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Review of recent experimental results on femtoscscopy with Levy sources

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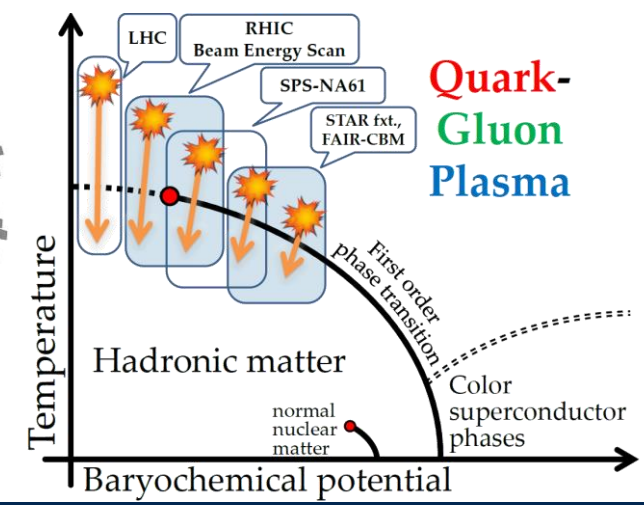
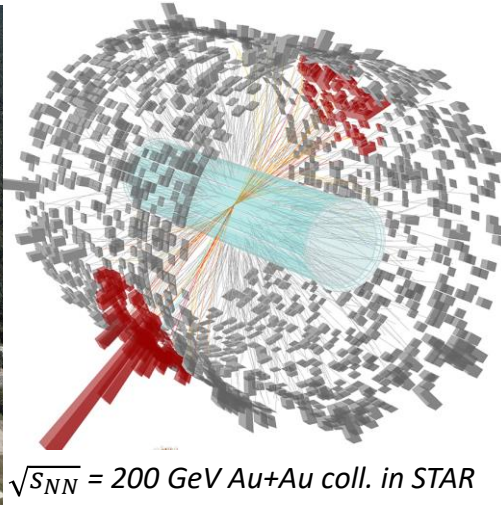
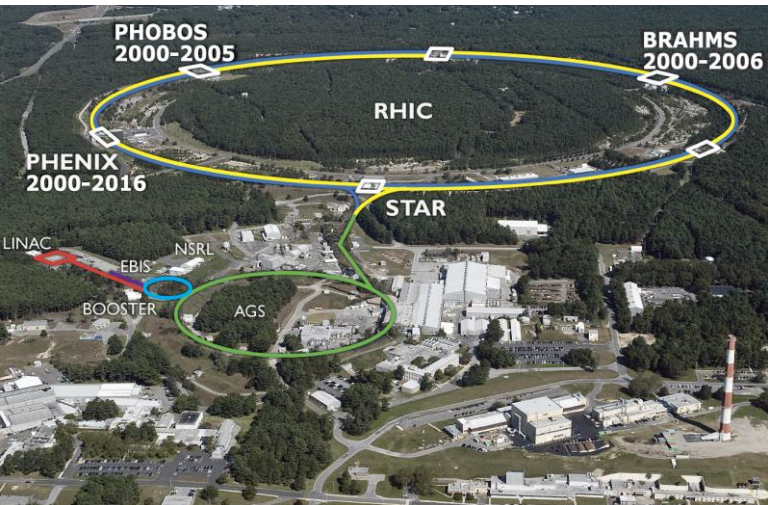
ELFT Particle Physics Summer School 2024

Part I.
Introduction, Motivation

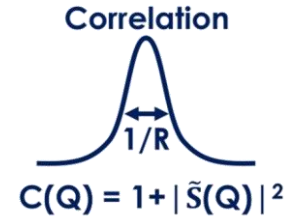


Exploring the properties of quark matter at RHIC

- **Relativistic Heavy Ion Collider (RHIC)** at Brookhaven National Laboratory (BNL)
- Strongly interacting **Quark-Gluon-Plasma (QGP)** discovered in the early 2000s
- Since then: ongoing extensive research program called **Beam Energy Scan (BES)**
- Many interesting measurements at RHIC, our focus: **femtoscopic correlations**



Basic definitions of femtoscopic correlation functions



s : Single particle phase-space density (emission func.)
 x : particle coordinate
 p : particle momentum

• Single particle momentum distribution: $N_1(p) = \int d^4x s(x, p)$

• Pair momentum distribution: $N_2(p_a, p_b) = \int d^4x_a d^4x_b s(x_a, p_a) s(x_b, p_b) |\psi_{p_a, p_b}(x_a, x_b)|^2$

• Correlation function:

$$C(p_a, p_b) = \frac{N_2(p_a, p_b)}{N_1(p_a)N_1(p_b)}$$

pair separation: $r = x_a - x_b$
 pair avg. mom.: $K = (p_a + p_b)/2$

• Pair source/spatial correlation:

$$D_K(\mathbf{r}) = \int d^4\rho s\left(\rho + \frac{\mathbf{r}}{2}, K\right) s\left(\rho - \frac{\mathbf{r}}{2}, K\right)$$

pair center-of-mass: $\rho = (x_a + x_b)/2$

relative pair momentum
 average pair momentum*

Pair wave function, containing FSI!

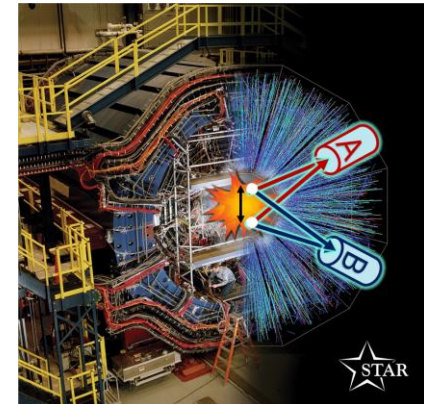
$$C(\mathbf{Q}, \mathbf{K}) = \int d^3r D_K(\mathbf{r}) |\psi_{\mathbf{Q}}(\mathbf{r})|^2$$

*Instead of K , m_T is often used:

$$m_T = \sqrt{k_T^2 + m_\pi^2}, k_T = \sqrt{K_x^2 + K_y^2}$$

• Experiments: measuring $C(\mathbf{Q}) \rightarrow$ information about $D(\mathbf{r})$ and FSI

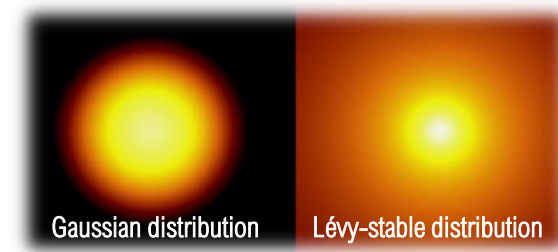
• Experimental (and phenomenological) indications:
 power-law tail for pions, **non-Gaussianity?**



Ann.Rev.Nucl.Part.Sci.55(2005) 357-402; Phys.Lett.B398 (1997), pp. 252-258.; Phys.Rev.Lett.98 (2007), p. 132301.

$$s(\mathbf{x}, \mathbf{p}) = \mathcal{L}(\alpha, R; \mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} e^{i\mathbf{q}\mathbf{x}} e^{-\frac{1}{2} |\mathbf{q}^T R^2 \mathbf{q}|^{\alpha/2}}$$

p dependence through α, R spherical symmetry: R^2 diagonal



Gaussian distribution

Lévy-stable distribution

What is the shape of the source? Gaussian & Lévy distributions in heavy-ion physics

- **Symmetric Lévy-stable distribution** *Eur.Phys.J.C 36 (2004) 67*

- From generalized central limit theorem (GCLT),
power-law tail (if $\alpha < 2$) $\sim r^{-(1+\alpha)}$

- $\alpha = 2$ Gaussian, $\alpha = 1$ Cauchy

$$s(\mathbf{x}, \mathbf{p}) = \mathcal{L}(\alpha, R; \mathbf{x})$$

↓

- Retains the same α under convolution $D_{\mathbf{K}}(\mathbf{r}) = \mathcal{L}(\alpha, 2^{1/\alpha} R; \mathbf{r})$

- **Experimental indications – Lévy source for pion pairs?**

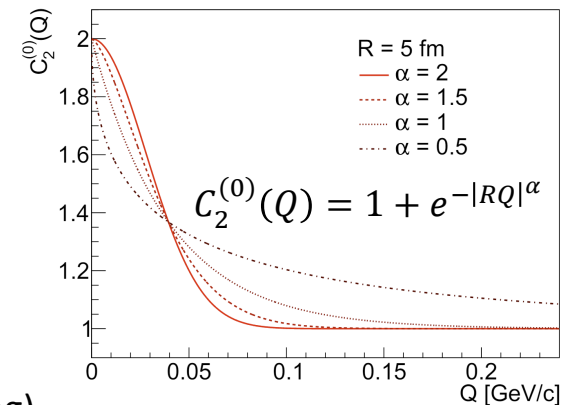
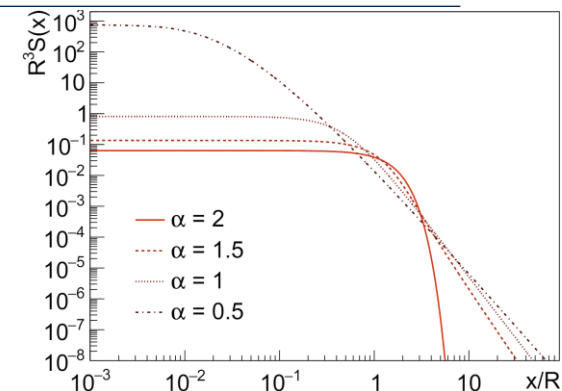
- **RHIC (PHENIX, STAR), LHC (CMS), SPS (NA61/SHINE)**

Phys.Rev.C 97 (2018) no.6, 064911; Universe 10 (2024) 3, 102

Phys.Rev.C 109 (2024) 2, 024914; Eur.Phys.J.C 83 (2024) 10, 919

- **Possible reasons for the $\alpha < 2$ Lévy exponent at RHIC:**

- Critical behavior, resonance decays, anomalous diffusion (rescattering)



New data analyses and phenomenological investigations are needed to gain a better understanding!

$$s(\mathbf{x}, \mathbf{p}) = \mathcal{L}(\alpha, R; \mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3q e^{iqx} e^{-\frac{1}{2}|\mathbf{q}^T R^2 \mathbf{q}| \alpha/2}$$

p dependence through α, R spherical symmetry: R^2 diagonal

Lévy source at RHIC energies not because of event averaging and 3D→1D conversion!

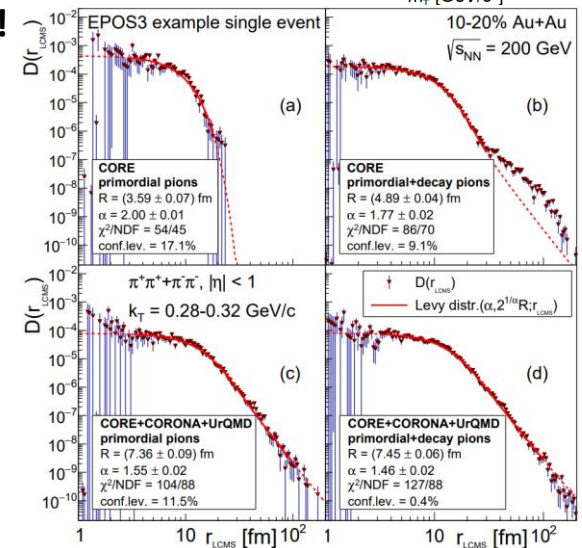
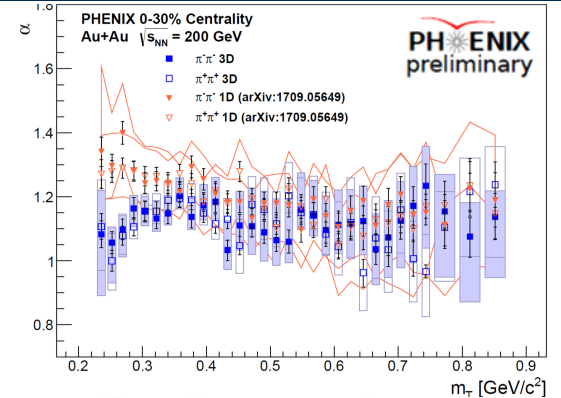
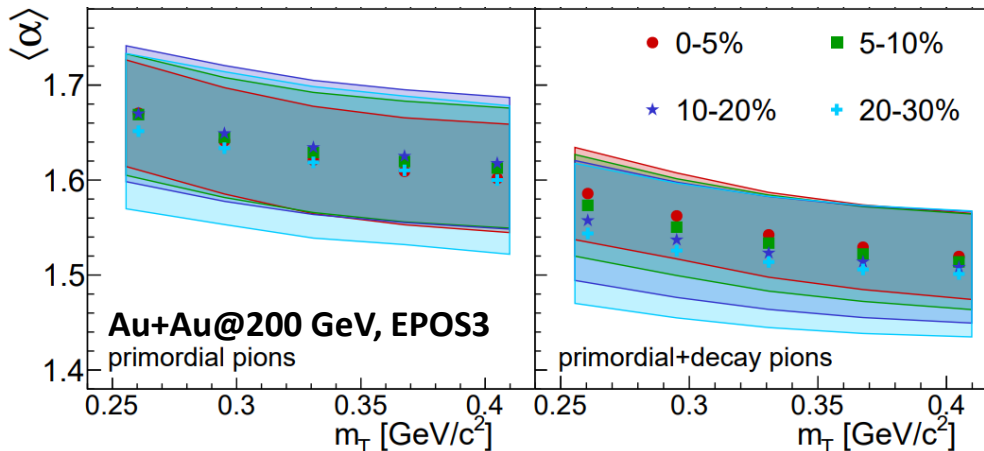
- Not spherically sym. source: 3D vs. 1D α compatible!
 $\alpha < 2$ in 1D analyses not because of angle averaging!

Kurgyis, Acta Phys. Pol. B Proc. Suppl. vol. 12 (2), 477 (2019)

$$R^2 = \begin{pmatrix} R_{out}^2 & 0 & 0 \\ 0 & R_{side}^2 & 0 \\ 0 & 0 & R_{long}^2 \end{pmatrix} \quad \mathbf{q} = \begin{pmatrix} q_{out} \\ q_{side} \\ q_{long} \end{pmatrix}$$

- Event-by-event pion pair-source analysis at 200 GeV:
 Decays and hadronic rescattering (UrQMD) play an important role!

Kincses, Stefaniak, Csanád, Entropy 24 (2022) 3, 308



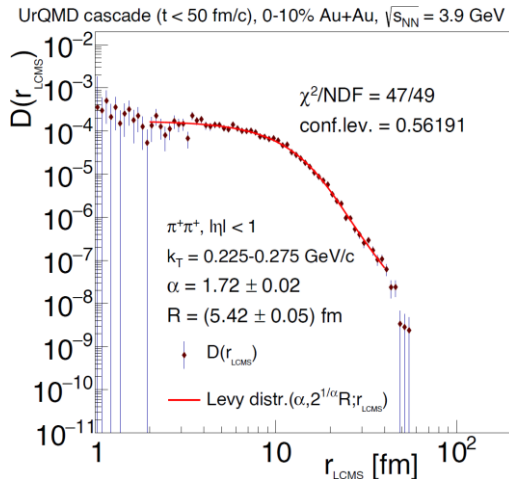
$$s(\mathbf{x}, \mathbf{p}) = \mathcal{L}(\alpha, R; \mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3q e^{i\mathbf{q}\mathbf{x}} e^{-\frac{1}{2}|\mathbf{q}^T R^2 \mathbf{q}|^{\alpha/2}}$$

p dependence through α, R *spherical symmetry: R² diagonal*

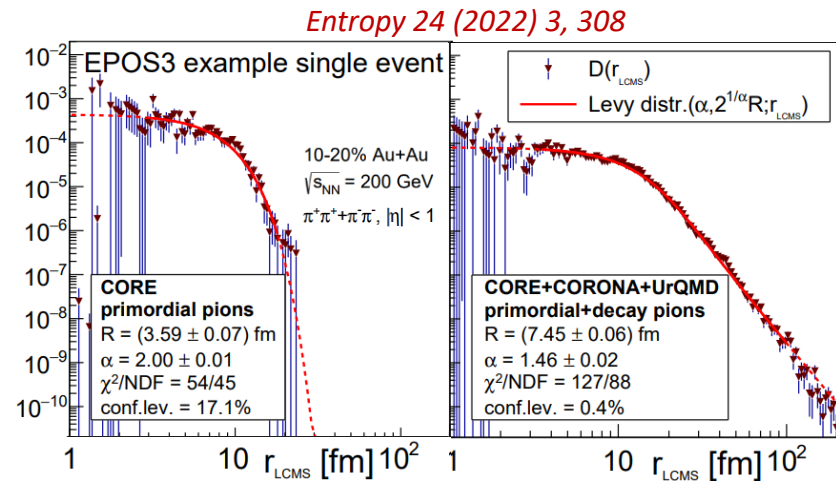
Lévy source at RHIC – rescattering and decays

- **Hadronic rescattering** → Convolution of many elementary processes → **Stable distribution**
- **Test from the phenomenology side:**
Direct source-function reconstruction in event generator models including decays and rescattering
- **Power-law source seen in event generators** from RHIC FXT up to LHC energies
- **Global trend: power-law exponent decreases with increasing collision energy**

UrQMD, 3.9 GeV, $\alpha \sim 1.7$



EPOS, 200 GeV, $\alpha \sim 1.5$



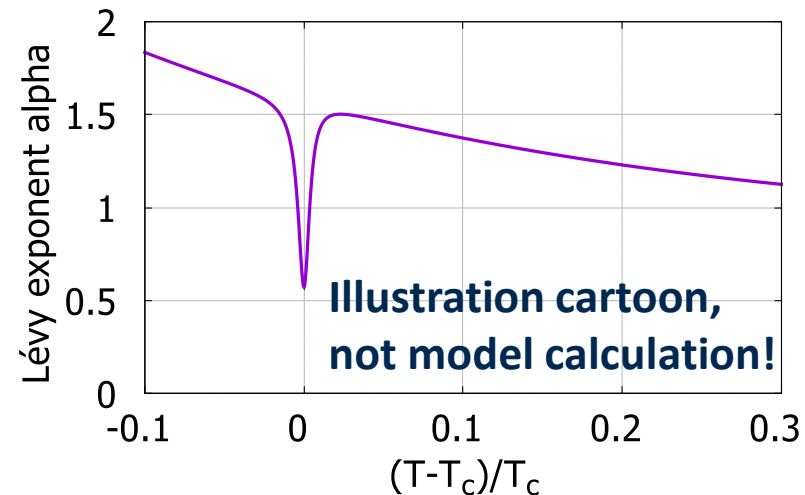
$$s(\mathbf{x}, \mathbf{p}) = \mathcal{L}(\alpha, R; \mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} e^{i\mathbf{q}\mathbf{x}} e^{-\frac{1}{2}|\mathbf{q}^T R^2 \mathbf{q}|^{\alpha/2}}$$

p dependence through α, R spherical symmetry: R^2 diagonal

Csörgő, Hegyi, Novák, Zajc, *AIP Conf.Proc.* 828 (2006) 1, 525-532

Lévy source at RHIC – critical behavior?

- Deviation from the trend predicted by rescattering → other effects in play!
- Order parameter: $c = \langle \bar{q}q \rangle$, correlation function: $\rho(r) = \langle c(r+R)c(r) \rangle - \langle c \rangle^2$
- Critical spatial correlation: $\rho(r) \sim r^{-(d-2+\eta)}$;
- 1D Lévy source: $D(r) \sim r^{-(1+\alpha)}$; $\alpha \Leftrightarrow \eta$?
- At the critical point:
 - Random field 3D Ising: $\eta = 0.50 \pm 0.05$
Rieger, Phys.Rev.B52 (1995) 6659
 - 3D Ising: $\eta = 0.03631(3)$
El-Showk et al., J.Stat.Phys.157 (4-5): 869
- Change in α_{Levy} - proximity of CEP?
- Note: finite-size effects are important, might not influence η that much



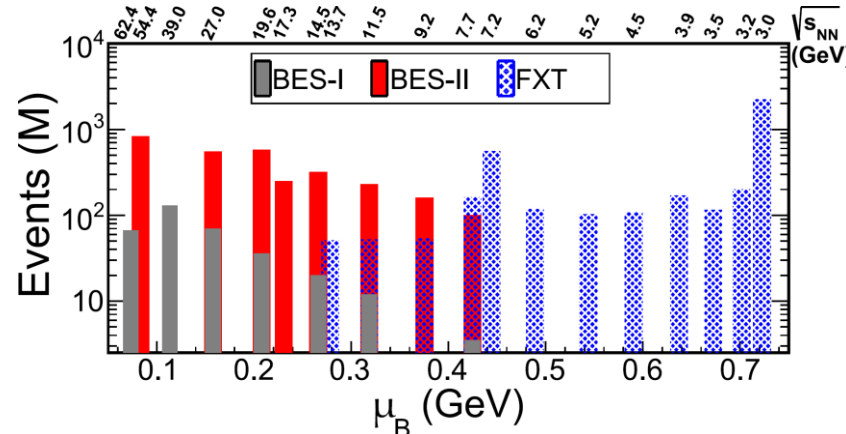
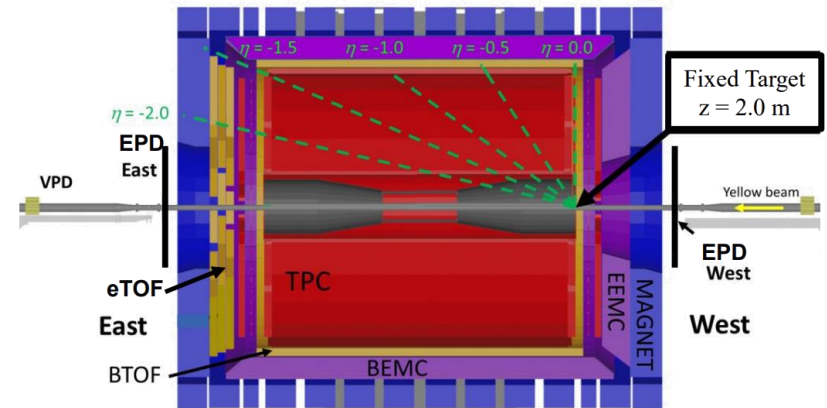
Phys. Lett. B 1996, 387, 125–131 *Phys. Rev. E* 2024, 108, 044146

Part II.
Measurement and fitting of
correlation functions
at STAR FXT energies



Lévy HBT analysis at STAR, Au+Au @ $\sqrt{s_{NN}} = 3.2 \text{ GeV}, 3.9 \text{ GeV}$

- **STAR FXT data analyzed**
3.9 GeV (7.3 AGeV), 3.2 GeV (4.59 AGeV)
- **Detectors used for the analysis:**
 - **TPC:** centrality, vertex position, tracking, dE/dx Particle Identification (PID)
 - **TOF:** time-of-flight PID
- **Event selection:**
 - Pile-up cuts using TOF vs. TPC multiplicity
 - Vertex cuts:
 $198 \text{ cm} < v_z^{TPC} < 202 \text{ cm}, |v_r^{TPC}| < 2 \text{ cm},$
 where $v_r^{TPC} = \sqrt{v_x^2 + (v_y + 2)^2}$



Measurement of two-pion correlation functions

- **Track-selection criteria**

- Combined PID using TPC $N\sigma$ (based on dE/dx) and TOF $N\sigma$ (based on time-of-flight):

$$\sqrt{N\sigma_{TOF,\pi}^2 + N\sigma_{TPC,\pi}^2} < 2.5$$
- Further single-track selection:
 - TPC number of hits > 20 , TPC number of hits/number of hits possible > 0.65 ,
 - $0.15 < p_T$ [GeV/c] < 1.0 , $-2.5 < \eta < -0.5$, Distance of Closest Approach (DCA) < 2 cm

- **Pair-selection criteria**

*J. Adams et al. (STAR Coll.),
Phys. Rev. C 71, 044906 (2005)*

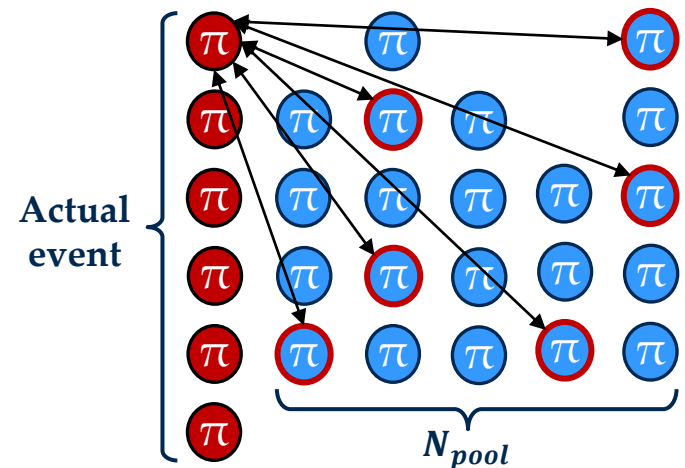
- Splitting level (SL) < 0.6
- Fraction of Merged Hits (FMH) $< 5\%$
- Average pair-separation (on TPC pad rows) $\Delta r > 3$ cm

- **Event mixing**

- Similarly to *Phys.Rev. C97 (2018) no.6, 064911*
- A(Q): pions from the same event
- B(Q): pions from different events
- $C(Q)=A(Q)/B(Q)$, appropriately normalized

- **Average transverse momentum k_T selection:**

- 8 k_T bins, (0.175 - 0.575) GeV/c



Fitting process with Lévy parametrization

- Lévy parametrization without final state effects:

$$C^{(0)}(Q) = 1 + \lambda \cdot e^{-|RQ|^\alpha}$$

LCMS three-momentum difference $Q = |q_{LCMS}| = \sqrt{(p_{1x} - p_{2x})^2 + (p_{1y} - p_{2y})^2 + q_{long,LCMS}^2}$

Lévy exponent α
 Lévy scale parameter R
 Intercept parameter (correlation strength) λ

- Formula used for fitting procedure:

$$C(Q) = \underbrace{(1 - \lambda + \lambda \cdot K(Q; \alpha, R))}_{\text{Coulomb correction}} \cdot \underbrace{(1 + e^{-|RQ|^\alpha})}_{\text{Possible linear background (usually negligible)}} \cdot N \cdot (1 + \varepsilon Q)$$

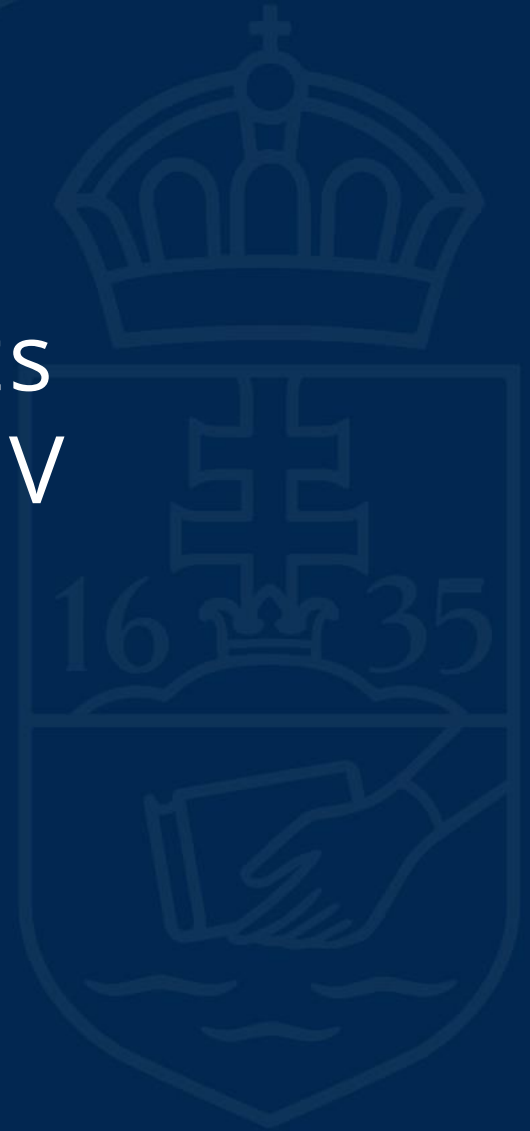
- Coulomb-correction:

$$K(Q; \alpha, R) = \frac{\int D_K(r) |\psi_Q^{Coul}(r)|^2 dr}{\int D_K(r) |\psi_Q^{(0)}(r)|^2 dr}$$

Two-particle wave function (with Coulomb interaction) $\psi_Q^{Coul}(r)$
 Two-particle wave function (plane wave) $\psi_Q^{(0)}(r)$
 Spatial correlations $D_K(r)$

→ **calculated semi-analytically**
Nagy, Purzsa, Csanád, Kincses, Eur. Phys. J. C 83, 1015 (2024)

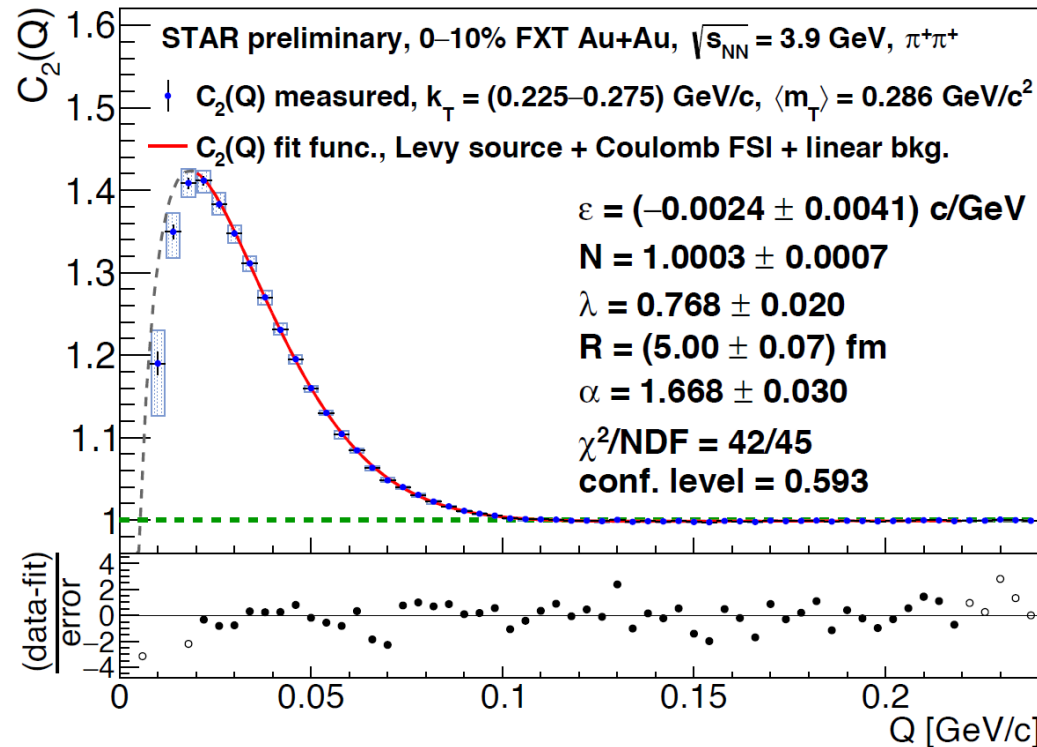
Part III.
STAR preliminary results
at $\sqrt{s_{NN}} = 3.9$ and 3.2 GeV
(m_T dependence)



$$C(Q) = (1 - \lambda + \lambda \cdot K(Q; \alpha, R) \cdot (1 + e^{-|RQ|^\alpha})) \cdot N \cdot (1 + \varepsilon Q)$$

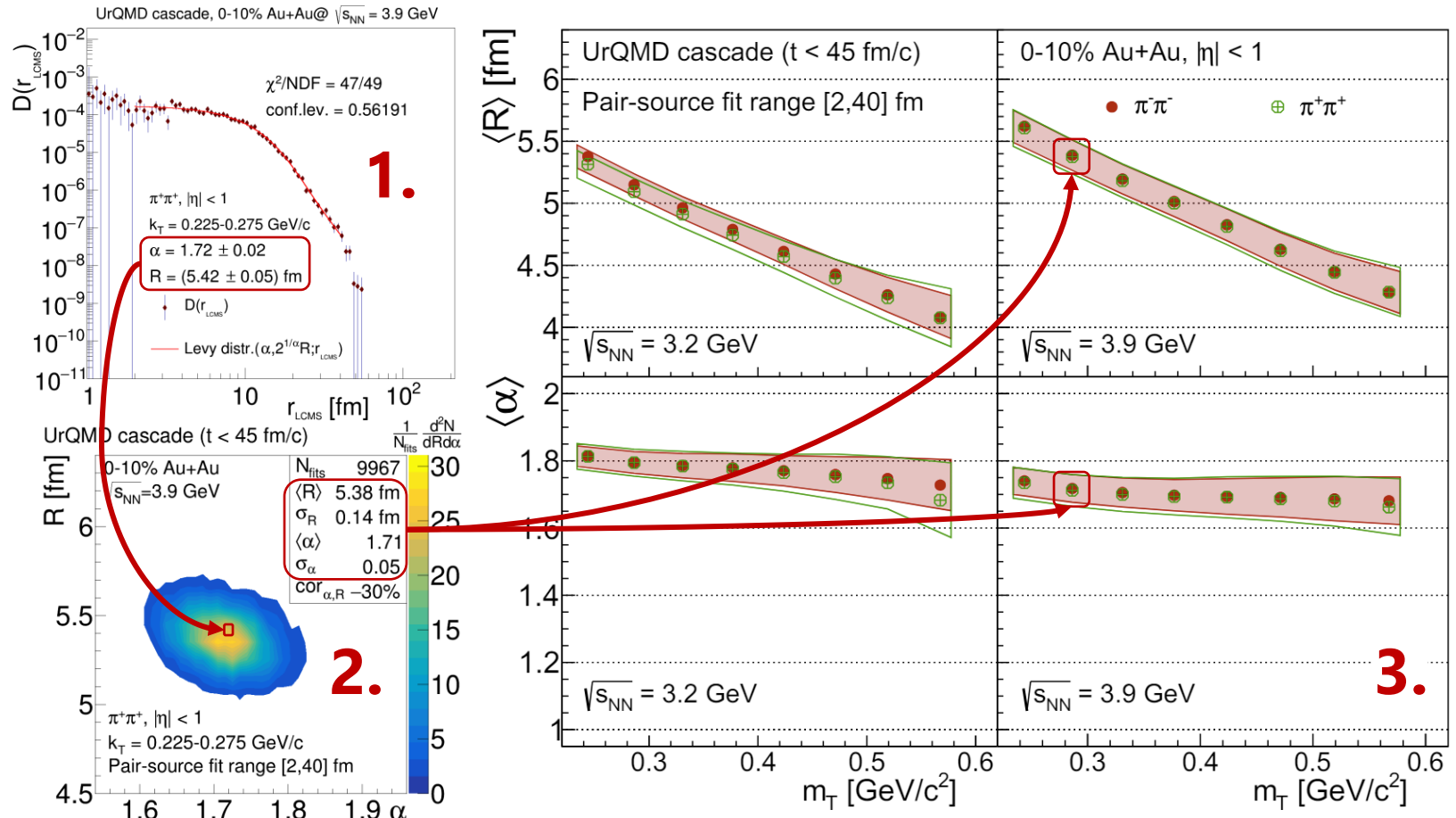
An example fit to a two-pion correlation function

- Example fit to a $\pi^+\pi^+$ corr.func., $k_T = (0.220-0.275)$ GeV/c
- **Iterative fitting method**, Coulomb FSI + Lévy-source
- Track and pair **sys. uncertainties illustrated with boxes**
- Total systematic uncertainties also include fit range study
- **Fits converged, conf.level > 0.1%**
- Confidence levels approx. uniformly distributed
- Similar fits done in 8 k_T bins, separately for $\pi^-\pi^-$ and $\pi^+\pi^+$, m_T dependence of the source parameters investigated at both energies



Before moving on to results: Model comparison – UrQMD cascade analysis

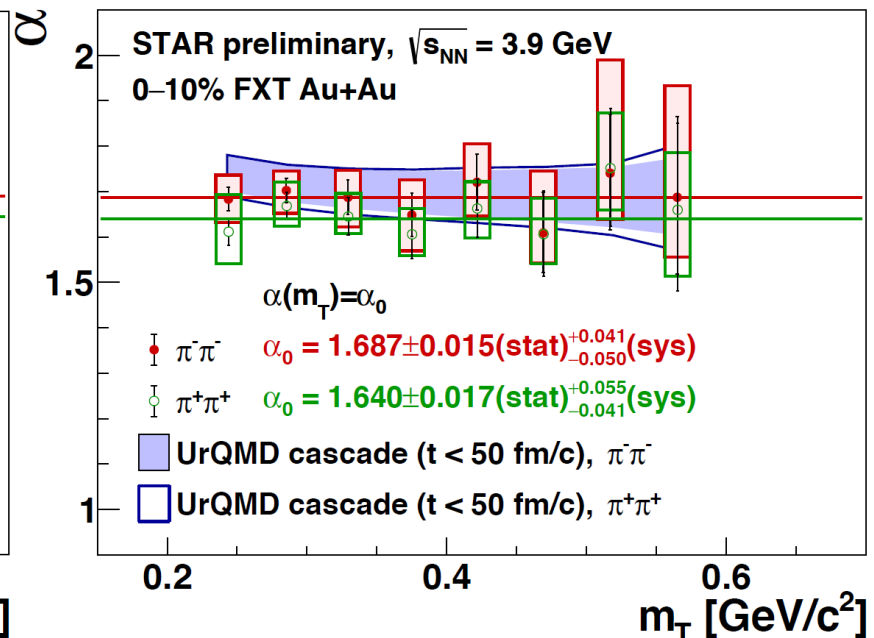
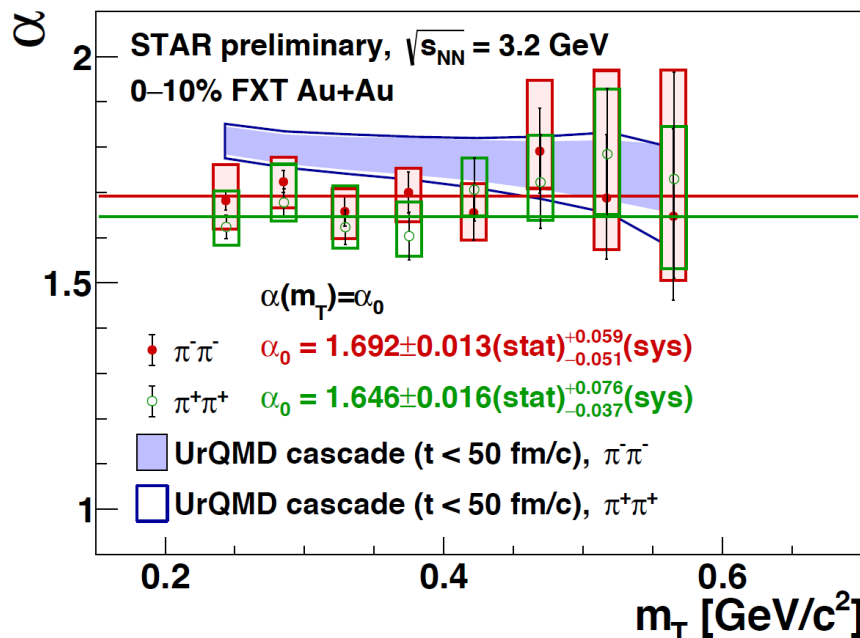
$D(r)$ event-by-event $\rightarrow (\alpha, R)$ distribution \rightarrow mean and std.dev vs m_T



$$C(Q; \lambda(m_T), R(m_T), \alpha(m_T)) = (1 - \lambda + \lambda \cdot K(Q; \alpha, R) \cdot (1 + e^{-|RQ|^\alpha}))$$

Lévy exponent α

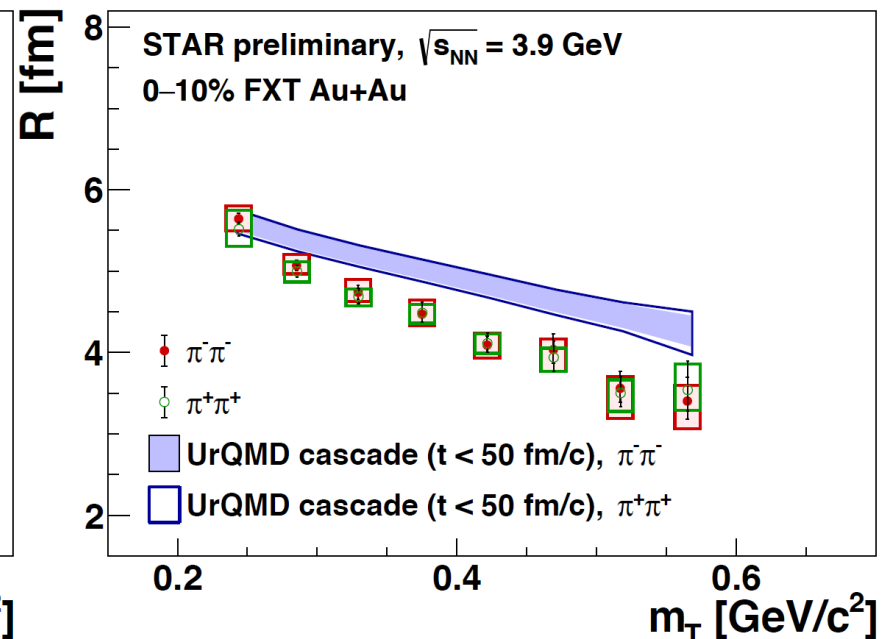
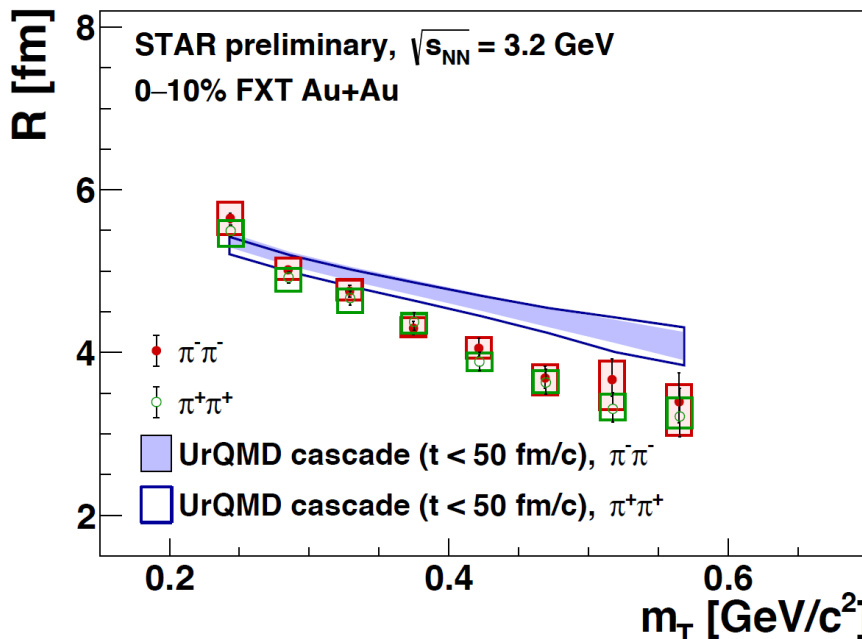
- Non-gaussian values ($\alpha < 2$)
- Small systematic difference between $\pi^- \pi^-$ and $\pi^+ \pi^+$ pairs
- 3.9 and 3.2 GeV compatible, no m_T dependence observed
- **UrQMD within uncert.** – no other effect but rescattering and decays, good agreement



$$C(Q; \lambda(m_T), R(m_T), \alpha(m_T)) = (1 - \lambda + \lambda \cdot K(Q; \alpha, R) \cdot (1 + e^{-|RQ|^\alpha}))$$

Lévy scale R

- Decreases towards higher m_T and lower energies
- Small systematic difference between $\pi^- \pi^-$ and $\pi^+ \pi^+$ pairs
- Both energies compatible
- UrQMD describes the trends qualitatively well, moderate quantitative mismatch

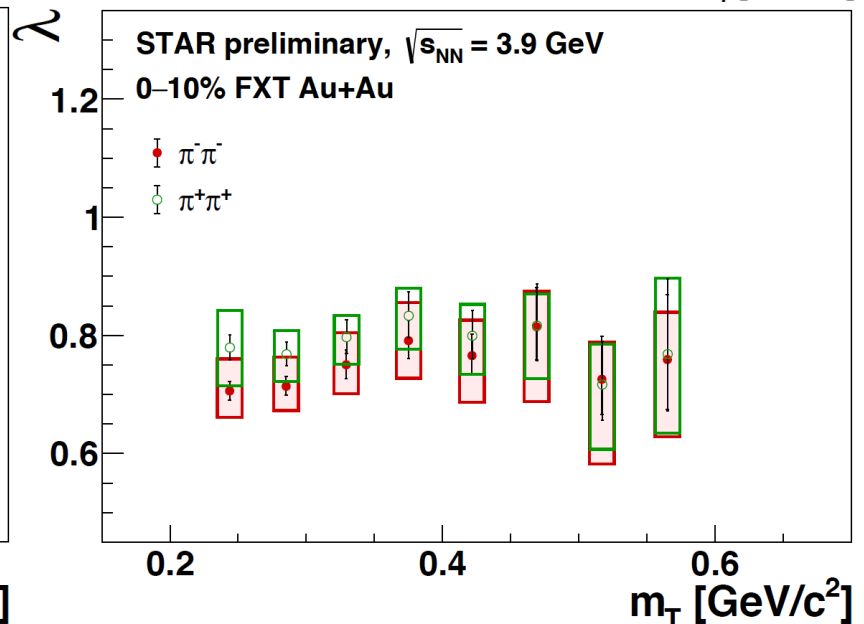
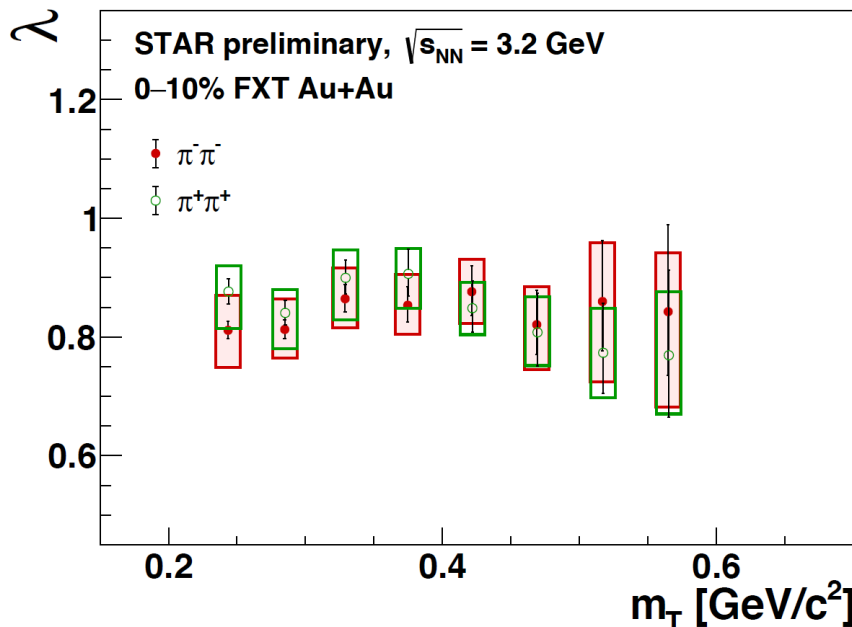
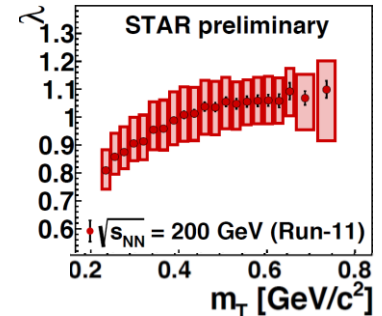


$$C(Q; \lambda(m_T), R(m_T), \alpha(m_T)) = (1 - \lambda + \lambda \cdot K(Q; \alpha, R) \cdot (1 + e^{-|RQ|^\alpha}))$$

Correlation strength λ

$$\lambda \equiv \lim_{Q \rightarrow 0} C_2^{(0)}(Q) - 1$$

- Small systematic difference between $\pi^- \pi^-$ and $\pi^+ \pi^+$ pairs
- Low- m_T decrease seen at higher energies not as pronounced (see next slide!)



Part IV.
STAR preliminary results at
top RHIC energy (200 GeV)

Universe 10 (2024) 3, 102



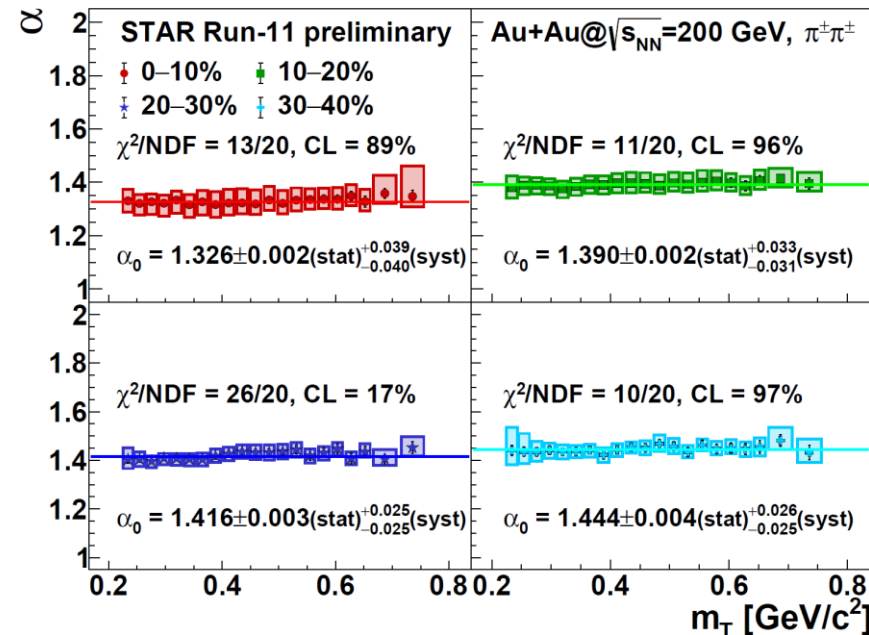
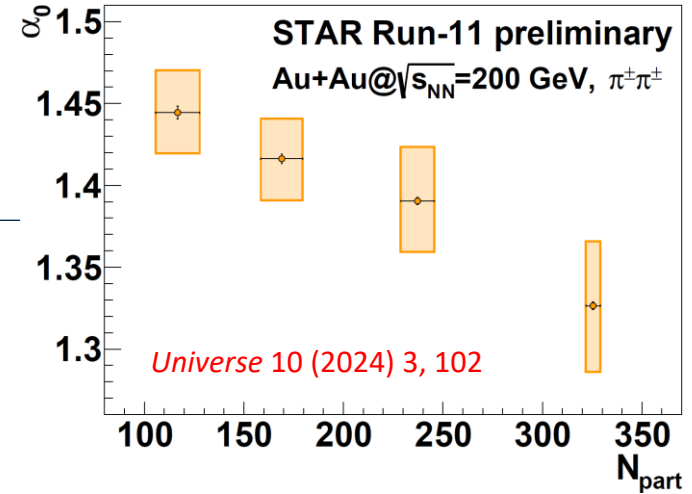
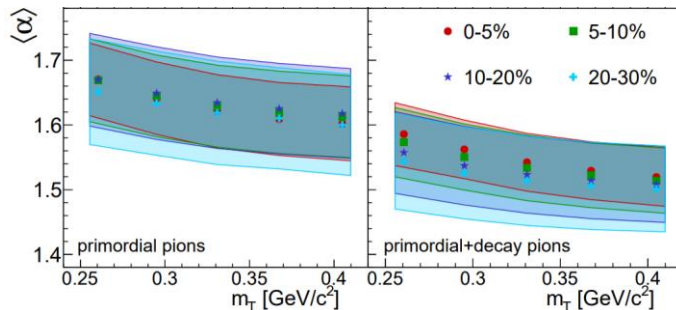
$$C(Q; \lambda(m_T), R(m_T), \alpha(m_T)) = (1 - \lambda + \lambda \cdot K(Q; \alpha, R) \cdot (1 + e^{-|RQ|^\alpha}))$$

$$\sqrt{s_{NN}} = 200 \text{ GeV}$$

Lévy exponent $\alpha(m_T, \text{centrality})$

- Non-gaussian values ($\alpha \ll 2$)
- No dependence on m_T , slight centrality dep.
- α_0 vs N_{part} (m_T average values from constant fit)
 - Decreasing trend due to anti-correlation with λ and R ?
 - CMS observed opposite trend, see *Phys.Rev.C* 109 (2024) 2, 024914
- EPOS model: slightly higher α values

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$$C(Q; \lambda(m_T), R(m_T), \alpha(m_T)) = (1 - \lambda + \lambda \cdot K(Q; \alpha, R) \cdot (1 + e^{-|RQ|^\alpha}))$$

$$\sqrt{s_{NN}} = 200 \text{ GeV}$$

Lévy scale $R(m_T, \text{centrality})$

- Decreasing trend with m_T , also with centrality
- Connection to flow and initial geometry?
- Fits with $R = R_0(Am_T + B)^{-1/\xi}$, $R_0 = 1 \text{ fm}$
- Hydro calculations (R_{Gauss}): $\xi = 2$

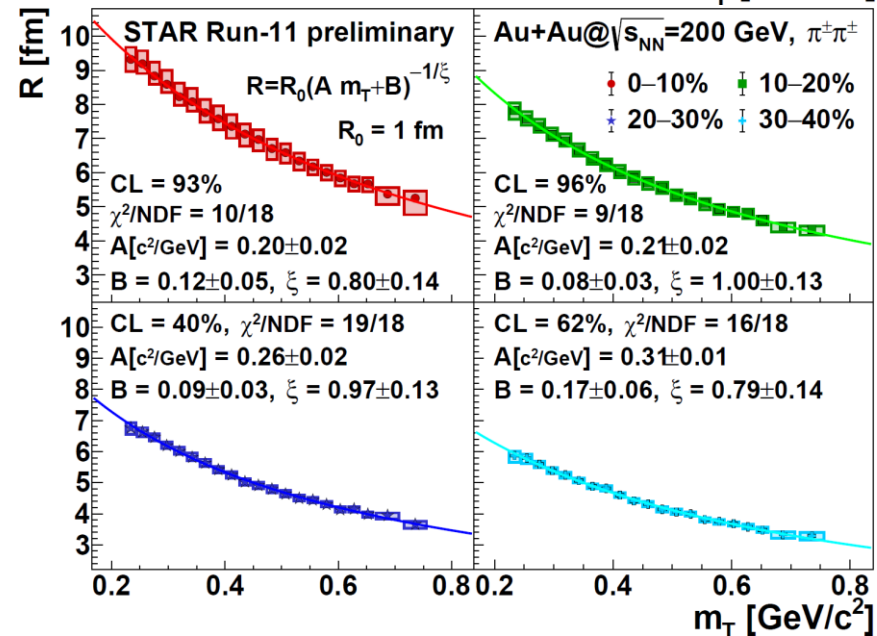
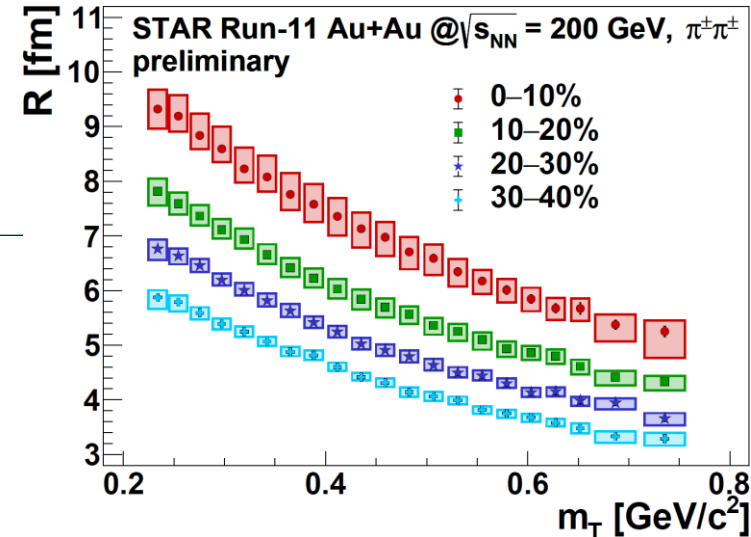
Makhlin, Sinyukov, Z. Phys. C 39, 69 (1988)

Csörgő, Lörstad, Phys. Rev. C 54, 1390 (1996)

Chapman, Scotto, Heinz, Acta Phys. Hung. A 1 (1995) 1-31

Csanád, Csörgő, Lörstad, Ster, J. Phys. G 30, S1079 (2004)

- Our case (R_{Levy}): ξ close to 1



$$C(Q; \lambda(m_T), R(m_T), \alpha(m_T)) = (1 - \lambda + \lambda \cdot K(Q; \alpha, R) \cdot (1 + e^{-|RQ|^\alpha}))$$

$$\sqrt{s_{NN}} = 200 \text{ GeV}$$

Correlation strength $\lambda(m_T, \text{centrality})$

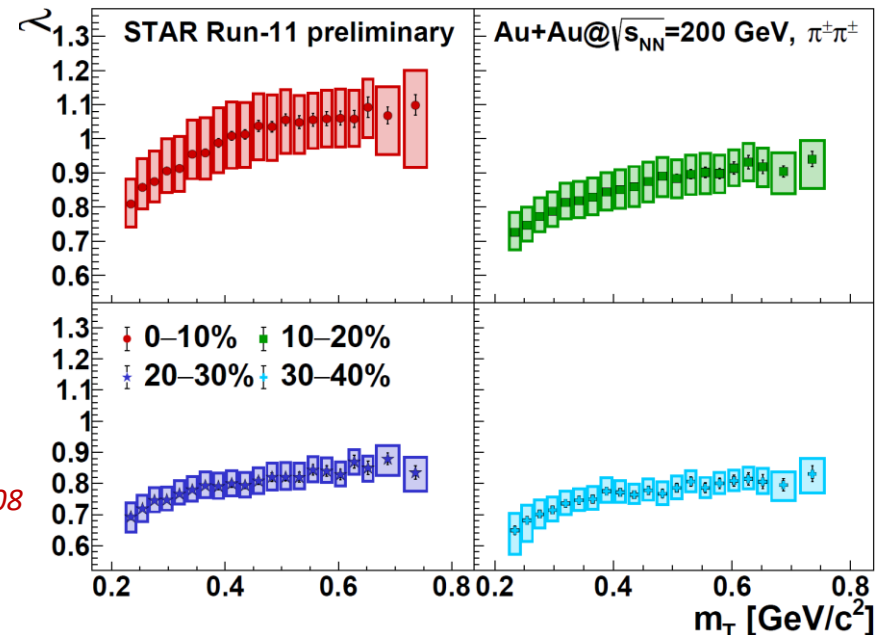
- Without FSI: $C_2^{(0)}(Q = 0) = 2 \rightarrow \lambda \equiv \lim_{Q \rightarrow 0} C_2^{(0)}(Q) - 1$, experimentally often $\lambda < 1$
- Core-halo picture** - two-component source: $S = S_{core} + S_{halo} \Rightarrow D = D_{(c,c)} + D_{(c,h)} + D_{(h,h)}$
 - Core: primordial + decays of short-lived resonances
 - Halo: decays of long-lived resonances \rightarrow large R \rightarrow small Q \rightarrow measurement limited

$$\lambda = N_{core}^2 / (N_{core} + N_{halo})^2$$

Csörgő, Lörstad, Zimányi, Z.Phys. C71 (1996) 491-497

Bolz et al, Phys.Rev. D47 (1993) 3860-3870;

- For power-law sources, more complicated picture!
- Increase from low to high m_T
 - More decay products at low m_T ?
 - In-medium mass modification of η '?
 - Partially coherent particle emission?
- Vance, Csörgő, Kharzeev, Phys. Rev. Lett.81 (1998), pp. 2205-2208*
- Bolz et al., Phys. Rev. D47 (1993), pp. 3860-3870*
- Decrease from central to peripheral



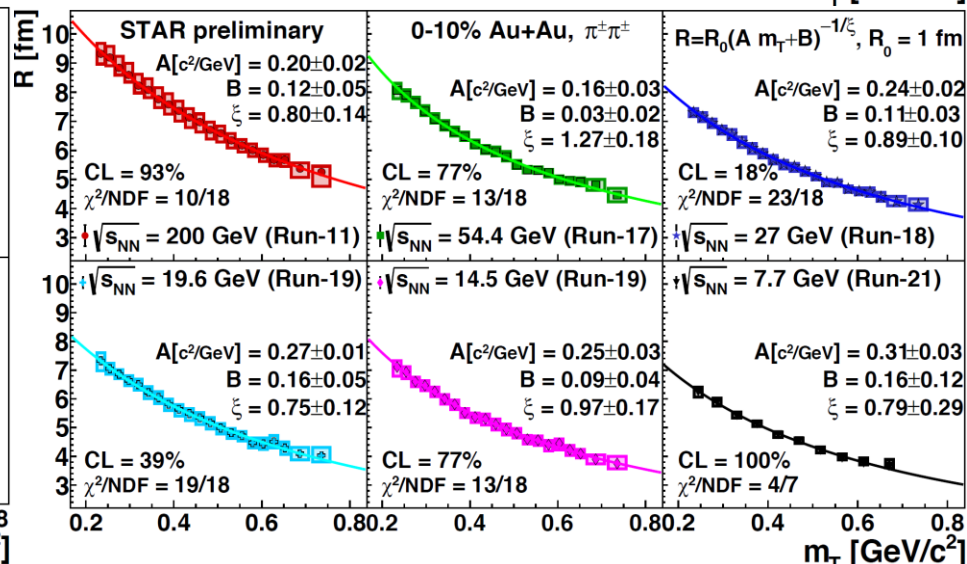
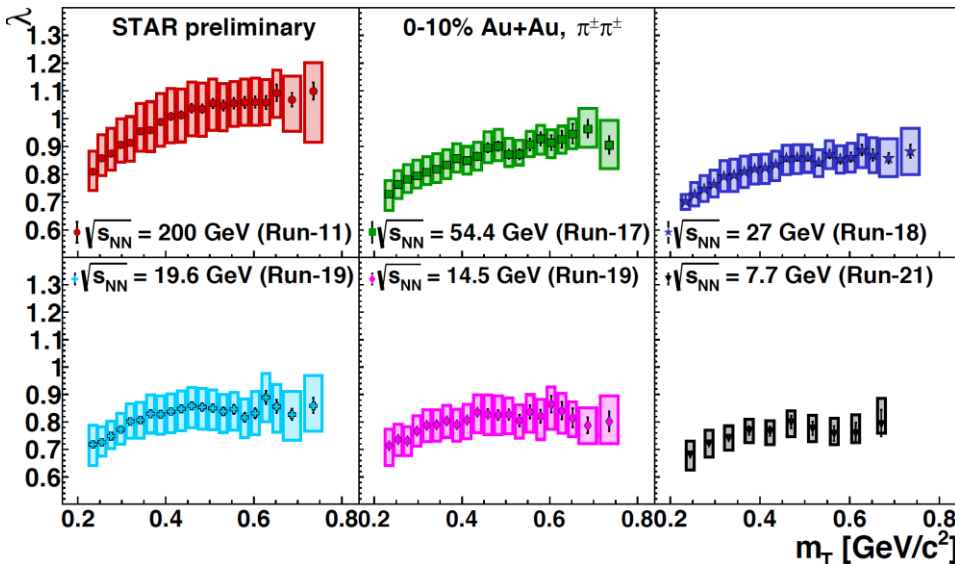
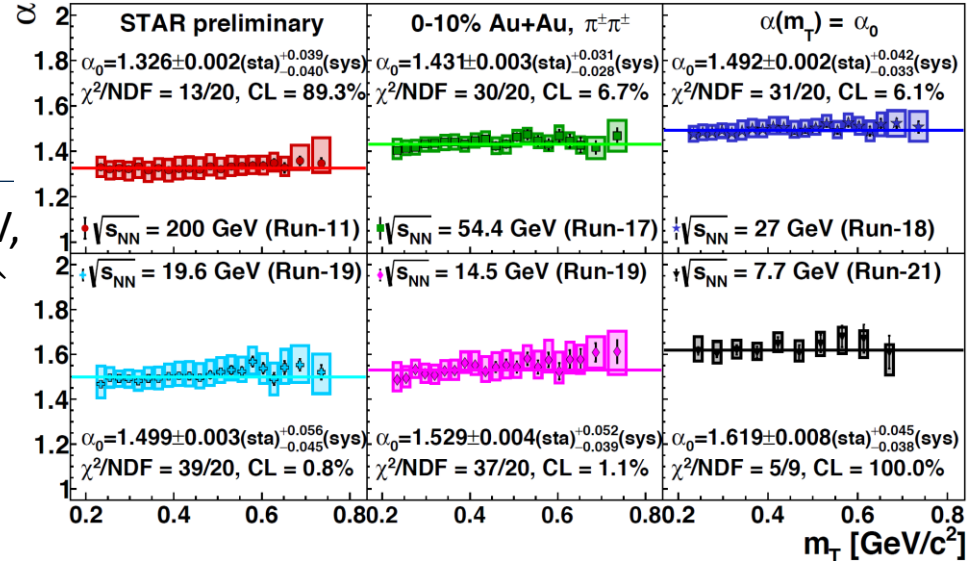
Part V.
STAR preliminary results at BES-II
($\sqrt{s_{NN}}$, m_T dependence)



$$C(Q; \lambda(m_T), R(m_T), \alpha(m_T)) = (1 - \lambda + \lambda \cdot K(Q; \alpha, R) \cdot (1 + e^{-|RQ|^\alpha}))$$

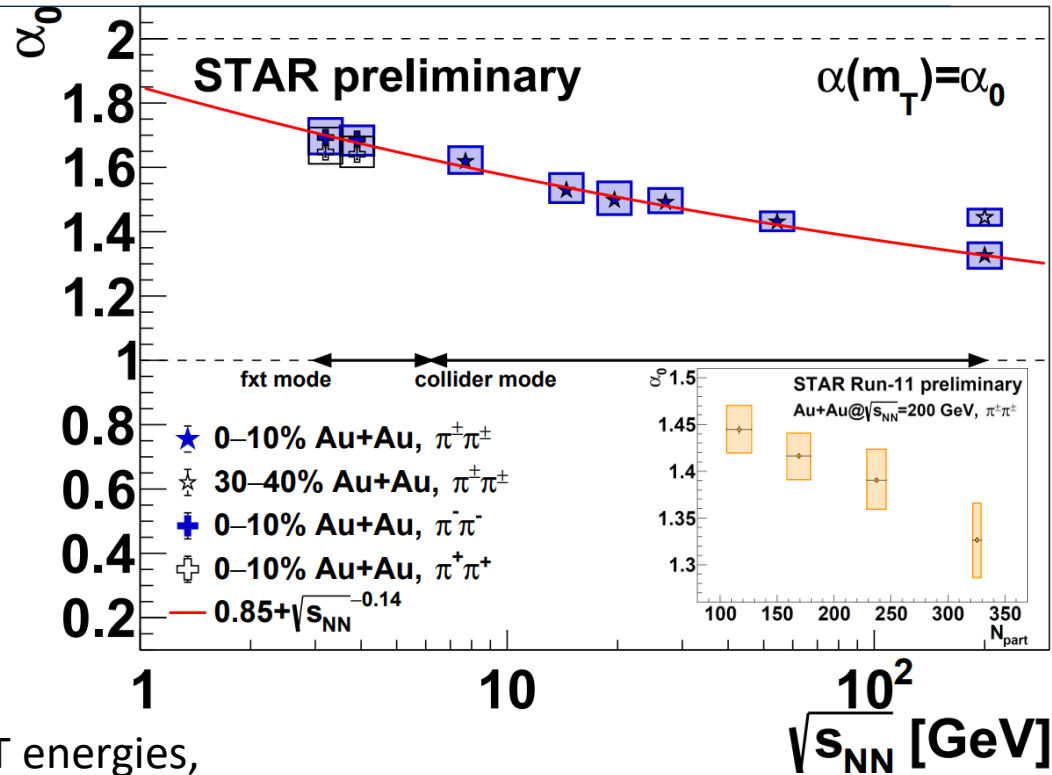
$\sqrt{s_{NN}}$ and m_T dependence of the source parameters

- $\sqrt{s_{NN}}$ dependence from 7.7 to 200 GeV, due to increasing density? $\alpha \downarrow, R \uparrow, \lambda \uparrow$
- m_T dependent trends at all energies: α const., $R \downarrow$ (connection to flow), $\lambda \uparrow$ (change in decay contribution?)



Excitation function of the m_T average Lévy exponent α_0

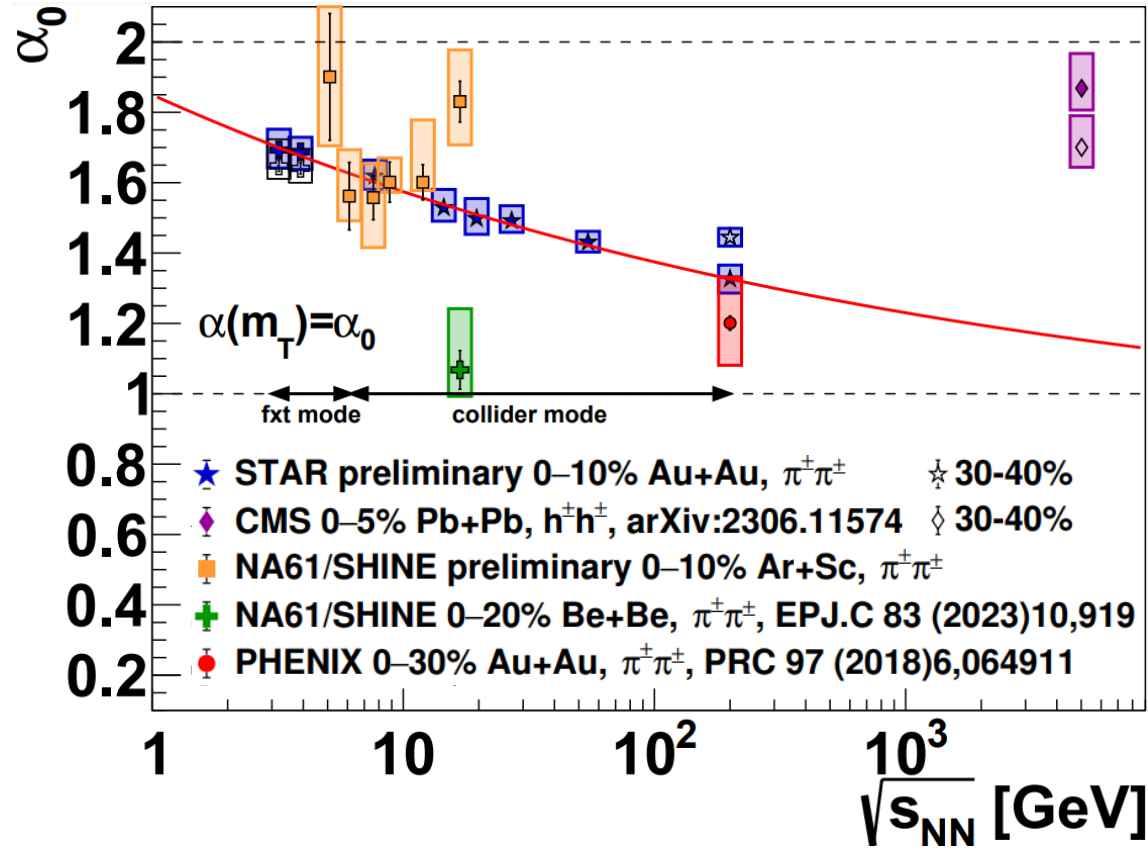
- Non-gaussian values ($\alpha \ll 2$)
- Increasing density \rightarrow rescattering decreases α ?
- 200 GeV centrality dependence: same trend!
- Trend illustrated by power-law: $\alpha_0 \sim 0.85 + \sqrt{s_{NN}}^{-0.14}$



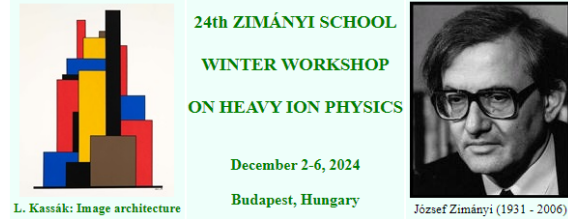
- Good description by UrQMD at FXT energies, a comprehensive energy scan is ongoing
- **No non-monotonic trend in α observed yet, no signs of criticality in Levy exponent**

Comparison with other experiments

- Opposite centrality trends?
CMS: peripheral < central
- Opposite energy dep. trends?
Increase from STAR to CMS??
Non-monotonicity in NA61??
- System size ordering?
BeBe < AuAu < ArSc ??
- Experimental differences might play an important role (momentum acceptance, PID)



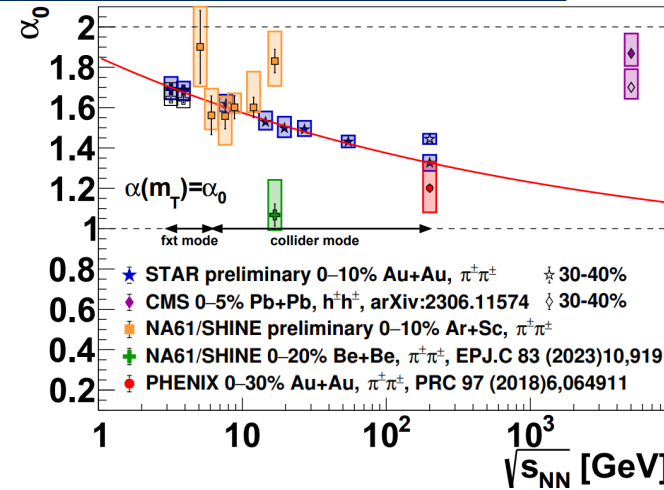
- Lots of open questions, more cross-checks and model studies needed



Summary

- 1-dim. two-pion correlation functions investigated
- Lévy-source + Coulomb FSI → good description
- **Observation at STAR:**
 - 0-10% Au+Au: **200 GeV → 3.2 GeV**
 - 200 GeV Au+Au: **central → peripheral**
- **Possible interpretation:**
 - $\alpha < 2$ connected to decays and rescattering
 - Increasing particle density → rescattering decreases α ?
- **Comparison with other experiments: different trends, other interpretation needed**
- **Next steps:**
 - even more energies (fxt&col), 3D analysis!
 - More model studies with hydro+transport codes

trends:
 $\alpha \uparrow, R \downarrow, \lambda \downarrow$



Further details, backup slides



$$\lambda \equiv \lim_{Q \rightarrow 0} C_2^{(0)}(Q) - 1$$

Correlation strength λ

- **Core-halo* picture** - two-component source: $S = S_{core} + S_{halo} \Rightarrow D = D_{(c,c)} + D_{(c,h)} + D_{(h,h)}$
 - Halo: decays of long-lived resonances \rightarrow unresolvable
 - For power-law sources, more complicated picture!

Single particle source:

$$S_{core}(x) = \mathcal{L}(\alpha, R; x),$$

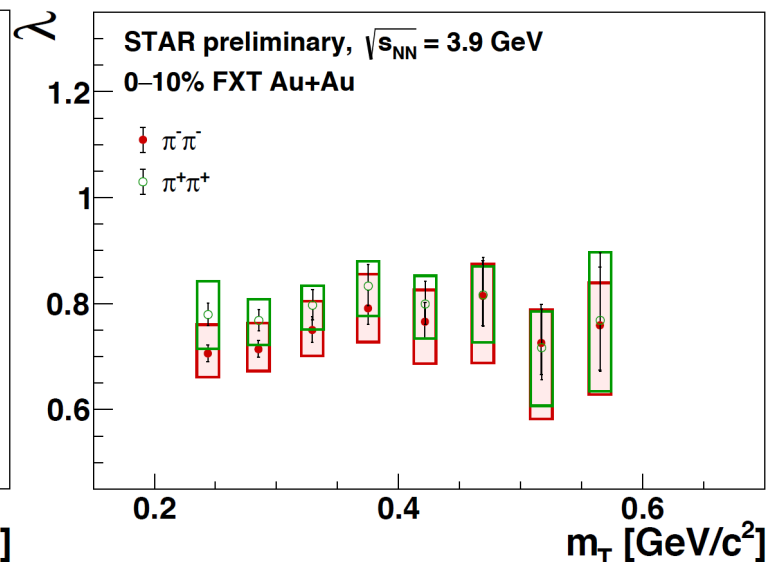
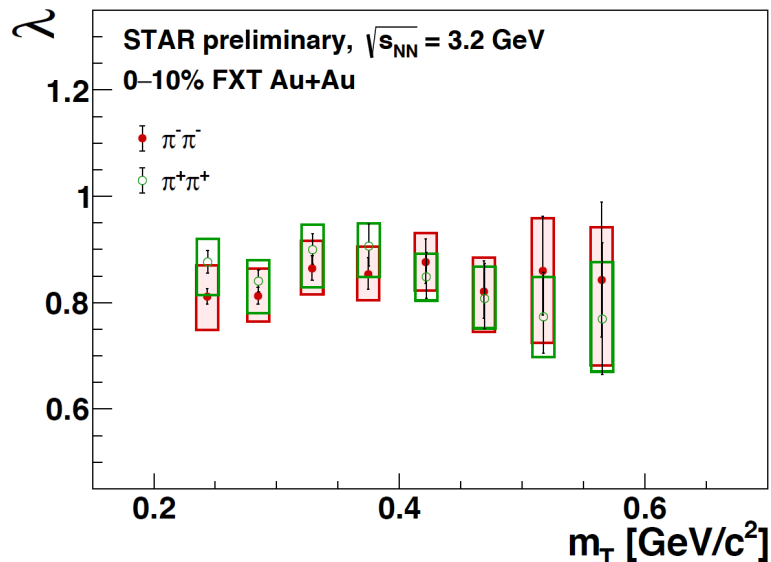
$$S_{halo}(x) = \mathcal{L}(\alpha_h, R_h; x)$$

Pair-source:

$$D_{(c,c)}(r) = \mathcal{L}(\alpha, 2^{1/\alpha} R; r)$$

$$R_h \gg R \Rightarrow D_{(c,h), (h,h)} \text{ unresolvable}$$

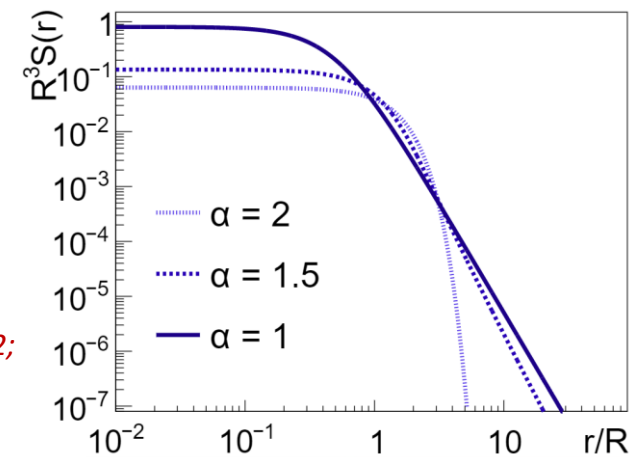
*note that core-halo here is not the same as EPOS core-corona, it is just a coincidence in nomenclature



What is the shape of the source?

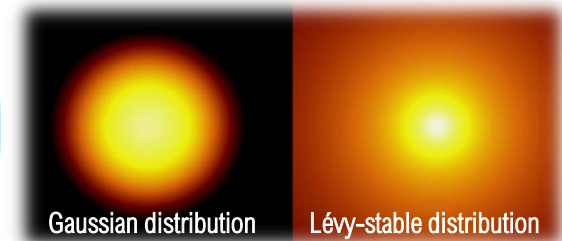
Gaussian & Lévy distributions in heavy-ion physics

- Possible interpretations of the α Lévy exponent based on:
 - Jet fragmentation *Csörgő, Hegyi, Novák, Zajc, Acta Phys.Polon. B36*
 - Critical behavior *Csörgő, Hegyi, Novák, Zajc, AIP Conf.Proc. 828*
 - Event averaging *Cimerman, Tomasik, Plumberg, Phys.Part.Nucl. 51 (2020) 3, 282*
 - Resonance decays *Kincses, Stefaniak, Csanád, Entropy 24 (2022) 3, 308*
 - Anomalous diffusion *Csanád, Csörgő, Nagy, Braz.J.Phys. 37 (2007) 1002;*



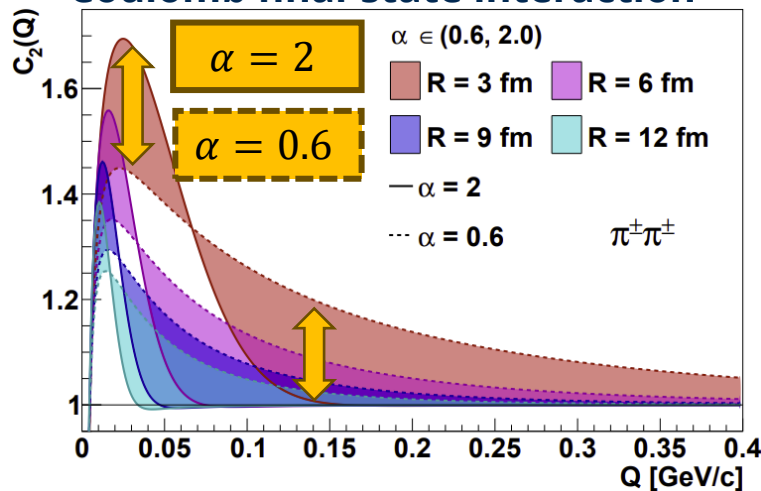
$$S(\mathbf{r}) = \mathcal{L}(\alpha, R; \mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} e^{i\mathbf{q}\mathbf{r}} e^{-\frac{1}{2} |\mathbf{q}^T \mathbf{R}^2 \mathbf{q}|^{\alpha/2}}$$

spherical sym.: $R_{ij}^2 = R^2 \delta_{ij}$



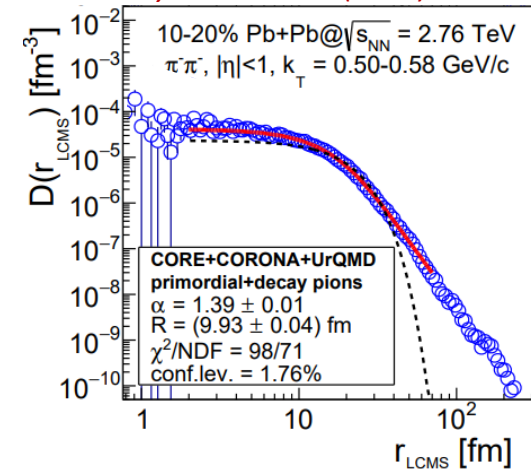
Recent phenomenological developments

- **Coulomb Corrections for Bose-Einstein Correlations from One- and Three-Dimensional Lévy-Type Source Functions** *Kurgyis, Kincses, Nagy, Csanád, Universe 9 (2024) 7, 328*
- **Event-by-Event Investigation of the Two-Particle Source Function in Heavy-Ion Collisions with EPOS** *Kincses, Stefaniak, Csanád, Entropy 24 (2022) 3, 308*
Kórodi, Kincses, Csanád, Phys. Lett. B 847 (2024) 138295
- **A novel method for calculating Bose-Einstein correlation functions with Coulomb final-state interaction** *Nagy, Purzsa, Csanád, Kincses, Eur. Phys. J. C 83, 1015 (2024)*



EPOS, 2.76 TeV, $\alpha \sim 1.4$

Phys. Lett. B 847 (2024) 138295

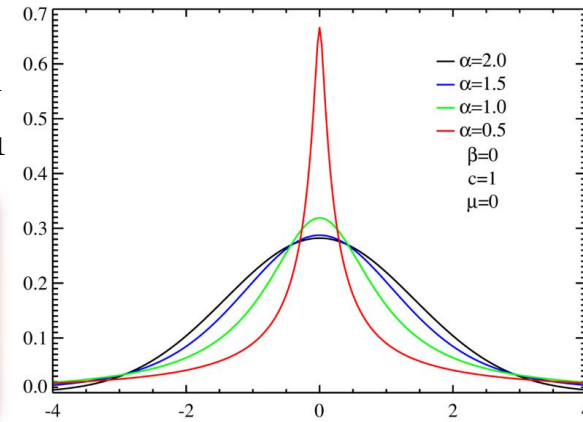
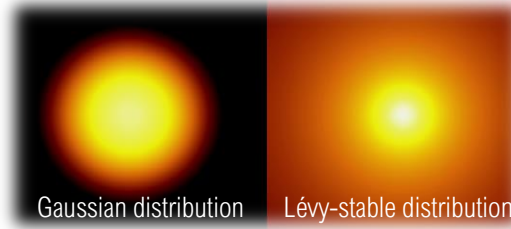


Properties of univariate stable distributions

- **Univariate stable distribution:** $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(q) e^{-ixq} dq$, where the characteristic function:

- $\varphi(q; \alpha, \beta, R, \mu) = \exp(iq\mu - |qR|^\alpha (1 - i\beta \operatorname{sgn}(q)\Phi))$
- α : index of stability
- β : skewness, symmetric if $\beta = 0$
- R : scale parameter
- μ : location, equals the median,
if $\alpha > 1$: $\mu = \text{mean}$

$$\Phi = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right), & \alpha \neq 1 \\ -\frac{2}{\pi} \log|q|, & \alpha = 1 \end{cases}$$



- **Important characteristics of stable distributions:**

- Retains same α and β under convolution of random variables
- Any moment greater than α isn't defined

In 3D: $\mathcal{L}(\mathbf{r}; \alpha, R) = \frac{1}{(2\pi)^3} \int d^3q e^{i\mathbf{q}\mathbf{r}} e^{-\frac{1}{2} |qRq|^\alpha}$

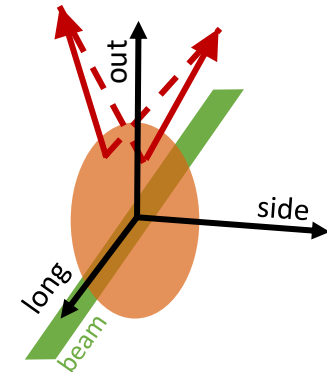
$$R_{\sigma\nu}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & R_{\text{out}}^2 & 0 & 0 \\ 0 & 0 & R_{\text{side}}^2 & 0 \\ 0 & 0 & 0 & R_{\text{long}}^2 \end{pmatrix}$$

Kinematic variables of the correlation function I.

- Smoothness approximation ($p_1 \approx p_2 \approx K$): $S(x_1, K - q/2) S(x_2, K + q/2) \approx S(x_1, K) S(x_2, K)$
 - $C_2(q, K) = \int d^4r D(r, K) \left| \psi_q^{(2)}(r) \right|^2$
 - Without any FSI $\left| \psi_q^{(2)}(r) \right|^2 = 1 + \cos(qr)$
- $$\left. \begin{array}{l} C_2(q, K) = \int d^4r D(r, K) \left| \psi_q^{(2)}(r) \right|^2 \\ \text{Without any FSI } \left| \psi_q^{(2)}(r) \right|^2 = 1 + \cos(qr) \end{array} \right\} C_2^{(0)}(q, K) \simeq 1 + \frac{\tilde{D}(q, K)}{\tilde{D}(0, K)}, \text{ where } \tilde{D}(q, K) = \int D(x, K) e^{iqx} d^4x$$
- **HBT correlation function in direct connection with Fourier transform of the pair-source function**
 - Important to determine the nature and dimensionality of the correlation function
 - Lorentz-product of $q = (q_0, \mathbf{q})$ and $K = (K_0, \mathbf{K})$ is zero, i.e.: $qK = q_0 K_0 - \mathbf{q}\mathbf{K} = 0$
 - Energy component of q can be expressed as $q_0 = \mathbf{q} \frac{K}{K_0}$
 - If the energy of the particles are similar, K is approximately on shell
 - **Correlation function can be measured as a function of three-momentum variables**

Kinematic variables of the correlation function II.

- $C_2(\mathbf{q}, \mathbf{K})$ as a function of three-momentum variables
- \mathbf{K} dependence is smoother, \mathbf{q} is the main kinematic variable
- Close to mid-rapidity one can use $k_T = \sqrt{K_x^2 + K_y^2}$, or $m_T = \sqrt{k_T^2 + m^2}$
- For any fixed value of m_T , the correlation function can be measured as a function of \mathbf{q} only
- Usual decomposition: **out-side-long or Bertsch-Pratt (BP) coordinate-system**
 - $\mathbf{q} \equiv (q_{out}, q_{side}, q_{long})$
 - long: beam direction
 - out: k_T direction
 - side: orthogonal to the others
 - Essentially a rotation in the transverse plane
- Customary to use a Lorentz-boost in the long direction and change to the **Longitudinal Co-Moving System (LCMS)** where the average longitudinal momentum of the pair is zero



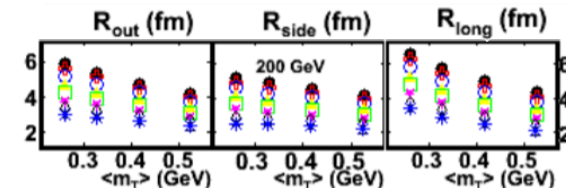
Kinematic variables of the correlation function III.

- Drawback of a 3D measurement: lack of statistics, difficulties of a precise shape analysis
- **What is the appropriate one-dimensional variable?**
- Lorentz-invariant relative momentum: $q_{inv} \equiv \sqrt{-q^\mu q_\mu} = \sqrt{q_x^2 + q_y^2 + q_z^2 - (E_1 - E_2)^2}$
- Equivalent to three-mom. diff. in Pair Co-Moving System (PCMS), where $E_1 = E_2$: $q_{inv} = |q_{PCMS}|$
- In LCMS using BP variables: $q_{inv} = \sqrt{(1 - \beta_T)^2 q_{out}^2 + q_{side}^2 + q_{long}^2}$ $\beta_T = 2k_T / (E_1 + E_2)$
- **Value of q_{inv} can be relatively small even when q_{out} is large!**
- Experimental indications: **in LCMS source is \approx spherically symmetric**
- Correlation function boosted to PCMS will not be spherically symmetric
- Let us introduce the following variable invariant to Lorentz boosts in the beam direction:

$$Q \equiv |q_{LCMS}| = \sqrt{(p_{1x} - p_{2x})^2 + (p_{1y} - p_{2y})^2 + q_{z,LCMS}^2}$$

$$\text{where } q_{z,LCMS}^2 = \frac{4(p_{1z}E_2 - p_{2z}E_1)^2}{(E_1 + E_2)^2 - (p_{1z} + p_{2z})^2}.$$

STAR, Phys.Rev.C 92 (2015) 1, 014904



Kinematic variables of the correlation function IV.

- Nature of the 1D variable in experiment: check correlation function in two dimensions!

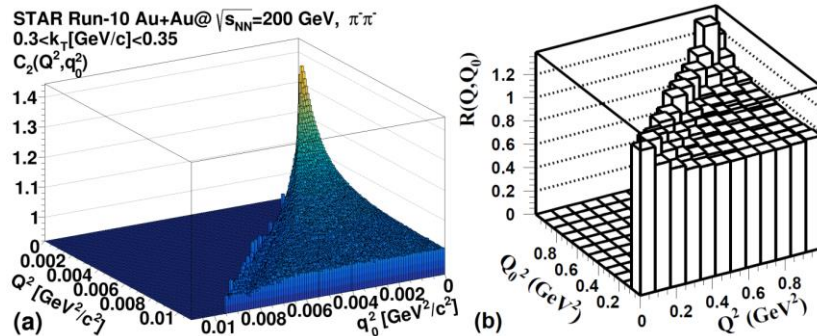


Figure 3.4: Example two-dimensional pion correlation functions for $\sqrt{s_{NN}} = 200$ GeV Au+Au collisions (a) and $\sqrt{s} = 91$ GeV e^+e^- collisions (b). The latter figure is taken from the thesis of Tamás Novák [161].

Q dep. corr.func.

q_{inv} dep. corr.func.

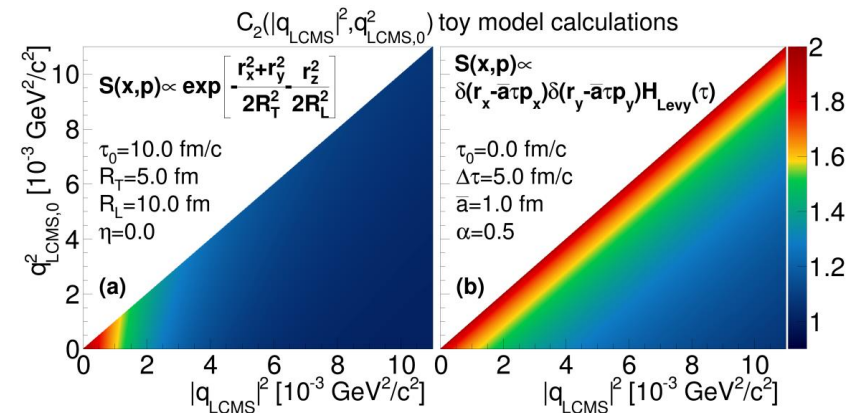
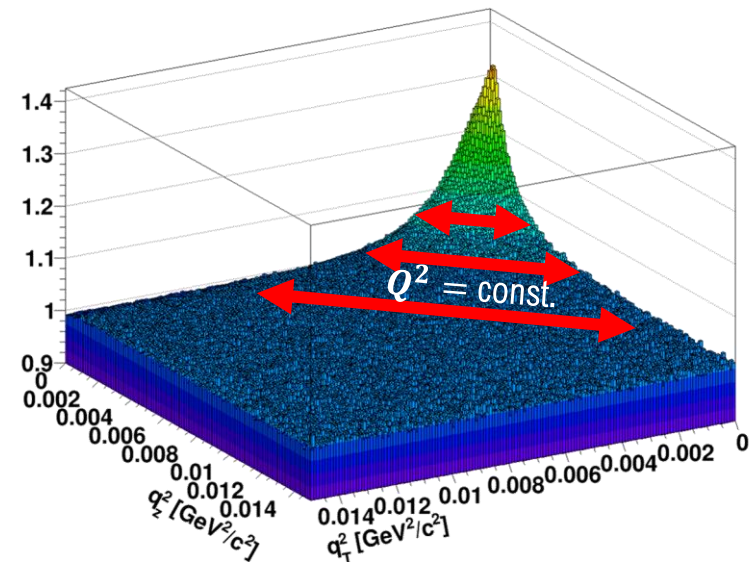
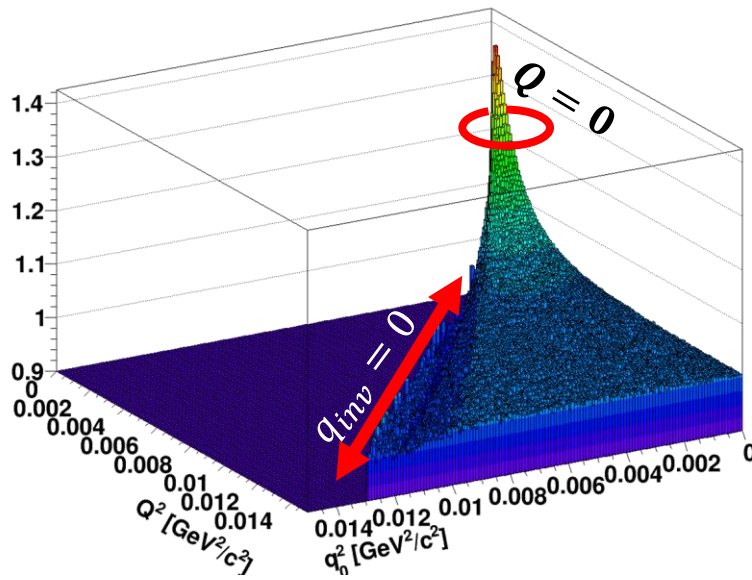


Figure 3.5: Toy model calculation for two different types of source functions. Taking a Gaussian source in both space and time leads to a correlation function that depends mostly on $|q_{LCMS}|$ (a), while a source that shows strong space-time and momentum space correlation leads to a q_{inv} dependent correlation function (b).

Kinematic variables of the correlation function V.

- Nature of the 1D variable in experiment: check correlation function in two dimensions!

$$Q = |\mathbf{q}_{LCMS}| = \sqrt{(p_{1x} - p_{2x})^2 + (p_{1y} - p_{2y})^2 + q_{long,LCMS}^2}$$



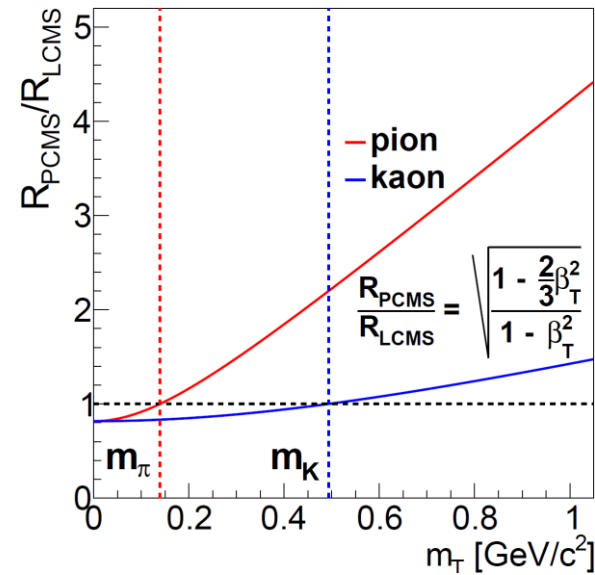
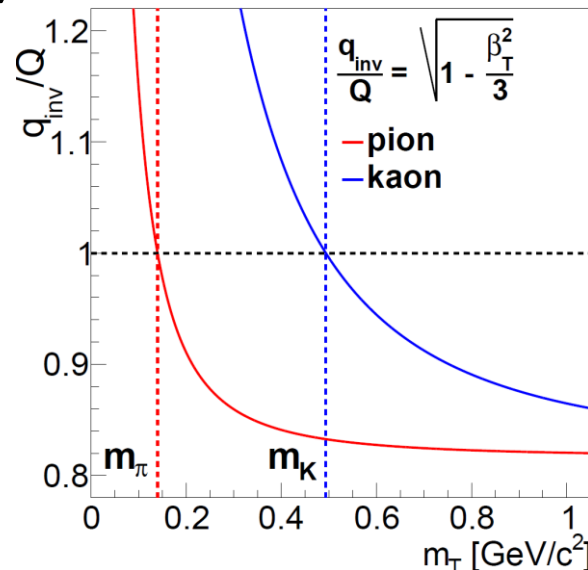
D. Kincses, Ph.D. thesis, [10.15476/ELTE.2022.164](https://doi.org/10.15476/ELTE.2022.164)

Kinematic variables of the correlation function VI.

- Correlation function measured in LCMS, Coulomb effect calculated in PCMS
- Approximation:
- (Note $m_T < m$ not physical of course)

$$q_{inv} \equiv q_{PCMS} \approx q_{LCMS} \cdot \sqrt{1 - \beta_T^2/3}$$

$$R_{PCMS} \approx R_{LCMS} \cdot \sqrt{\frac{1 - 2\beta_T^2/3}{1 - \beta_T^2}}$$



Coulomb correction and fitting of the corr. function

- Core-Halo model, Bowler-Sinyukov method: $C_2(Q, k_T) = 1 - \lambda + \lambda \int d^3r D_{(c,e)}(\mathbf{r}, k_T) |\Psi_Q^{(2)}(\mathbf{r})|^2$
- Neglecting FSI and using a Lévy-stable source function: $C_2^{(0)}(Q, k_T) = 1 + \lambda e^{-|RQ|^\alpha}$
- **Using numerical integral calculation as fit function results in numerically fluctuating χ^2 landscape**
- Treat FSI as correction factor: $K(Q, k_T) = \frac{C_2(Q, k_T)}{C_2^{(0)}(Q, k_T)}$
- An iterative method can be used: $C_2^{(fit)}(Q; \lambda, R, \alpha) = C_2^{(0)}(Q; \lambda, R, \alpha) \cdot K(Q; \lambda_0, R_0, \alpha_0)$
- Procedure continued until $\Delta_{iteration} = \sqrt{\frac{(\lambda_{n+1} - \lambda_n)^2}{\lambda_n^2} + \frac{(R_{n+1} - R_n)^2}{R_n^2} + \frac{(\alpha_{n+1} - \alpha_n)^2}{\alpha_n^2}} < 0.01$
- **Iterations usually converge within 2-3 rounds, fit parameters can be reliably extracted**

Coulomb correction and fitting of the corr. function

- Lévy-type correlation function without final state effects: $C^{(0)}(Q) = 1 + \lambda \cdot e^{-|RQ|^\alpha}$

- Bowler-Sinyukov method:

$$C(Q_{LCMS}; \lambda, R_{LCMS}, \alpha) = (1 - \lambda + \lambda \cdot K(q_{inv}; \alpha, R_{PCMS}) \cdot (1 + e^{-|R_{LCMS} Q_{LCMS}|^\alpha})) \cdot N \cdot (1 + \varepsilon Q_{LCMS})$$

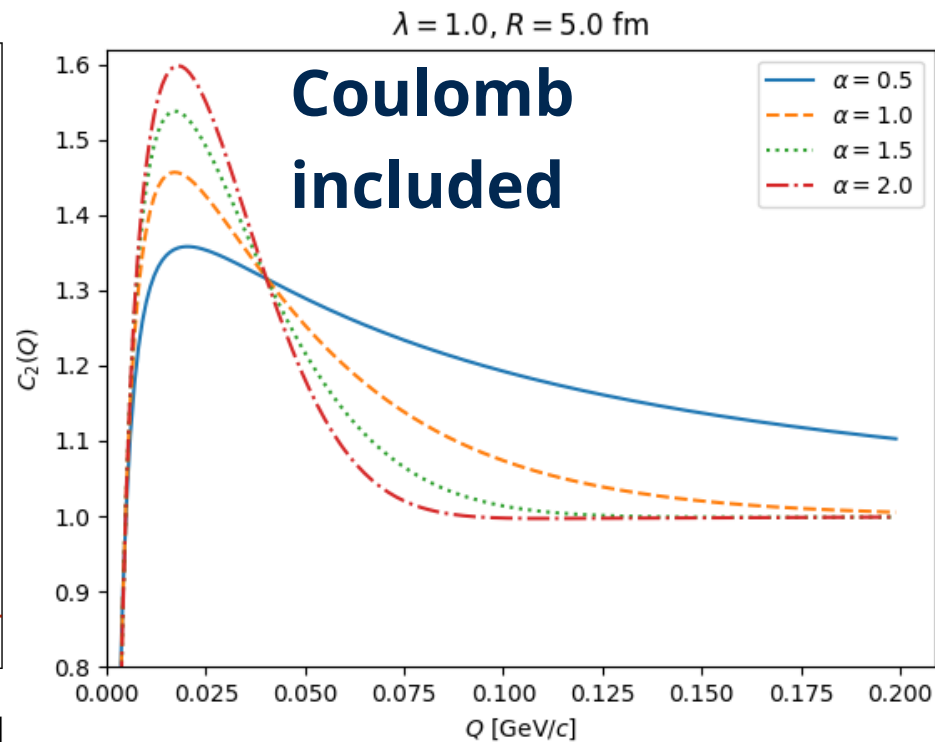
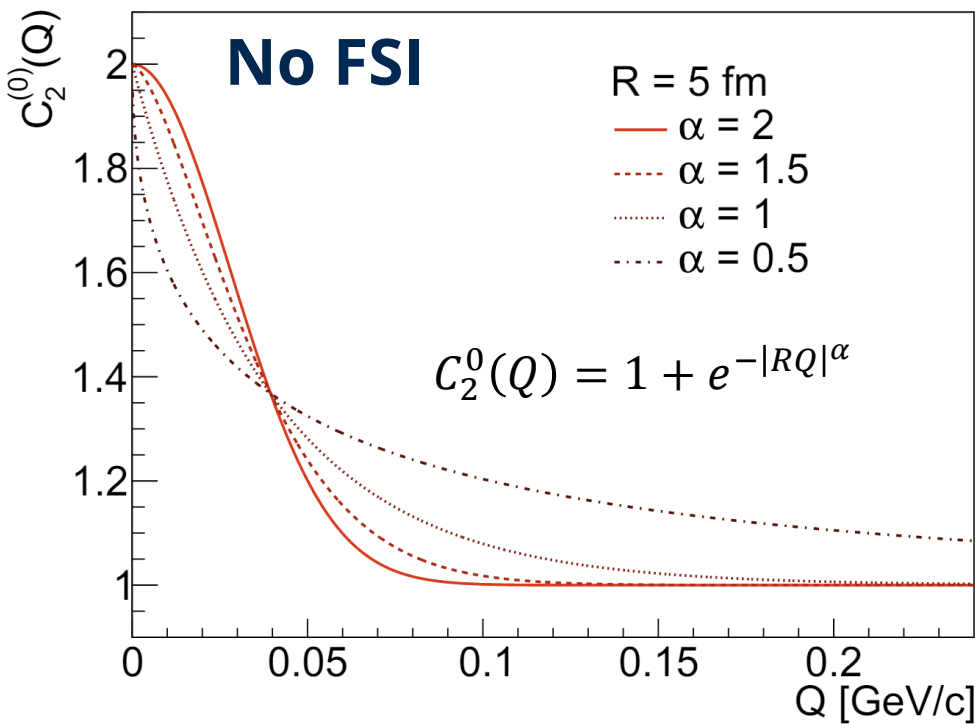
Intercept parameter (correlation strength)
Lévy scale parameter
Possible linear background (usually negligible)

Coulomb correction
Lévy exponent

- Coulomb-correction calculated numerically (in PCMS)

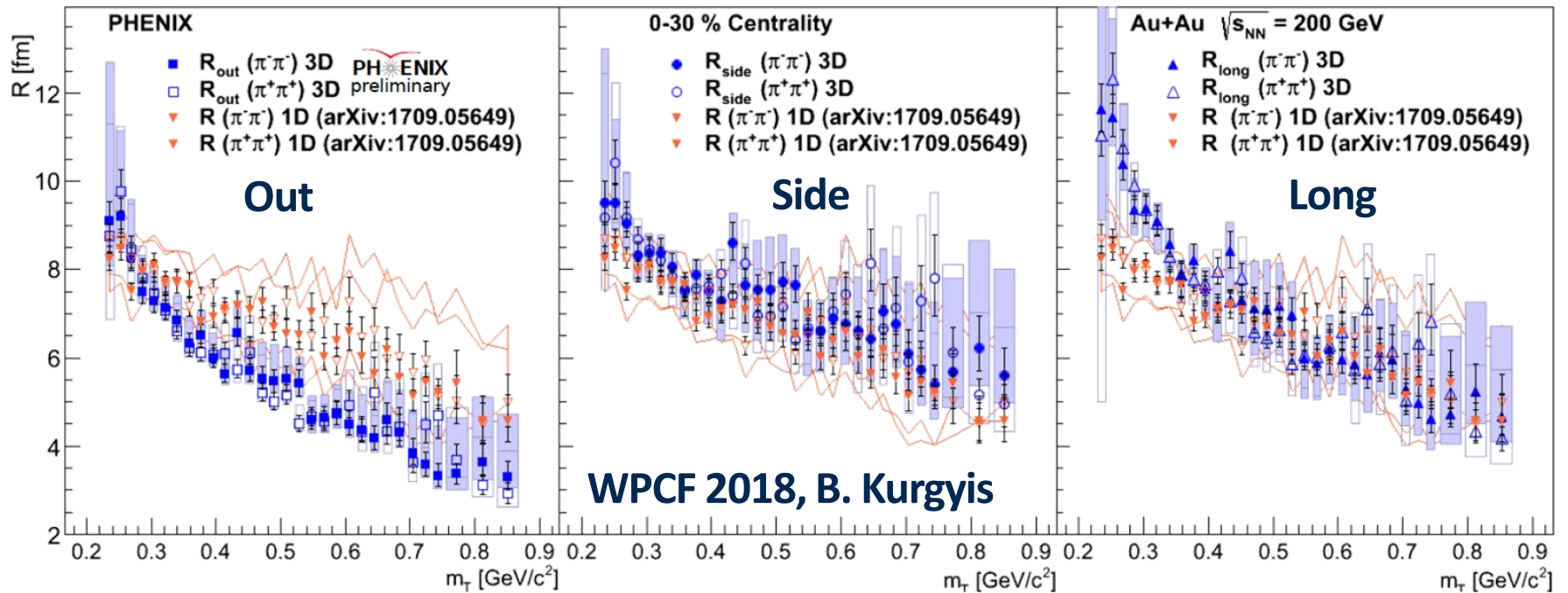
$$q_{inv} \equiv q_{PCMS} \approx q_{LCMS} \cdot \sqrt{1 - \beta_T^2/3} \qquad R_{PCMS} \approx R_{LCMS} \cdot \sqrt{\frac{1 - 2\beta_T^2/3}{1 - \beta_T^2}}$$

Shape of the correlation function

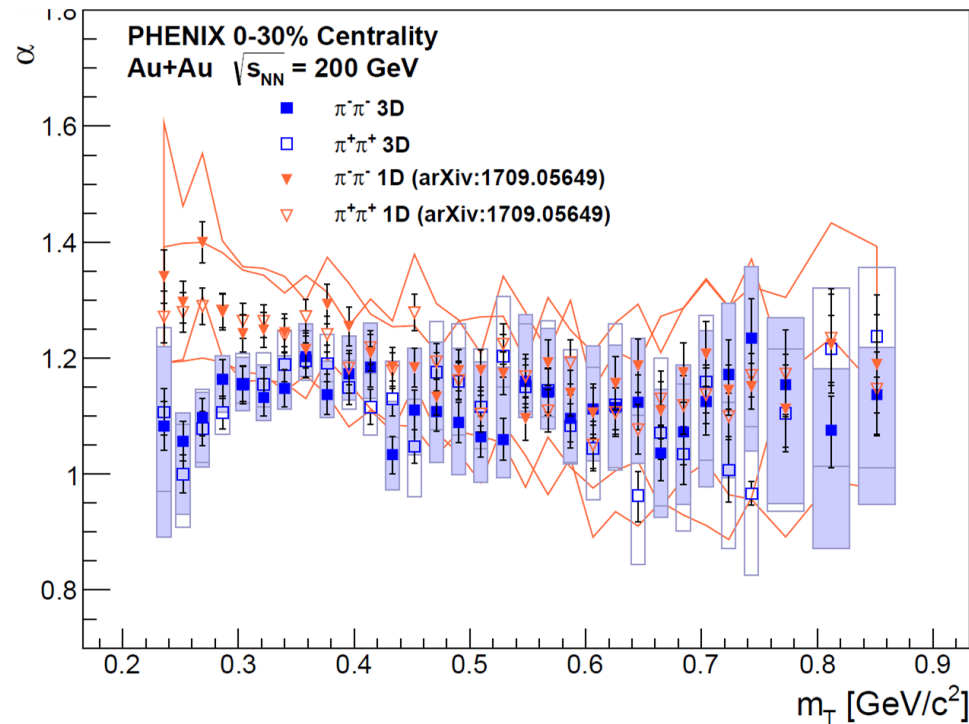


Cross-check with 3D analysis – PHENIX preliminary

$$C(Q) = \left(1 - \lambda + \lambda \cdot K(q_{inv}; \alpha, R_{inv}) \cdot \left(1 + e^{-|R_o^2 q_o^2 + R_s^2 q_s^2 + R_l^2 q_l^2|^{\alpha/2}} \right) \right) \cdot N \cdot (1 + \varepsilon Q)$$



Cross-check with 3D analysis – PHENIX preliminary



- **Compatible with 1D (Q_{LCMS}) measurement** of Phys. Rev. C 97, 064911 (2018)
- Small discrepancy at small m_T : due to large R_{long} at small m_T ?

3D Gaussian vs 1D Levy

- **Angle averaged 3D Gaussian \neq 1D Levy!**
 - Difference: several percent
 - Available experimental precision: much better than this difference

