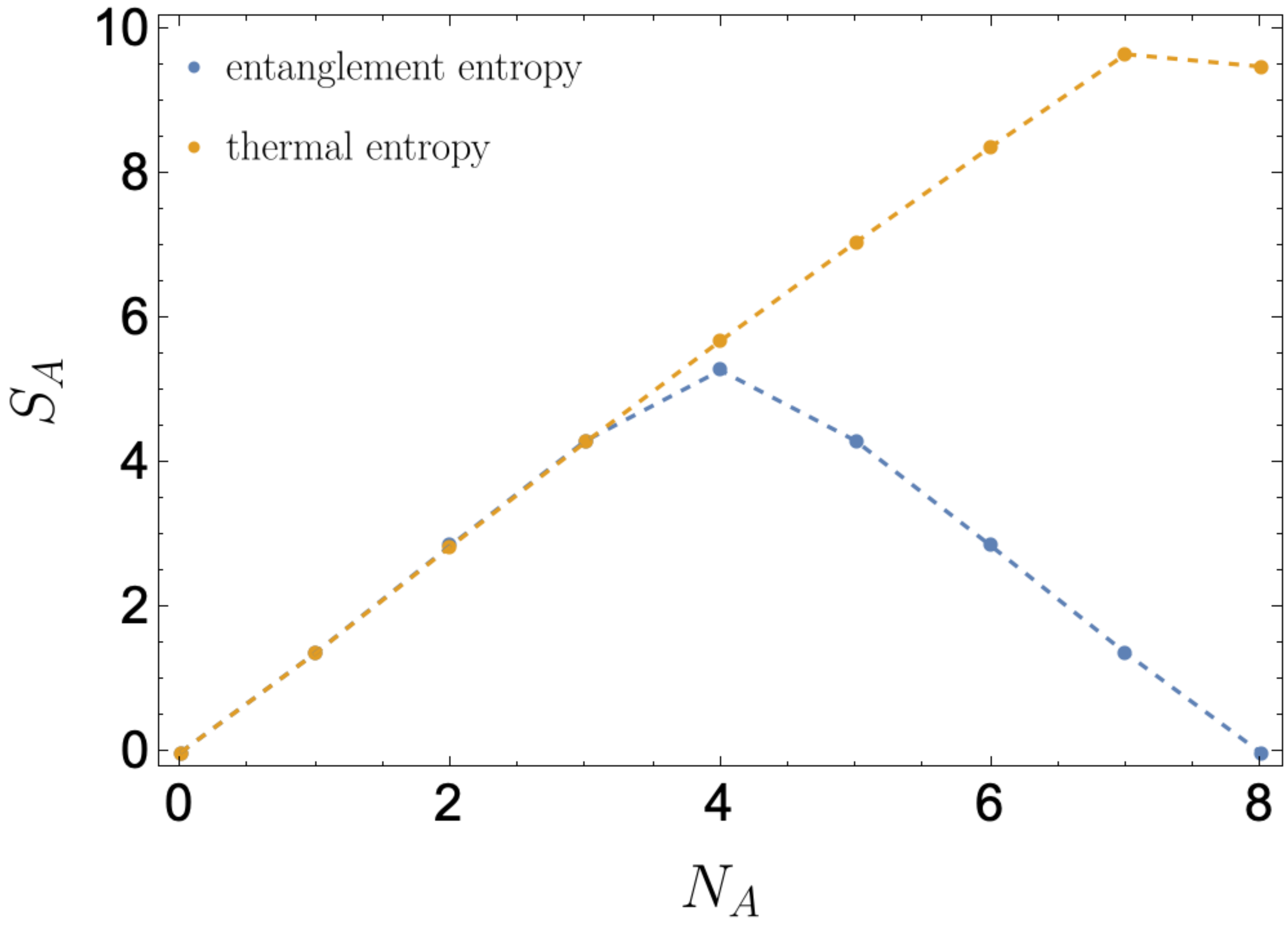
Page curve

$$S(N_A) = c_0 + \frac{c}{3}\ln(c_1) + \frac{c}{3}\left[\ln[\sinh(c_1^{-1}N_A)]\theta\left(\frac{N}{2} - N_A\right) + \ln\{\sinh[c_1^{-1}(N - N_A)]\}\theta\left(N_A - \frac{N}{2}\right)\right]$$





S_{EE} versus S_{th}



$$S_{\mathrm{EE}}(A) \approx \min\left(S_{\mathrm{th}}(A), S_{\mathrm{th}}(\bar{A})\right)$$

using

$$T = \left[\frac{dS_{\mathrm{mc}}(E)}{dE}\right]^{-1}$$

Summary

We obtained extensive numerical evidence for ETH in (2+1)-dim. SU(2) lattice gauge theory.
Three cases studied by direct diagonalization of the KS Hamiltonian:

(1) long chain with $j_{\max} = 1/2$

(2) short chain with $j_{\max} = 7/2$ and fully converged spectrum

(3) 2D honeycomb with $j_{\max} = 1/2$

We found:

- Wigner-Dyson level spacing statistics ✓
- Clustering of diagonal matrix elements around micro canonical average ✓
- Random matrix behavior of off-diagonal matrix elements ✓
- Transport peak in spectral function at small $|\omega|$ ✓
- Page curve for energy eigenstates ✓

Future Plans

There are many possible directions for future research, e.g.:

(1) (2+1)-D honeycomb with higher $j_{\max}$

(2) (3+1)-D SU(2) on triamond lattice

(3) SU(3) gauge theory

(4) Include fermions

(5) Implementation on a quantum computer

Extent of further investigations will depend on availability of computing resources.
More efficient algorithms than full diagonalization of H_{KS} must also be explored.