Renormalization in quantum gravity

Sándor Nagy

University of Debrecen, Department of Theoretical Physics, Hungary

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Outline

- functional renormalization group method
  - motivation
  - global renormalization
  - effective action, 1-loop
  - effective average action, Wetterich equation
  - Wegner-Houghton equation
  - applications
    - fixed points
    - $O(N)$ model
    - sine-Gordon model
    - Gross-Neveu model
    - nonlinear σ model

Outline

- gravity
  - classical gravity
  - quantum gravity (?)
  - truncations
  - evolution equations
  - phase space
  - conformally reduced gravity
  - region of general relativity, trajectory chosen by nature
  - GFP scaling
  - IR scaling

Motivation

- Can we understand the macroscopic behavior from knowing the microscopic interactions?
- When do we need RG? see hydrodynamics
- Many scales are important, e.g. critical phenomena

Landau ϕ^4 model
describing the ferromagnetic behavior with free energy

$$F = V\left(\frac{1}{2}g_2\phi^2 + \frac{1}{24}g_4\phi^4\right)$$

ϕ – magnetization, V – volume, g_2 and g_4 – parameters.

- $g_4 > 0$
- $g_2 > 0$ trivial minimum
- $g_2 < 0$ nontrivial minimum at ϕ_0

it can describe the phase transition but the exponents are wrong.

Motivation

– an improve: **Ginsburg-Landau theory**, $\phi(x)$

$$F = \int d^3x \left((\nabla\phi(x))^2 + \frac{1}{2} g_2 \phi^2(x) + \frac{1}{24} g_4 \phi^4(x) - j(x)\phi(x) \right)$$

– new terms: kinetic energy, source

– higher orders in ϕ and ∇ (gradient expansion)

it does not help:

– the improvement modifies only the small fluctuations

solution

– we introduce scale dependent g_2 and g_4

$$F = \int d^3x \left((\nabla\phi(x))^2 + \frac{1}{2} g_2(k) \phi^2(x) + \frac{1}{24} g_4(k) \phi^4(x) - j(x)\phi(x) \right)$$

k actual energy scale

– it gives correct exponents

Motivation

The renormalization group method is useful in many areas in modern physics, with varying energy scale

UC: 10^{-7} meV, ultracold atoms

CM: 1 eV for conduction electrons in solids
origin of RG: Kadanoff blocking for spin systems

QCD: 1 GeV for QCD, comparable with lattice simulations

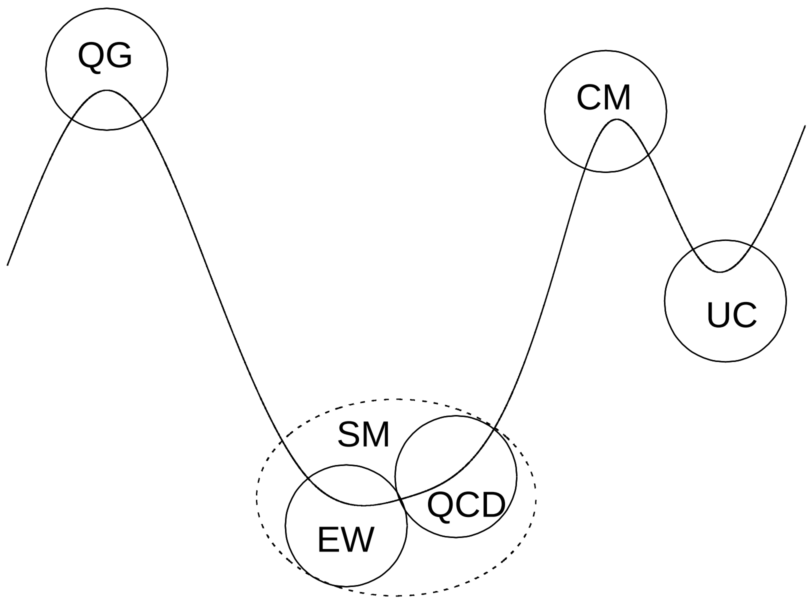
EW: 125 GeV, Higgs particle, electroweak theory

QG: At the Planck scale, at 10^{19} GeV, we expect the enter of gravity

RG can

- map out phase structures of models
- find relations among models: global picture of RG

Global renormalization



- problem of triviality
- immanent vs transcendent
- perturbative vs functional RG

Actions

ingredients of RG

- QFT
- action(s), relations
- small parameter

action	blocked	effective	effective average
symbol	S_Λ	Γ	Γ_k
variable	field	average for Ω	average for k^{-1}
limits	$\lim_{k\to\infty}\Gamma_k$	$\lim_{k\to 0}\Gamma_k$	–
couplings	bare	–	renormalized
RG	Wegner-Houghton	–	Wetterich

Effective action

– action for a scalar field

$$S[\phi] = \int_x \mathcal{L}, \ \ \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi), \quad V(\phi) = \frac{1}{2}m^2\phi^2 - \frac{g}{24}\phi^4$$

– the minimum configuration is at $\frac{\partial V}{\partial \phi}\Big|_{\phi=\phi_0} = 0$

– generating functional (Minkowski)

$$Z[J] = e^{iW[j]} \int \mathcal{D}\phi^{iS[\phi]+iJ\cdot\phi}$$

– the expectation value of ϕ is the classical field ϕ_c: $\frac{\delta W}{\delta J} = \langle 0|\phi|0\rangle \equiv \phi_c$

– the effective action is

$$\Gamma[\phi_c] = W[J] - J \cdot \phi_c$$

– Legendre transform variables $J \leftrightarrow \varphi$, furthermore

$$\frac{\delta \Gamma}{\delta \phi_c(x)} = -J(x) \ \text{ if } \ J(x) = 0 \ \to \phi_c = const, \quad \frac{d\Gamma[\phi_c]}{d\phi_c]}\Bigg|_{\langle\phi\rangle} = 0$$

Effective action

– expansion of the effective action with vertex functions

$$\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int dx_1 \dots dx_n \Gamma(n)(x_1, \dots, x_n)\phi_c(x_1)\dots\phi_c(x_n)$$

– gradient expansion (translational invariance $\rightarrow$ momentum conservation)

$$\Gamma[\phi_c] = \int_x [-U(\phi_c(x)) + \frac{1}{2}(\partial_\mu \phi_c(x))^2 Z(\phi_c(x)) + \dots]$$

– U is the effective potential, if $\phi_c(x) = \phi_c \rightarrow \Gamma[\phi_c(x)] = -\Omega U(\phi_c)$ and

$$U(\phi_c) = -\sum_n \frac{1}{n!}\phi_c^n \Gamma^{(n)}(p_i = 0)$$

– the vertices have connections to the physical quantities

$$\Gamma^{(2)}(p_i = 0) = \frac{\partial^2 U}{\partial^2 \varphi} = m^2 \quad i\Gamma^{(4)}(p_i = 0) = -i\frac{\partial^4 U}{\partial^4 \varphi} = g$$

m^2 : physical mass, g : renormalized coupling constant

1-loop effective potential

– expanding the action around ϕ_0 the minimum of the potential V

$$S[\phi] = S[\phi_0] + \eta \cdot J - \frac{1}{2}\eta \cdot (\Box + V''(\phi_0)) \cdot \eta$$

– integration over the fluctuations around the saddle point

$$\begin{aligned} e^{iW} &= e^{iS[\phi_0]} \int \mathcal{D}\eta e^{-\frac{i}{2}\eta \cdot (\Box + V''(\phi_0)) \cdot \eta} \\ &= e^{iS[\phi_0]} [\det(\Box + V''(\phi_0))]^{-1/2} \end{aligned}$$

– giving

$$W[J] = S[\phi_0] + \phi_0 \cdot J + \frac{i}{2} \mathrm{Tr} \ln[\Box + V''(\phi_0)]$$

– relating $S[\phi_c]$ and $S[\phi_0]$, $\phi_0 = \phi_c - \phi_1$

$$\begin{aligned} S[\phi_0] &= S[\phi_c - \phi_1] = S[\phi_c] - \phi_1 \cdot \left.\frac{\delta S}{\delta \phi}\right|_{\phi_c} + \mathcal{O}(\hbar^2) \\ S[\phi_0] &= S[\phi_c] + \mathcal{O}(\hbar^2) \end{aligned}$$

when $J \to 0$

1-loop effective potential

effective action

$$\Gamma[\phi_c] = S[\phi_c] + \frac{i}{2}\mathrm{Tr}\ln[\Box + V''(\phi_c)]$$

effective potential (ϕ_c =const.)

$$\begin{aligned} U(\phi_c) &= V(\phi_c) - \frac{i}{2\Omega}\mathrm{Tr}\ln[\Box + V''(\phi_c)] \\ &= V(\phi_c) + \frac{1}{2}\int_k \ln[k^2 + V''(\phi_c)] \end{aligned}$$

Euclidean case

$$U(\phi_c) = V(\phi_c) + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \ln\left[k^2 + m^2 - \frac{g\phi_c^2}{2}\right]$$

- $m^2 > 0 \to \phi_c = 0$
- $m^2 < 0 \to \phi_c^2 = -\frac{6m^2}{g}$

Effective average action

– effective average action $\phi_c \to \phi$

$$e^{-\Gamma_k[\phi]} = \int \mathcal{D}\varphi \prod_x \delta(\phi_k(x) - \phi(x)) e^{-S[\varphi]}$$

restriction to the integration variable

– the constraint

$$\phi_k(x) = \frac{1}{V_k} \int_{V_k} d^d y \varphi(x+y)$$

1. is the average of φ over a volume $V_k \sim k^{-d}$
2. transition from the microscopic φ to macroscopic ϕ

– similarity to the Ising model

$$Z = \prod_m \int ds_m 2\delta(s_m^2 - 1) e^{K \sum_n \sum_i s_n s_{n+i}}$$

– creating a Gaussian model (from discrete to continuous spin variable)

$$Z = \prod_m \int ds_m 2 e^{-\frac{1}{2} b s_m^2} e^{K \sum_n \sum_i s_n s_{n+i}}$$

Effective average action

– scale dependent effective potential

$$e^{-\Omega U(\phi])} = \int \mathcal{D}\varphi \prod_x \delta(\phi_k(x) - \phi) e^{-S[\varphi]}$$

– similarly to the Ising $\rightarrow$ Gaussian models

$$\prod \delta(\phi_k(x) - \phi(x)) \; \rightarrow \; e^{-\int_x [\nu(\phi_k(x) - \phi(x))]}$$

– some fluctuations of ϕ_k around ϕ is allowed in order to guarantee continuous description
– we define a continuous version of the average field

$$\phi(x)_k = \int_y f_k(x-y)\varphi(x), \qquad f_k(x) = \pi^{-d/2} k^d e^{-k^2 x_\mu x_\mu}$$

– it decreases rapidly if $|x - y| > k^{-1}$
– in momentum space

$$\phi(x) = \sum_q f_k(q)\varphi(q) e^{-iq^\mu x_\mu} \qquad \phi_q = f_k(q)\varphi(q)$$

Effective average action

– $f_k(q)$ should obey

$$
\begin{aligned}
&0 < f_k(q) \leq 1 \\
&f_k(q) < 1 \ \text{ for } \ q^2 > 0 \\
&\lim_{k\to\infty} f_k(q) = 1 \\
&\lim_{k\to 0} f_k(q) = 0 \ \text{ for } \ q \neq 0
\end{aligned}
$$

– example

$$
f_k(q) = e^{-a(q^2/k^2)^\beta}
$$

$\beta = 1$ gives the Gaussian curve in coordinate space

– the averaging gives an extra term to the action

$$
S \ \rightarrow \ S + \mathcal{R}_k[\varphi], \quad \mathcal{R}_k[\varphi] = \frac{1}{2}\varphi_q \frac{q^2 f_k^2(q)}{1 - f_k^2(q)} \varphi_q \equiv \frac{1}{2}\varphi_q \mathcal{R}_k(q^2)\varphi_q
$$

Effective average action

properties

- $$\lim_{k\to 0}\Gamma_k = \Gamma$$

  the average action goes to the effective action when $k \to 0$
- the average action is not convex, therefore it is more suitable to discuss symmetry breaking ($\phi_0 \neq 0$)
- $$\lim_{k\to\infty}\Gamma_k = S$$

  the average action goes to the classical action when $k \to \infty$
- if the Λ dependence cannot be removed (non-renormalizable theories), then

  $$\Gamma_k - S = \frac{1}{2}\mathrm{Tr}\ln[(S'' + \mathcal{R})M^{-2}]$$

  1-loop type difference, does not vanish

Wetterich equation

– can we derive a differential equation for Γ_k, which evolves from S to Γ?

– differentiating the modified generating functional w.r.t k

$$\dot{Z}_k = -\dot{W}_k[J]e^{-W_k[J]} = \int \mathcal{D}[\phi] \left(-\dot{\mathcal{R}}_k[\phi]\right) e^{-\frac{1}{\hbar}(S_\Lambda + \mathcal{R}_k[\phi] - J\cdot\phi)}$$

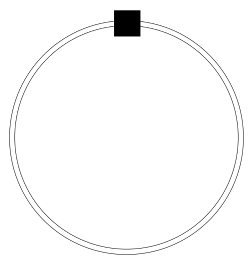
with $\dot{} = k\partial/\partial k$

$$\dot{W}_k[J] = \frac{1}{2} e^{\frac{1}{\hbar}W_k[J]} \int \dot{\mathcal{R}}_k \left(-\hbar \frac{\delta^2 W_k[J]}{\delta J^2} + \frac{\delta W_k[J]}{\delta J}\frac{\delta W_k[J]}{\delta J}\right) e^{\frac{1}{\hbar}W_k[J]}$$

– using $\phi = \frac{\delta W_k[J]}{\delta J}$, $\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi}\frac{\delta^2 W_k[J]}{\delta J \delta J} = -1$, and $\Gamma_k[\phi] + \mathcal{R}_k[\phi] \to \Gamma_k$

– the **Wetterich equation** is

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\frac{\dot{\mathcal{R}}_k}{\mathcal{R}_k + \Gamma_k''} = \frac{1}{2}$$



Wetterich equation

– gradient expansion

$$\Gamma_k = \int d^d x \left[U_k(\phi_x) + \frac{1}{2} Z_k(\phi_x)(\partial_\mu \phi_x)^2 + H_1(\phi_x)(\partial_\mu \phi_x)^4 + H_2(\phi_x)(\Box \phi_x)^2 + \ldots \right]$$

– evolution equations

$$\dot{U}_k = \frac{1}{2} \int_p \frac{\dot{\mathcal{R}}_k}{Z_k p^2 + \mathcal{R}_k + \tilde{V}_k''}$$

– if $Z_k(\phi, p) = Z_k(\phi) \equiv Z_k$ then

$$\begin{aligned}
\dot{Z}_k \quad &= \quad \frac{1}{2} \int_p \dot{\mathcal{R}}_k \Big[- \frac{Z_k''}{[p^2 Z_k + \mathcal{R}_k + V_k'']^2} + \frac{\frac{2}{d} Z_k'^2 p^2 + 4 Z_k' (Z_k' p^2 + V_k''')}{(p^2 Z_k + \mathcal{R}_k + V_k'')^3} \\
&+ \frac{\frac{8}{d} p^2 (Z_k' p^2 + V_k''')^2 + \Big(Z_k + \partial_{p^2} \mathcal{R}_k \Big)^2}{(p^2 Z_k + \mathcal{R}_k + V_k'')^5} \\
&- 2 \frac{\left(Z_k' p^2 + V_k'''\right)^2 \Big(Z_k + \partial_{p^2} \mathcal{R}_k + \frac{2}{d} p^2 \partial^2_{p^2} \mathcal{R}_k \Big)}{(p^2 Z_k + \mathcal{R}_k + V_k'')^4} \\
&- \frac{\frac{2}{d} Z_k' p^2 \left(Z_k' p^2 + V_k''' \right) \Big(Z_k + \partial_{p^2} \mathcal{R}_k \Big)}{(p^2 Z_k + \mathcal{R}_k + V_k'')^4} \Big]
\end{aligned}$$

Wetterich equation

$$U_k = \sum_n \frac{g_{2n}}{(2n)!} \phi^{2n}$$

– the beta funtions: $\dot{g}_i = \beta_i(g_j, k)$

– in LPA

$$\beta_i(g_j, k) = \partial_\phi^i \left(\frac{1}{2} \int_p \frac{\dot{\mathcal{R}}_k}{p^2 + \mathcal{R}_k + V_k''} \right) \Bigg|_{\phi=0}$$

$$\begin{aligned}
\beta_2 &= -\int_p \frac{\dot{R}_k g_4}{(p^2 + R_k + g_2)^2} \\
\beta_4 &= \int_p \dot{R}_k \left(\frac{6g_4^2}{(p^2 + R_k + g_2)^3} - \frac{g_6}{(p^2 + R_k + g_2)^2} \right) \\
\beta_6 &= \int_p \dot{R}_k \left(-\frac{90g_4^3}{(p^2 + R_k + g_2)^4} + \frac{30g_4 g_6}{(p^2 + R_k + g_2)^3} - \frac{g_8}{(p^2 + R_k + g_2)^2} \right) \\
&\vdots
\end{aligned}$$

Wetterich equation

– to get the phase structure and fixed points we need dimensionless quantities
– the action is dimensionless if $\hbar = 1$:

$$\left[\int d^d x (\partial_\mu \phi)^2\right] = 0$$

– the dimension of the field variable

$$-d + 2 + 2[\phi] = 0 \;\; \rightarrow \;\; [\phi] = \frac{d-2}{2} \;\; \rightarrow \;\; \phi = k^{(d-2)/2}\tilde{\phi}$$

$\tilde{\mathcal{O}}$ denotes dimensionless quantities
– the potential is

$$\left[\int d^d x U_k\right] = 0 \;\; \rightarrow \;\; [U_k] = d$$

– its derivative

$$\begin{aligned}
\dot{U}_k &= k\partial_k U_k[\phi] = k\partial_k(k^d \tilde{U}_k[\tilde{\phi}]) = dk^d \partial_k \tilde{U}_k + k(\partial_k \tilde{\phi})\partial_{\tilde{\phi}} \tilde{U}_k + k^d k \partial_k \tilde{U}_k \\
&= dk^d \partial_k \tilde{U}_k + k(\partial_k k^{-(d-2)/2} \phi)\partial_{\tilde{\phi}} \tilde{U}_k + k^d k \partial_k \tilde{U}_k \\
&= k^d (d - \frac{d-2}{2} \tilde{\phi} \partial_{\tilde{\phi}} + k\partial_k) \tilde{U}_k
\end{aligned}$$

Wetterich equation

– the dimensionless potential is

$$\tilde{U}_k = \sum_n \frac{1}{(2n)!} \tilde{g}_{2n} \tilde{\phi}^{2n}$$

– the equations for the couplings

$$\begin{aligned}
(d - 2n\frac{d-2}{2} + k\partial_k)\tilde{g}_{2n} &= \beta_{2n}(\tilde{g}_i) \\
\dot{\tilde{g}}_{2n} &= (-d + n(d-2))\tilde{g}_{2n} + \beta_{2n}(\tilde{g}_i)
\end{aligned}$$

– concretely

$$\begin{aligned}
\dot{\tilde{g}}_2 &= -2\tilde{g}_2 + \beta_2(\tilde{g}_i) \\
\dot{\tilde{g}}_4 &= (d-4)\tilde{g}_4 + \beta_4(\tilde{g}_i) \\
\dot{\tilde{g}}_6 &= (2d-6)\tilde{g}_6 + \beta_6(\tilde{g}_i) \\
&\vdots
\end{aligned}$$

Wetterich equation

– introducing the dimensionless $\tilde{\beta}$ functions

$$\dot{\tilde{g}}_i = -d_i \tilde{g}_i + \beta_i(\tilde{g}_j) \equiv \tilde{\beta}_i(\tilde{g}_j)$$

– the connection between dimensionful and dimensionless couplings

$$\tilde{g}_i = k^{-d_i} g_i$$

dimensionless Wetterich equation

– dimensionless regulator

$$\begin{aligned} R &= p^2 r, \;\; \frac{R}{k^2} = yr \\ \dot{r} &= = r'(-2y) \;\rightarrow\; \dot{R}_k = -2p^2 y r' \end{aligned}$$

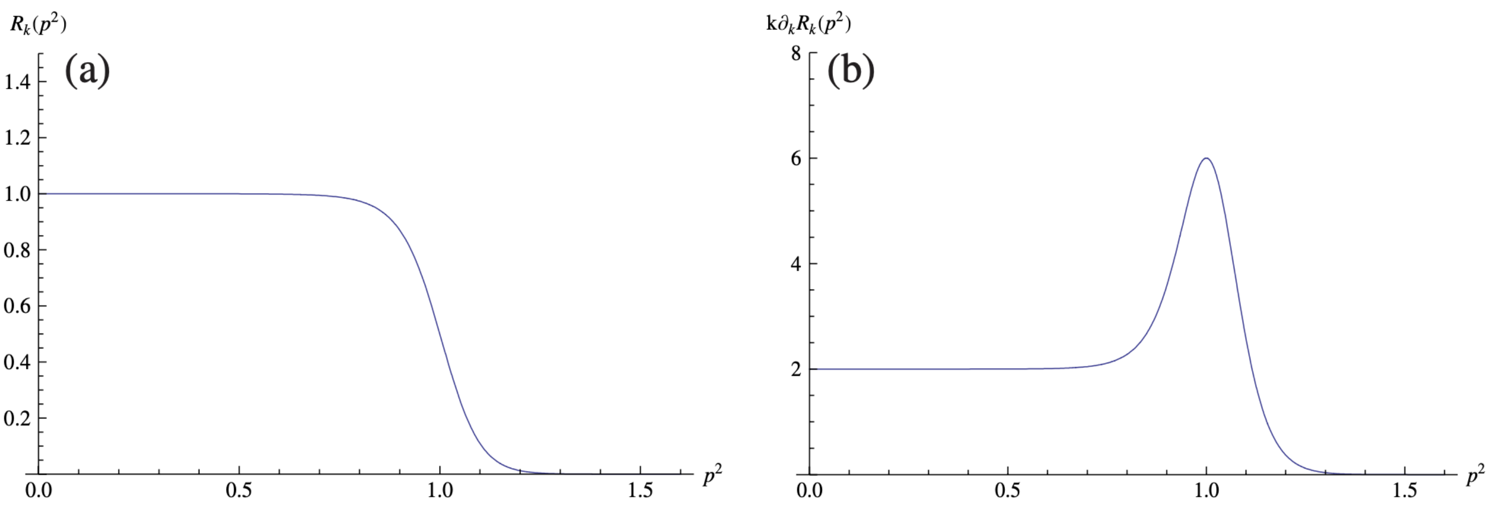
$$\begin{aligned} \dot{V}_k &= \frac{1}{2}\int_p \frac{\dot{\mathcal{R}}_k}{p^2 + \mathcal{R}_k + V_k''} = \frac{1}{4}\alpha_d k^d \int_y y^{d/2-1} \frac{-2p^2 y r'}{p^2 + \mathcal{R}_k + V_k''} \\ &= -\frac{1}{2}\alpha_d k^d \int_y y^{d/2+1} \frac{r'}{y(1+r) + \tilde{V}_k''} \end{aligned}$$

Wetterich equation

regulator properties

1. $\lim_{p^2/k^2 \to 0} R_k > 0$, i.e. removes IR divergences
2. $\lim_{k^2/p^2 \to 0} \mathcal{R}_k \to 0$, the effective action limit
3. $\lim_{k^2 \to \infty} \mathcal{R}_k \to \infty$, the classical action limit

regulator form



Wetterich equation

typical regulators

– css regulator

$$r_{css} = \frac{s_1}{\exp[s_1 y^b/(1-s_2 y^b)] - 1}\theta(1-s_2 y^b)$$

where $y = p^2/k^2$, $b \geq 1$ and s_1, s_2 positive parameters

– limits

$$\begin{aligned}
\lim_{s_1\to 0} r_{css} &= \left(\frac{1}{y^b} - s_2\right)\theta(1-s_2 y^b) \ \text{Litim} \\
\lim_{s_1\to 0, s_2\to 0} r_{css} &= \frac{1}{y^b} \ \text{power law} \\
\lim_{s_2\to 0} r_{css} &= \frac{s_1}{\exp[s_1 y^b] - 1} \ \text{exponential}
\end{aligned}$$

– Litim regulator

$$\begin{aligned}
r &= \quad = \left(\frac{1}{y} - 1\right)\theta(1-y) \\
r' &= \frac{dr}{dy} = -\frac{1}{y^2}\theta(1-y) + \left(\frac{1}{y} - 1\right)\delta(1-y)
\end{aligned}$$

Wetterich equation

– the Litim regulator is optimized, it provides the fastest evolution

– the evolution of the effective potential $V_k (== U_k)$ is

$$d - \frac{d-2}{2}\tilde{\phi}\partial_{\tilde{\phi}} + k\partial_k \tilde{V}_k = -\alpha_d k^d \int_y y^{d/2+1} \frac{-\frac{1}{y^2}\theta(1-y)}{1+\tilde{V}_k''}$$
$$= \alpha_d \frac{2}{d} \frac{1}{1+\tilde{V}_k''}$$

– evolution equations in $d = 3$

$$\tilde{\beta}_2 = -2\tilde{g}_2 - \frac{\tilde{g}_4}{4\pi^2(1+\tilde{g}_2)^2},$$
$$\tilde{\beta}_4 = -\tilde{g}_4 + \frac{3\tilde{g}_4^2}{2\pi^2(1+\tilde{g}_2)^3}$$

Wegner-Houghton equation

– the blocking in momentum space, the lowering $k \to k - \Delta k$ of the UV cutoff, is supposed to preserve the generating functional

$$Z = \int D[\phi] e^{-S_k[\phi]}$$

– $S_{k-\Delta k}$ is found by integrating out the modes with wave vector $k - \Delta k < |p| < k$

$$e^{-S_{k-\Delta k}[\phi]} = \int D[\varphi] e^{-S_k[\phi+\varphi]}$$

for $\phi(x)$: $|p| < k - \Delta k$, for $\varphi(x)$: $k - \Delta k < |p| < k$

– we expand the exponent in φ

$$e^{-S_{k-\Delta k}(\phi)} \approx \int D[\varphi] e^{-S_k[\phi] - \frac{1}{2}\varphi S_k''[\phi]\varphi}$$

– after performing the Gaussian integral we get

$$e^{-\hbar S_{k-\Delta k}(\phi)} = e^{-S_k[\phi] - \frac{1}{2}\ln\det S_k''[\phi]}$$

Wegner-Houghton equation

– **Wegner-Houghton equation**

$$\dot{S}_k[\phi] = -\frac{k}{2\Delta k}\mathrm{Tr}\ln S''_k[\phi]$$

– the Tr is performed on a shell of thickness Δk, $|q| = k$,

$$\dot{S}_k[\phi] = -\frac{k}{2\Delta k}\Delta k \int_q \delta(q-k)\ln S''_q[\phi]$$

– usually

$$S''_q = D_q^{-1} = q^2 + U''$$

– after performing the integral we get the WH equation

$$\dot{U} = -\frac{1}{2}\alpha_d k^d \ln(k^2 + U'')$$

– with

$$\alpha_d = \frac{\Omega_d}{(2\pi)^d}, \;\; \Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

WH – Wetterich comparison

1. S_k contains the bare couplings at $k(=\Lambda)$ and the WH equation can tell their values at $k-\Delta k$, starting from Λ to 0
2. Γ_k contains renormalized couplings at k and the Wetterich equation gives Γ at $k=0$, therefore the Wetterich equation gives a chain of effective average actions with different regulator parameter k
3. in principle the effective action does not depend on the regulator, however the effective average action does
4. the regulator modifies the propagator, therefore the original dispersion relations change, the regulator dependence is the strongest at k
5. the regulator dependence should be checked
6. WH equation $\to$ LPA (nonlocality?)
7. Wetterich equation is compatible with the gradient expansion

Fixed points

– usually no analytic solutions $\rightarrow$ looking for fixed points

– the fixed point equations are

$$\dot{\tilde{g}}_i = 0$$

– we linearize the RG equations around the fixed point and get

$$\dot{y}_i = M_{ij} y_j$$

with $y_i = \tilde{g}_i - \tilde{g}_i^*$ and the matrix

$$M_{ij} = \frac{\partial \tilde{\beta}_i}{\partial \tilde{g}_j}$$

– after diagonalizing S: $S_{ik}^{-1} M_{kl} S_{ln} = \delta_{in} s_n$, and introducing $z_i = S_{ik}^{-1} y_k$

$$\dot{z}_i = s_i z_i$$

– its solution reads as

$$z_i = z_i(k_\Lambda) e^{s_i t} = z_i(0) \left(\frac{k}{k_\Lambda} \right)^{s_i}$$

Fixed points

assuming that there is two couplings, the possibilities are ($k \to \infty$, UV)

1. the eigenvalues are real, $s_1, s_2 \in \mathbb{R}$ and they are negative, $s_1, s_2 < 0$
   the trajectory approaches the fixed point, **attractive fixed point**
2. $s_1, s_2 \in \mathbb{R}$ and $s_1, s_2 > 0$
   the trajectory goes away from the fixed point, **repulsive fixed point**
3. $s_1, s_2 \in \mathbb{R}$ and with opposite signs
   a direction flows into the fixed point, another one is repelled, **hyperbolic point or a saddle point**
4. complex eigenvalues, $s_1, s_2 \in \mathbb{C}$, complex conjugate pairs
   if $\Re s_1, \Re s_2 < 0$, **attractive focal point**
5. $s_1, s_2 \in \mathbb{C}$ and $\Re s_1, \Re s_2 > 0$
   Then the trajectory is repelled by the fixed point, **repulsive focal point**
6. $s_1, s_2 \in \mathbb{C}$ and the real part is zero
   elliptic point, a specific form of a limit cycle.

Truncation and fixed points

– the d-dimensional dimensionless potential $\tilde{V}_k$ with Litim regulator ($\tilde{V} \sim \tilde{U}$)

$$\dot{\tilde{V}}_k = -d\tilde{V}_k + \frac{d-2}{2}\tilde{\phi}\partial_{\tilde{\phi}}\tilde{V}_k + \alpha_d \frac{2}{d}\frac{1}{1+\tilde{V}_k''}$$

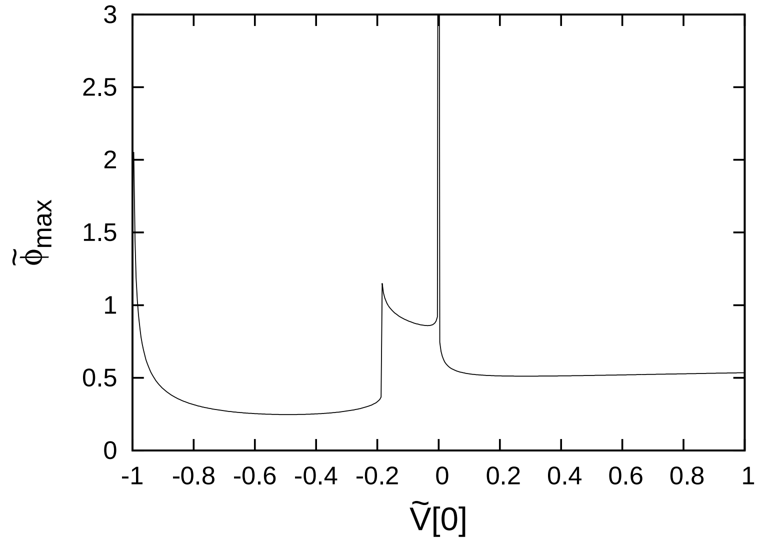
– the fixed point equation is $\dot{\tilde{V}}^* = 0$, which provides the fixed point potential

– usually $\tilde{V} \to \infty$ at $\tilde{\phi}_{max}$, but if $\tilde{\phi}_{max} \to \infty$ then we get a fixed point potential

– we solve the equation with the initial conditions $\tilde{V}'[0] = 0$ and $\tilde{V}[0]$, parameterized as

$$\tilde{V}[0] \equiv a(x) = -\frac{2\alpha_d}{d^2}\frac{1}{1+x}$$

- $d = 3$
- GFP at $a = 0$
- Wilson-Fisher, $a < 0$
- IR, when $a \to -1$?



Gaussian fixed point

– GFP corresponds to the origin of the theory space, $\tilde{g}_i^* = 0$

– a free theory for massless particles

– the linearization of the flow equations in the vicinity of the GFP

– Taylor expansion of β functions around the origin

$$\tilde{\beta}_i = -d_i \tilde{g}_i + a_i \tilde{g}_i + a_{ijk} \tilde{g}_j \tilde{g}_k \ldots$$

– the matrix M is

$$M_{ij} = -d_{ij} + a_{ij}$$

– it turns out that $a_{ij} = 0$, when $i < j$, $\rightarrow$

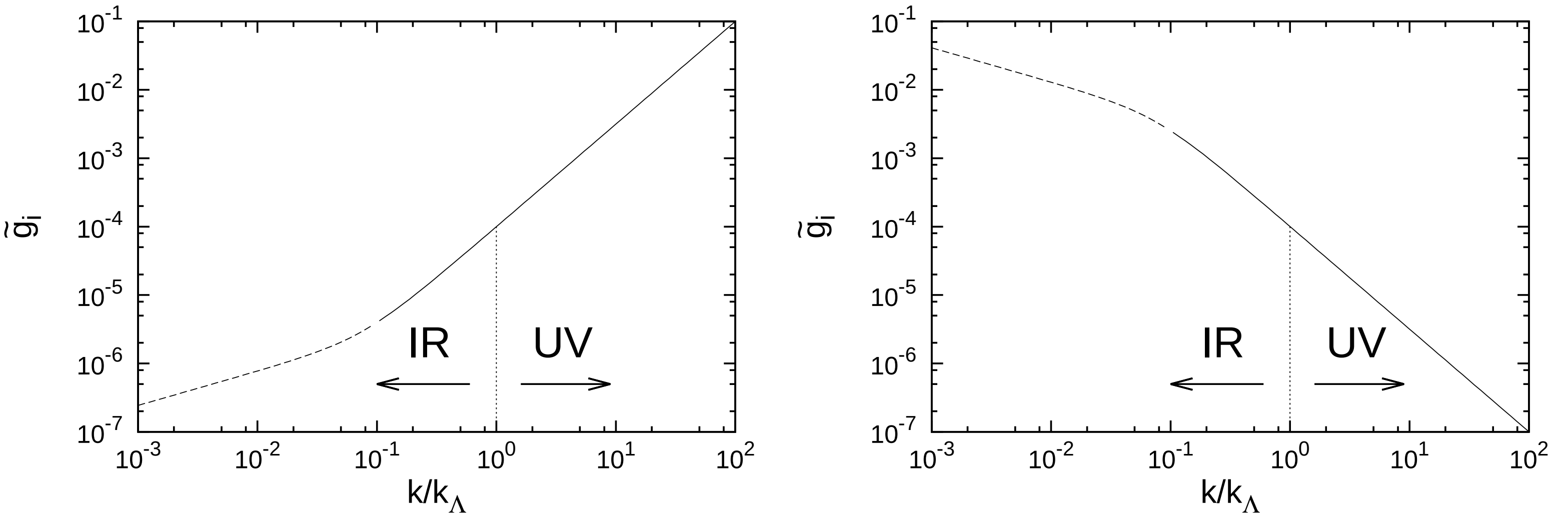
$$s_i = -d_i$$

– the eigenvalues are real in the GFP

Gaussian fixed point

– if $s_i > 0 \to d_i < 0$ then $z_i \to \infty$ implying that $\tilde{g}_i \to \infty$ so the trajectory is repelled by the fixed point

– if $s_i < 0$ then trajectory is attracted by the fixed point



- **relevant**: increases in the IR, decreases in the UV
- **irrelevant**: decreases in the IR, increases in the UV
- **asymptotic freedom**: all couplings are relevant in the GFP
- **asymptotic safety**: all couplings are relevant in the NGFP

$O(N)$ **model**

usage of d-dimensional $O(N)$ model

$N = 0$ polymers,

$N = 1$ liquid-vapour transition, or uniaxial (Ising) ferromagnets,

$N = 2$ He^2 superfliud phase transition,

$N = 3$ Heisenberg ferromagnets,

$N = 4$ chiral phase transition for two quark flavors.

The 3d $O(1)$ or 3d ϕ^4 model with power law regulator ($\mathcal{R} = p^2(k^2/p^2)^b$) at $b = 1$ ($\mathcal{R} = k^2$, Callan-Symanzik)

$$\dot{V} = -\frac{k^2}{4\pi}\sqrt{k^2 + V''}$$

β functions in their dimensionless forms are

$$\begin{aligned}\tilde{\beta}_2 &= -2\tilde{g}_2 - \frac{\tilde{g}_4}{8\pi(1+\tilde{g}_2)^{1/2}}, \\ \tilde{\beta}_4 &= -\tilde{g}_4 + \frac{3\tilde{g}_4^2}{16\pi(1+\tilde{g}_2)^{3/2}}\end{aligned}$$

$O(N)$ model

– fixed point equations: $\tilde{\beta}_2 = 0$ and $\tilde{\beta}_4 = 0$
– the model has two fixed points

- the derivative or stability matrix

$$M = \begin{pmatrix} -2 + \frac{\tilde{g}_4}{16\pi(1+\tilde{g}_2)^{3/2}} & -\frac{1}{8\pi(1+\tilde{g}_2)^{1/2}} \\ -\frac{9\tilde{g}_4^2}{32\pi(1+\tilde{g}_2)^{5/2}} & -1 + \frac{3\tilde{g}_4}{8\pi(1+\tilde{g}_2)^{1/2}} \end{pmatrix}$$

- GFP: $\tilde{g}_2^* = \tilde{g}_4^* = 0$

$$M_{\tilde{g}_2^*=0,\tilde{g}_4^*=0} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

so $s_1 = -2$ and $s_2 = -1$

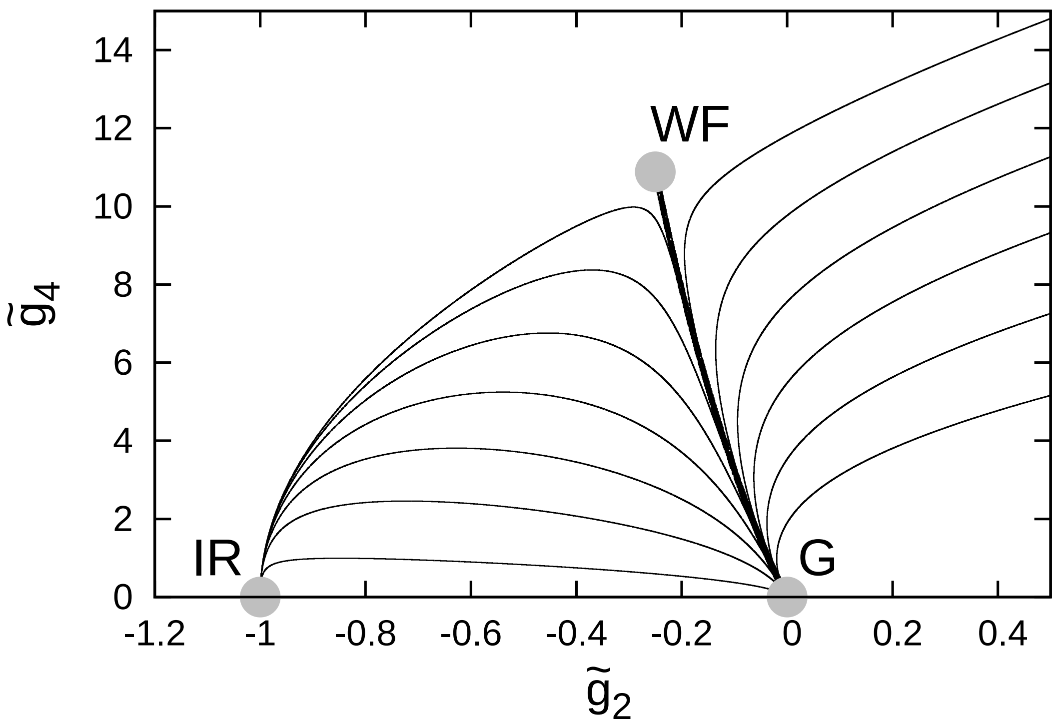
- Wilson-Fisher fixed point (WFFP): $\tilde{g}_2^* = -1/4$ and $\tilde{g}_4^* = 2\sqrt{3}\pi$

$$M_{\tilde{g}_2^*=-1/4,\tilde{g}_4^*=2\sqrt{3}\pi} = \begin{pmatrix} -\frac{5}{3} & -\frac{1}{4\sqrt{3}\pi} \\ -4\sqrt{3}\pi & 1 \end{pmatrix}$$

– giving $s_1 = -2$ and $s_2 = 4/3$, a saddle point or a hyperbolic point

$O(N)$ model

- WFFP appears in ϕ^4 model in dimension $2 < d < 4$
- when $d \to 4$ then the WF fixed point tends to the origin and in $d = 4$ it melts into the GFP
- for the critical exponent ν, $\nu = -1/s_1 = 1/2$, mean field
- the 3d ϕ^4 model has two phases, $(\phi \leftrightarrow -\phi)$ is broken



The 2d sine–Gordon model

– its effective action contains a sinusoidal potential of the form

$$\Gamma_k = \int \left[\frac{z}{2}(\partial_\mu \phi)^2 + u \cos\phi\right]$$

where z is the field independent wave-function renormalization and u is the coupling.

– the RG evolution equations for the couplings are

$$\begin{aligned}
\dot{u} &= \frac{1}{2}\mathcal{P}_1 \int_p \dot{R} G \\
\dot{z} &= \frac{1}{2}\mathcal{P}_0 \int_p \dot{R}\Big[-Z'' G^2 + \left(\frac{2}{d} Z'^2 p^2 + 4 Z' V'''\right) G^3 \\
&\quad -2\left[V'''^2\left(\partial_{p^2} P + \frac{2}{d} p^2 \partial^2 P\right) + \frac{4}{d} Z' p^2 V''' \partial_{p^2} P\right] G^4 \\
&\quad + \frac{8}{d} p^2 V'''^2 \partial_{p^2} P^2 G^5\Big]
\end{aligned}$$

with $G = 1/(zp^2 + R + V'')$, $P = zp^2 + R$

– projections: $\mathcal{P}_1 = \int_\phi \cos(\phi)/\pi$ and $\mathcal{P}_0 = \int_\phi /2\pi$

The 2d sine–Gordon model

Symmetries

- Z_2
- periodicity

– the conditions imply that the the effective (dimensionful) potential is zero

– what does the RG method say?

– the linearized flow equation in LPA is

$$\dot{\tilde{u}} = \tilde{u}\left(-2 + \frac{1}{4\pi z}\right) + \mathcal{O}(\tilde{u}^2),$$

with any regulator

– the equation can be solved analytically

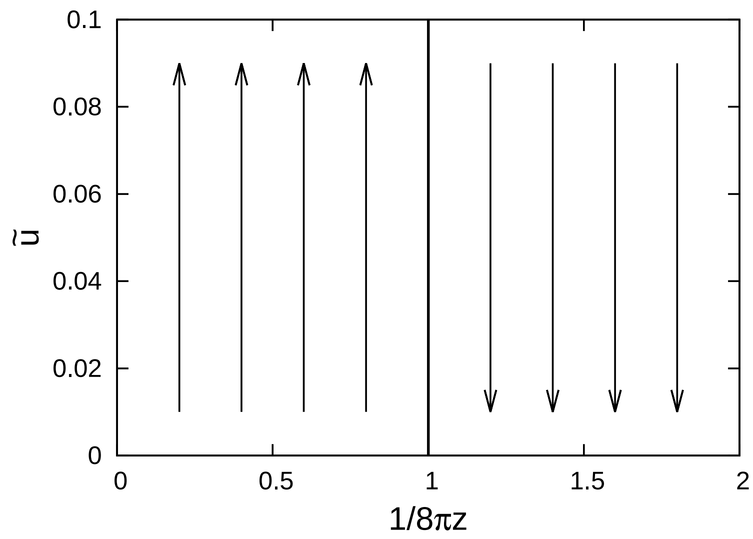
$$\tilde{u} = \tilde{u}(k_\Lambda)\left(\frac{k}{k_\Lambda}\right)^{\frac{1}{4\pi z}-2}.$$

– the fixed point solution is $\tilde{u}^* = 0$ and z^* arbitrary

The 2d sine–Gordon model

The SG model has **two phases**:

- $\frac{1}{z} > 8\pi \leftrightarrow$ **symmetric phase**. The coupling $\tilde{u}$ is irrelevant, the SG model is perturbatively nonrenormalizable
- $\frac{1}{z} < 8\pi \leftrightarrow$ (spontaneously) **broken** (symmetric) **phase**. The coupling $\tilde{u}$ is relevant, the SG model is perturbatively renormalizable



How one can distinguish the phases in the model?

$\Rightarrow$ The dimensionful coupling $\tilde{u}$ tends to zero, but the dimensionless one does not.

This idea can be generalized when we take into account the upper harmonics:

- **symmetric phase:**

$$\tilde{V}_{k\to 0}(\phi) = 0$$

- **broken phase:**

$$\tilde{V}_{k\to 0}(\phi) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(n\phi)}{n^2} = -\frac{1}{2}\phi^2, \quad \phi \in [-\pi, \pi]$$

a concave function, which is repeated periodically in the field variable.

Local potential approximation

The 'exact' evolution equation is

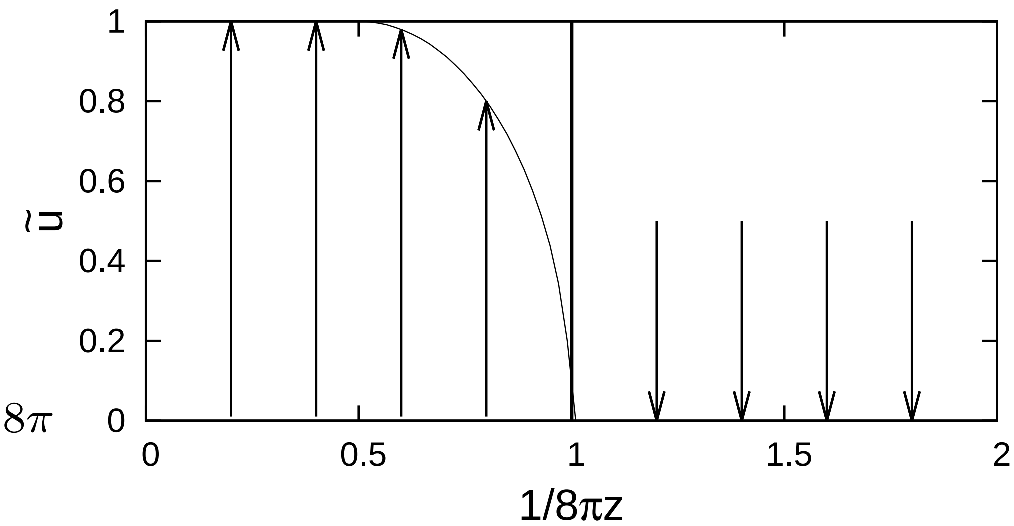
$$\dot{\tilde{u}} = -2\tilde{u} + \frac{1}{2\pi\tilde{u}z}\left[1 - \sqrt{1-\tilde{u}^2}\right],$$

with a constant z. The fixed points are

$$\tilde{u}^* = 1, \quad \text{when} \quad 0 < \frac{1}{z} < 4\pi$$

$$\tilde{u}^{*2} = \frac{1}{2\pi z}\left(1 - \frac{1}{8\pi z}\right) \quad \text{when} \quad 4\pi < \frac{1}{z} < 8\pi$$

$$\tilde{u}^* = 0 \quad \text{when} \quad \frac{1}{z} > 8\pi$$



Coleman point: $\tilde{u}^* = 0$ and $z_c^* = \frac{1}{8\pi}$

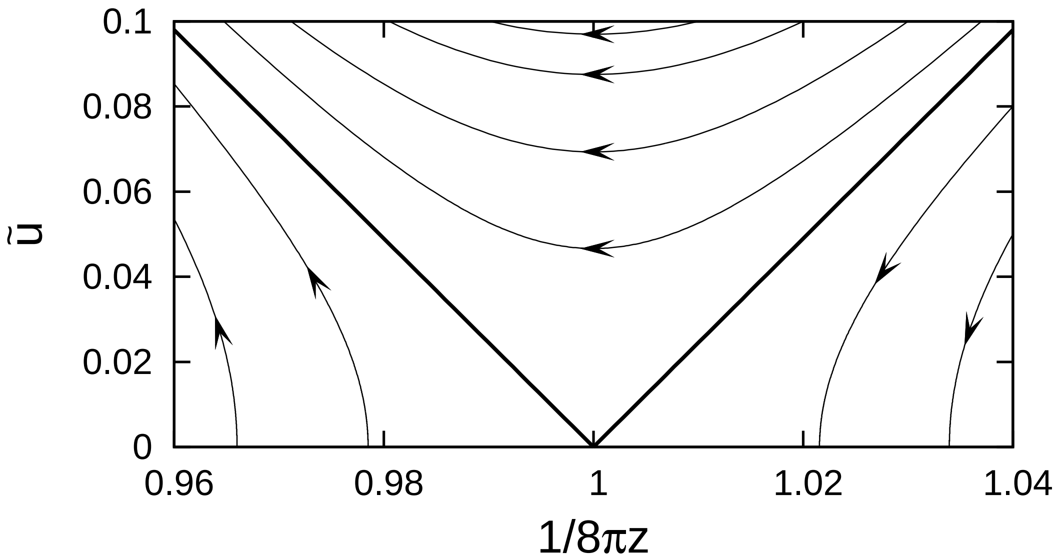
- in the symmetric phase the irrelevant scaling makes the model perturbatively nonrenormalizable
- in the broken phase we have finite IR values for the coupling $\tilde{u}$

Wave-function renormalization

The linearized RG equations are

$$\dot{\tilde{u}} = -2\tilde{u} + \frac{1}{4\pi z}\tilde{u},$$
$$\dot{z} = -\frac{\tilde{u}^2}{z^{2-2/b}} c_b,$$

with $c_b = \frac{b}{48\pi}\Gamma\left(3 - \frac{2}{b}\right)\Gamma\left(1 + \frac{1}{b}\right)$.



The RG trajectories are hyperbolas

$$\tilde{u}^2 = \frac{2}{(8\pi)^{1-2/b} c_b}\left(z - \frac{1}{8\pi}\right)^2 + \tilde{u}^{*2},$$

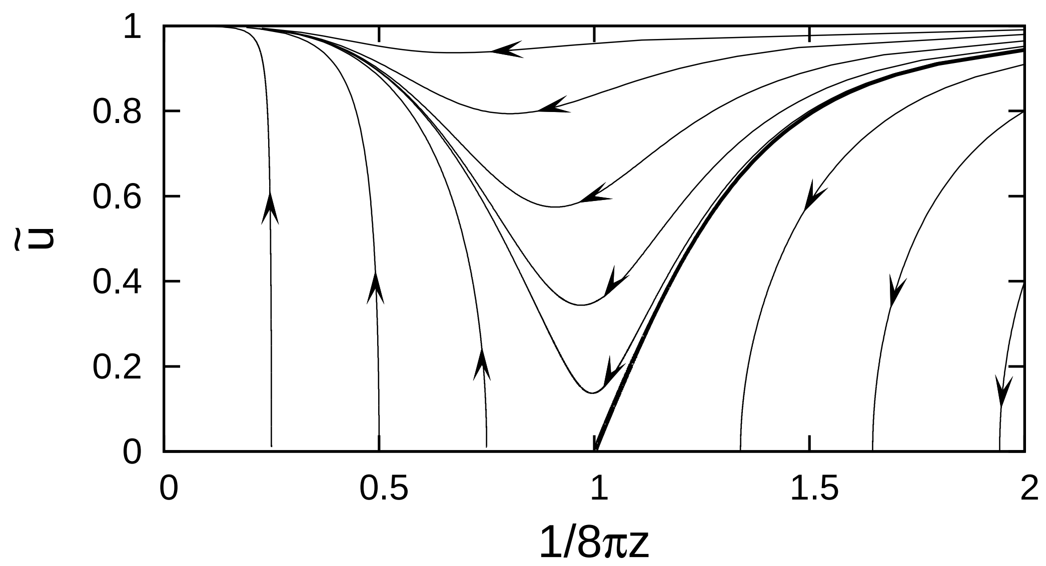
The correlation length ξ is identified as $k_c \sim 1/\xi$ (singularity points). One obtains

$$\log \xi \approx \frac{\sqrt{\pi}}{8\sqrt{c_b}} \frac{1}{\tilde{u}^*} + \mathcal{O}(\tilde{u}^*), \quad \text{furthermore} \quad \tilde{u}^{*2} = kt + \mathcal{O}(t^2)$$

where the reduced temperature is $t \sim z(\Lambda) - z_s(\Lambda)$ ($z_s(\Lambda)$ is a point of the separatrix). We get

$$\boxed{\log \xi \propto t^{-\nu}} \quad \text{with} \quad \nu = \frac{1}{2}$$ **KT type phase transition**

Wave-function renormalization



The exact RG equations are $(b = 1)$

$$
\begin{aligned}
(2 + k\partial_k)\tilde{u} &= \frac{1}{2\pi\tilde{u}z}\left[1 - \sqrt{1 - \tilde{u}^2}\right] \\
k\partial_k z &= -\frac{1}{24\pi}\frac{\tilde{u}^2}{(1 - \tilde{u}^2)^{3/2}}
\end{aligned}
$$

There are seemingly no fixed points.

- Taylor expanding in $\tilde{u}$ we get $\tilde{u}^* = 0$, z (line of fixed points).
  - $1/z < 8\pi$ UV attractive
  - $1/z > 8\pi$ IR attractive
- Rescaling equations with $(\omega = \sqrt{1 - \tilde{u}^2}$, $\chi = 1/z\omega$ and $\partial_\tau = \omega^2 k\partial_k)$

$$
\begin{aligned}
\partial_\tau \omega &= 2\omega(1 - \omega^2) - \frac{\omega^2\chi}{2\pi}(1 - \omega), \\
\partial_\tau \chi &= \chi^2\frac{1 - \omega^2}{24\pi} - 2\chi(1 - \omega^2) + \frac{\omega\chi^2}{2\pi}(1 - \omega).
\end{aligned}
$$

We got an IR attractive fixed point at $\tilde{u}^* = 1$, $1/z^* = 0$.

Scheme dependence, IR divergences

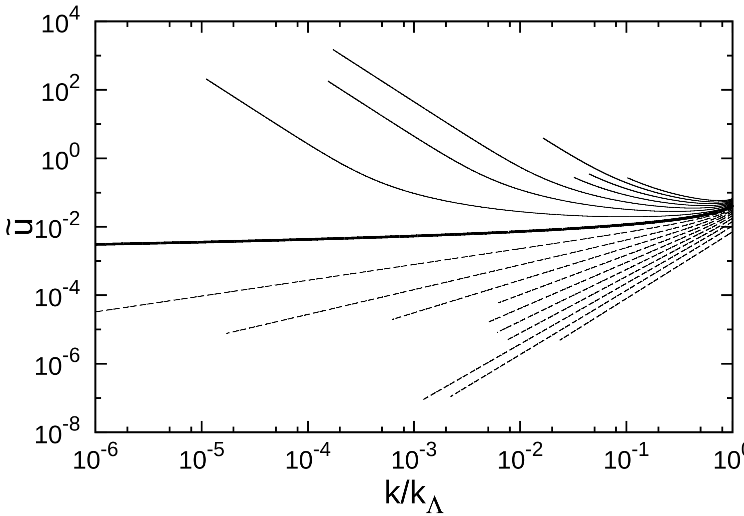
- we introduce $\bar{k} = \min(zp^2 + R)$
- for the power law IR regulator $R = p^2(k^2/p^2)^b$, with $b \geq 1$ we can calculate $\bar{k}$ analytically
- the corresponding renormalization scale is

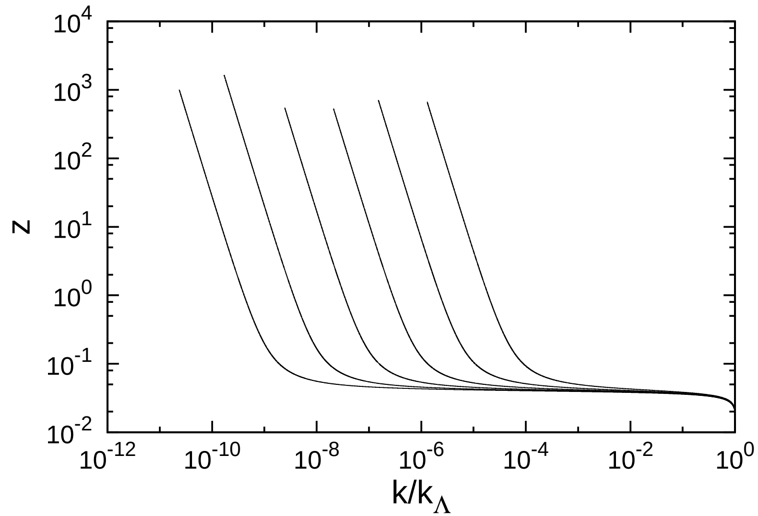
$$\bar{k}^2 = bk^2 \left(\frac{z}{b-1}\right)^{1-1/b}$$

- when $b = 1$, then $\bar{k} = k$
- we can remove the dimension of the coupling u by k or by $\bar{k}$

$$\tilde{u} = \frac{u}{k^2} \text{ and } \bar{u} = \frac{u}{\bar{k}^2}.$$

Scheme dependence, flow of the couplings



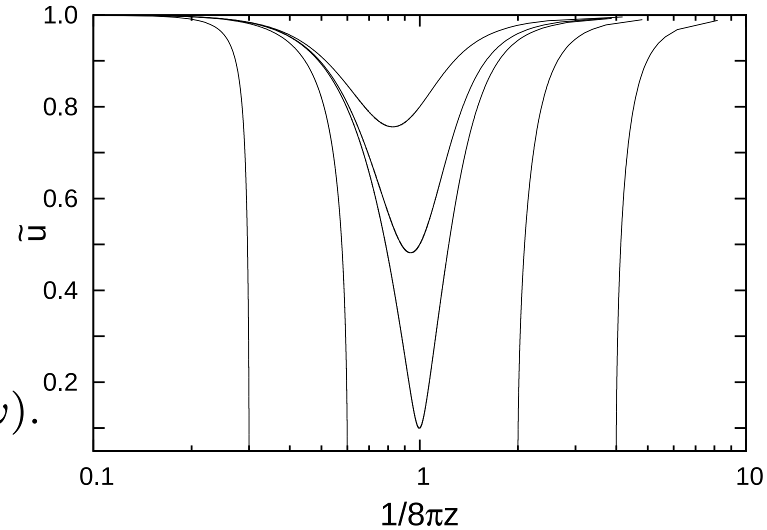


- $b = 2$
- the dashed (solid) lines represent the trajectories belonging to the (broken) symmetric phase, respectively, the wide line denotes the separatrix between the phases
- the couplings $\tilde{u}$ and z scales according to $k^{-\alpha}$ in the IR region (IR scaling regime exists)
- **symmetric phase**
  - the coupling $\tilde{u}$ tends to zero (α is negative and b dependent)
  - z is constant (not plotted) $\rightarrow$ LPA is a good approximation
- **broken phase**
  - the coupling $\tilde{u}$ diverges (α is positive and b dependent)
  - z also diverges

Asymptotic safety

Rescaling:
$\omega = \sqrt{1-\tilde{u}^2}$, $\zeta = z\omega$ and $\partial_\tau = z\omega^2 k\partial_k$.

$$\partial_\tau \omega = 2\zeta\omega(1-\omega^2) - \frac{\omega^2}{2\pi}(1-\omega),$$
$$\partial_\tau \zeta = \left(2\zeta^2 - \frac{\zeta}{24\pi}\right)(1-\omega^2) - \frac{\omega\zeta}{2\pi}(1-\omega).$$



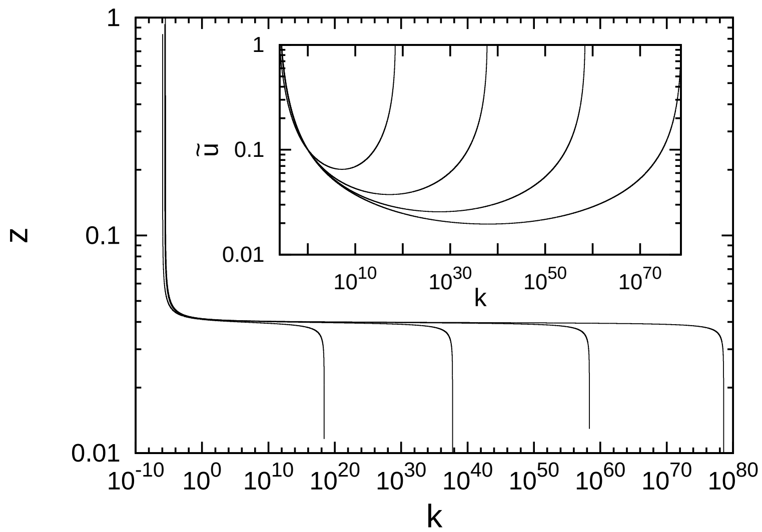
New fixed point can found at $z \to 0$ and $\tilde{u} \to 1$. The fixed point is UV attractive.
The fixed point of the 2d sine–Gordon model

- $\tilde{u}^* = 0$, z (line of fixed points)
  - $1/z < 8\pi$ UV attractive **GFP**
  - $1/z > 8\pi$ IR attractive
  - $1/z = 8\pi$ Coleman point
- $\tilde{u}^* = 1$, $1/z^* = 0$ IR attractive
- $\tilde{u}^* = 1$, $z^* = 0$ UV attractive **NGFP**

The model shows **asymptotic freedom** and **asymptotic safety**.

Asymptotic safety

- both in the IR and in the UV limits we get $\tilde{u} \to 1$.
- when $k \to 1$ then $z \to \infty$
- when $k \to \infty$ then $z \to 0$. The kinetic term tends to zero. Similar appears in the confining mechanism.



- The singularities shows up the limitation of the applicability of the models. New degrees of freedom appear.

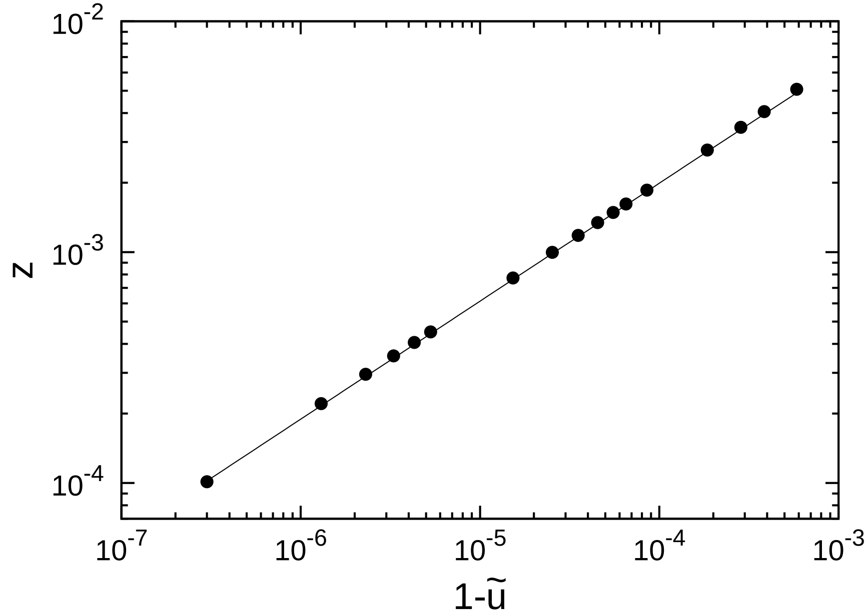
  IR: low energy limit, condensate

  UV: high energy limit, presumably instead of vortices we have single spins
- around the UV NGFP we can also identify $\xi = 1/k_c$ and we get

$$\log \xi \propto t^{-\nu} \quad \nu = \frac{1}{2}.$$

KT type phase transition. It originates from the Coleman point.

Asymptotic safety



- The phase space does not show singularity.
- The sudden increase of $\tilde{u}$ and the sudden decrease of z compensate each other giving regular flows.
- around the UV NGFP we have $z = (1 - \tilde{u})^{3/2}$

Asymptotic freedom vs. asymptotic safety

	asymptotic freedom	asymptotic safety
relevant couplings	finite number	finite number
irrelevant couplings	set to zero	set to zero
fixed point	gaussian	non-gaussian
UV limit	free, massless	interacting
examples	ϕ^4 QCD	3d Gross-Neveu 3d nonlinear sigma 2d sine-Gordon AS gravity

Gross-Neveu model

Gross-Neveu (GN) model: interaction via a four-fermionic term

- the model is asymptotically free in $d = 2$
- in $d = 3$ the model is not asymptotically free
- the Euclidean effective action of the GN model has the form

$$S[\bar\psi, \psi] = \int_x \left[Z_\psi \bar\psi i \partial\!\!\!/ \psi + \frac{\bar g}{2N_f} (\bar\psi \psi)^2 \right]$$

- the dimensionless g from dimensionful $\bar g$ is $g = Z_\psi^{-2} k^{d-2} \bar g$. ($Z_\psi = 1$ in LPA), the RG equations of the GN model is

$$\beta_g = (d - 2 + 2\eta_\psi) g - 4 d_\gamma v_d l_1^F(0) g^2$$

in the $N_f \to \infty$ limit

- a GFP at $g^* = 0$, scaling exponent $s_G = d - 2$
- the NGFP in $d = 3$ becomes $g^* = 3\pi^2/4$, scaling exponent $s_{UV} = -1$
- it is relevant when $d > 2$, so it is an UV attractive NGFP
- in the case of $d = 2$ the model is asymptotically free, perturbatively renormalizable

Gross-Neveu model

partially bosonised version of GN model

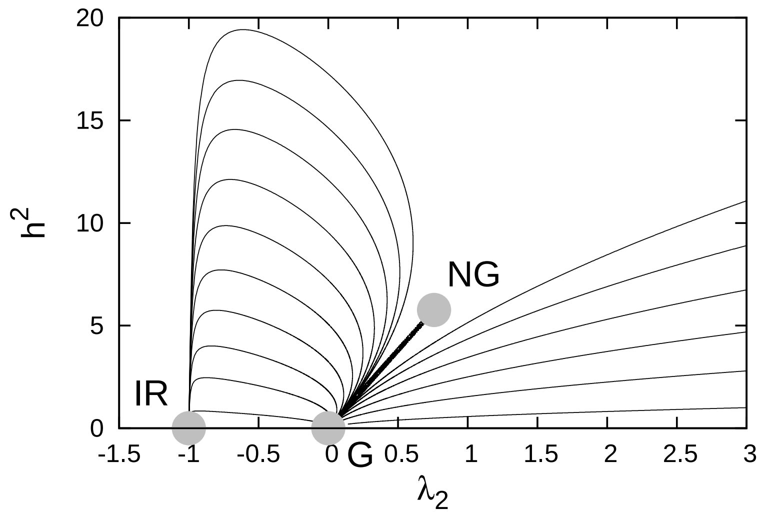
$$\dot{u} = -du + (d-2+\eta_\sigma)u'\rho - 2d_\gamma v_d l_0^{(F)d}(2h^2\rho;\eta_\psi) + \frac{1}{N_f} 2v_d l_0^d(u'+2\rho u'';\eta_\sigma)$$

with $u(\rho) = \sum_{n=0}^{\infty} \frac{\lambda_{2n}}{n!}\rho^n$ and $g = h^2/\lambda_2$ the flow equations are $(d = 3)$

$$\dot{\lambda}_2 = -2\lambda_2 + \frac{4}{3\pi^2}h^2 + \frac{5}{3\pi^2}h^2\lambda_2,$$

$$\dot{h}^2 = -h^2 + \frac{5}{3\pi^2}h^4 + \frac{2h^4(2+\lambda_2) - \frac{2}{9\pi^2}h^6}{N_f 3\pi^2(1+\lambda_2)^2}$$

- G: $h_G^{2*} = 0$ and $\lambda_{2G}^* = 0$, attractive
- NG: $h_{NG}^{2*} = 5.764$ and $\lambda_{2NG}^* = 0.758$, saddle point



The nonlinear σ model

the dynamics of a map φ from a d-dim. manifold $\mathcal{M}$ to a N-dim. manifold $\mathcal{N}$
analogy with quantum gravity:

- nonpolynomial action
- the dimension of the couplings are the same
- background field RG equations

derivative interactions in the action

$$S = \frac{1}{2}\zeta \int d^d x \partial_\mu \varphi^\alpha \partial^\mu \varphi^\beta h_{\alpha\beta}(\varphi)$$

$h_{\alpha\beta}$ is the dimensionless metric, $\zeta = 1/g_0^2$, with $g_0 \sim k^{(2-d)/2}$
properties:

- in $d = 2$ it is asymptotically free
- beyond $d = 2$ it becomes nonrenormalizable, the UV GFP becomes a hyperbolic
- a nontrivial UV fixed point arises, the model becomes asymptotically safe

The nonlinear σ model

perturbative RG equations

$$\beta_{g_0} = \frac{d-2}{2}\tilde{g}_0 - c_d \frac{R}{D}\tilde{g}_0^3,$$
$$c_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}$$

fixed points

GFP: $g_G^* = 0$, $s_G = (d-2)/2$, if $d > 2$ $\rightarrow$ nonrenormalizable

NGFP: $g_{UV}^{*2} = (d-2)D/(2c_d D)$, $s_G = 2-d$, if $d > 2 \rightarrow$ UV NGFP $\rightarrow$ AS

exact RG equations

$$\beta_g = \frac{d-2}{2}\tilde{g} - \frac{c_d \frac{R}{D}\tilde{g}^3}{1 - 2c_d \frac{R}{D(d+2)}\tilde{g}^2}$$

fixed points

GFP: $g_G^* = 0$, $s_G = (d-2)/2$, if $d > 2$ $\rightarrow$ nonrenormalizable

NGFP: $g_{UV}^{*2} = D(d^2-4)/(4c_d dR)$,
$s_G = -2d(d-2)/(d+2)$,
if $d > 2 \rightarrow$ UV NGFP $\rightarrow$ AS,
$s_{UV} = -2d(d-2)/(d+2)$,
if $d = 3 \rightarrow \nu = -1/s_{UV} = 5/6$.

they are qualitatively the same

The nonlinear σ model

more terms in the action give more couplings

$$\begin{aligned}\tilde{\beta}_{\tilde{g}_0} &= -\tilde{g}_0 + \tilde{g}_0(N-2)\tilde{Q}_{d/2,2} + d\tilde{g}_1(N-2)\tilde{Q}_{d/2+1,2}, \\ \tilde{\beta}_{\tilde{g}_1} &= \tilde{g}_1 + \tilde{g}_1(N-2)\tilde{Q}_{d/2,2},\end{aligned}$$

with

$$\tilde{Q}_{n,l} = \frac{1}{(4\pi)^{d/2}\Gamma(n)}\left(\frac{(2n+2+\partial_t)\tilde{g}_0}{n(n+1)(\tilde{g}_0+\tilde{g}_1)^l} + \frac{2(2n+4+\partial_t)\tilde{g}_1}{n(n+2)(\tilde{g}_0+\tilde{g}_1)^l}\right).$$

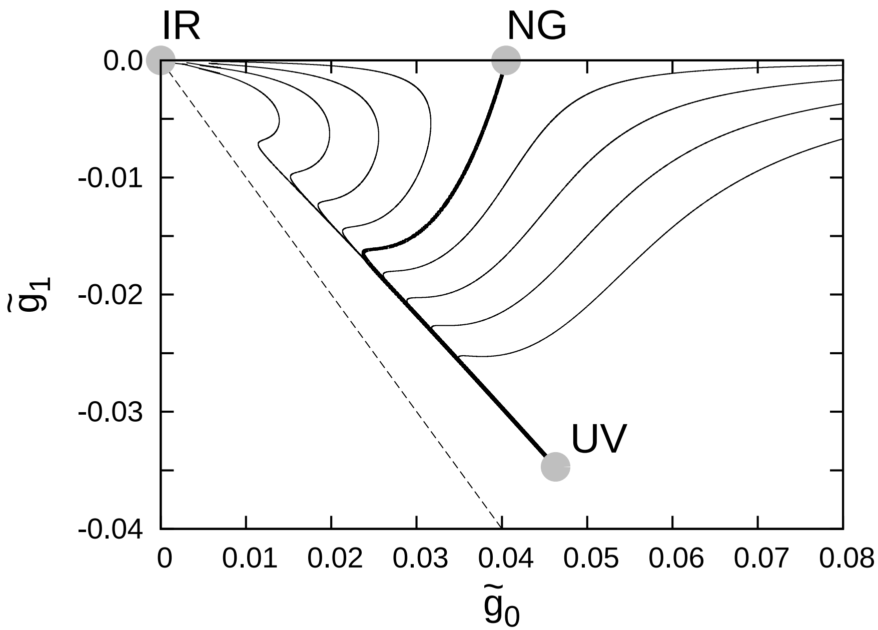
fixed points

NG: $\tilde{g}_{0NG} = 2/5\pi^2$ and $\tilde{g}_{1NG} = 0$,
saddle point ($\tilde{g}_0$ attractive, $\tilde{g}_1$ repulsive)
scaling exponents: $s_{NG0} = -6/5$ and $s_{NG1} = 2 \rightarrow \nu = -1/s_{NG0} = 5/6$.

UV: $\tilde{g}^*_{UV0} = 16/35\pi^2$ and $\tilde{g}^*_{UV1} = -12/35\pi^2$,
scaling exponents: $s_{UV0} = -0.457$ and $s_{UV1} = -13.11$,
the UV fixed point makes the model asymptotically safe

The nonlinear σ model

phase structure, $d = 3$



- $\tilde{g}_0 = -\tilde{g}_1$: singularity limit
- two phases
- NG-IR trajectory: separatrix
- NG: saddle point
- UV: UV attractive fixed point
- the model is asymptotically safe

Classical gravity

– it is one of the fundamental interactions, acts between massive particles. According to Newton

$$V = -G\frac{m_1 m_2}{r}$$

m_1, m_2 masses, r distance, a G Newton's constant, $G = 6.67 \times 10^{-11}$

– anomalous precession of the perihelion of Mercury

– general theory of relativity, its Einstein-Hilbert (EH) action

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{g}(2\Lambda - R)$$

– **Einstein equation**

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$g_{\mu\nu}$ a metric tensor, a $R_{\mu\nu}$ Ricci tensor, $g = \det(g)$, $T_{\mu\nu}$ energy-momentum tensor

– the curvature (or metrics) becomes dynamical variable

– Λ cosmological constant, $\Lambda \approx 10^{-52}\mathrm{m}^{-2}$

Gravity

- AS gravity at the UV scale?
- QM + gravity = ?
- problem with quantization: the Newton's constant is irrelevant $\rightarrow$ gravity is not renormalizable perturbatively (around the GFP)
- we cannot imagine the UV gravity without G
- not sure that quantum gravity is renormalizable (e.g. SM, ϕ^4 theory), open question
- Causal Dynamical Triangulations and Euclidean Dynamical Triangulations use Monte Carlo techniques to investigate the phase space of quantum geometries resulting from the gravitational path integral. Reuter fixed point
- alternatively, the Reuter fixed point can manifest itself in (approximate) solutions of the Wetterich equation

Gravity

– it is expected that at the Planck scale

$$M_{Planck} = \sqrt{\frac{\hbar c}{G}} \approx 10^{19}\ \text{GeV}$$

– the general relativity is not applicable, a new *quantum* gravitational theory arises. There the quantum ($\hbar$), the relativistic (c) and gravitational (G) effects are equally important
– no data at Planck scale $\rightarrow$ no check
– a possible formalism for quantum gravity: path integral
– the metrics is the dynamical variable playing the role of the fluctuating quantum field
– two problems

- the RG scale is defines a certain length, i.e. the blocking steps needs the concept of length
- gravity behaves as a gauge theory, the EH action is invariant under coordinate transformations

$$\delta g_{\mu\nu} = \mathcal{L}_v g_{\mu\nu} = v^\rho \partial_\rho g_{\mu\nu} + (\partial_\mu v^\rho) g_{\rho\nu} + (\partial_\nu v^\rho) g_{\rho\mu}$$

(Lie derivative along a vector field v^μ) (physically equivalent configurations)
need a gauge fixing term which chooses certain configurations

– **solution:** background field method

– how can we define the scale k?

– Laplacian: $\Delta = -\bar{g}^{\mu\nu}\bar{D}_\mu \bar{D}_\nu$

– a set of eigenmodes

$$\Delta h^n_{\mu\nu} = E_n h^n_{\mu\nu}, \;\; E_0 \leq E_1 \leq E_2 \ldots$$

– long range: $E_n \leq k^2$ short range: $E_n \geq k^2$

– evolution equations

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left[\frac{\dot{\mathcal{R}}_k}{\Gamma^{(2)}_k + \mathcal{R}_k}\right]$$

– expansion in a basis of monomials $\mathcal{O}_i$ of the effective action

$$\Gamma_k = \sum_i \bar{u}^i(k)\mathcal{O}_i$$

– dimensionfull couplings $\bar{u}^i(k)$, dimensionless ones: $u^i(k) = \bar{u}^i(k)k^{-d_i}$, $d_i = [\bar{u}^i]$ mass dimension

Truncation

– Einstein-Hilbert truncation

$$\mathcal{O}_1 = \int dx \sqrt{g}, \;\; \mathcal{O}_2 = \int dx \sqrt{g} R$$

$\sqrt{g} = \sqrt{\det(g)}$, R Ricci scalar

– other monomials

$$\begin{aligned} \text{Taylor expansion in } R: \quad & 1, \;\; R, \;\; R^2 \ldots \\ \text{gradient expansion}: \quad & R\Delta R, \;\; R\Delta^2 R \ldots \end{aligned}$$

– Riemann basis

$$\mathcal{O}_i[g] = \mathcal{O}_i[\sqrt{g}, R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, D_\mu]$$

with Ricci scalar, Ricci tensor, Riemann tensor, covariant derivative

– Weyl basis

$$\mathcal{O}_i[g] = \mathcal{O}_i[\sqrt{g}, R, R_{\mu\nu}, C_{\mu\nu\rho\sigma}, D_\mu]$$

with

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{d-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \frac{2}{(d-1)(d-2)} g_{\mu[\rho}R_{\sigma]\nu}$$

Truncation

– multiplications, e.g. $R^2 R_{\mu\nu} R^{\mu\nu}$, $C_{\mu\nu\rho\sigma} \Delta C^{\mu\nu\rho\sigma}$

– field dependent functions: $f(R)$

– furthermore $E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu\nu} R^{\mu\nu} + R^2$ Gauss-Bonnet term, etc.

– avoid Ostrogradski instability due to the polinomial momentum dependence of the inverse propagator $\rightarrow$

form factors: collect the covariant derivative dependent terms into operator-valued functions

– e.g.

$$\sum_i \bar{u}^i(k) R \Delta^n R \quad \rightarrow \quad R W_k^R(\Delta) R$$

$$\sum_i \bar{u}^i(k) C_{\mu\nu\rho\sigma} \Delta^n C^{\mu\nu\rho\sigma} \quad \rightarrow \quad C_{\mu\nu\rho\sigma} W_k^C(\Delta) C^{\mu\nu\rho\sigma}$$

– they become momentum dependent, $f(p)$

approximaton of Γ_k	**RG flow**	**FP**
finite number of $\mathcal{O}_i$	ODE	algebraic
field dependent functions, $f(R)$	PDE	PDE
momentum dependent form factors, $f(p)$	IDE	IDE

EH truncation

$$\Gamma_k^{EH}[g] = \frac{1}{16\pi G_k}\int d^4x\sqrt{g}(2\Lambda_k - R)$$

dimensionless couplings $\lambda = \Lambda k^{-2}$ $g = Gk^2$, $\eta = \dot{G}_k/G_k$

– full effective action

$$\Gamma_k[h,\bar{C},C;\bar{g}] = \Gamma_k^{EH}[g] + \Gamma_k^{gf}[h;\bar{g}] + \Gamma_k^{gh}[h,\bar{C},C;\bar{g}]$$

– gauge fixing term

$$\Gamma_k^{gf}[h;\bar{g}] = \frac{1}{32\pi G_k \alpha}\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu}F_\mu F_\nu, \quad F_\mu = [\delta_\mu^\sigma \bar{D}^\rho - \beta\bar{g}^{\sigma\rho}\bar{D}_\mu]h_{\sigma\rho}$$

– ghost term

$$\Gamma_k^{gh}[h,\bar{C},C;\bar{g}] = -\sqrt{2}\int d^dx\sqrt{\bar{g}}\bar{C}_\mu\mathcal{M}[g,\bar{g}]^\mu{}_\nu C^\nu$$

with the Faddeev-Popov operator

$$\mathcal{M}[g,\bar{g}]^\mu{}_\nu = \bar{g}^{\mu\rho}\bar{D}^\sigma(g_{\rho\nu}D_\sigma + g_{\sigma\nu}D_\rho) - 2\beta\bar{g}^{\rho\sigma}\bar{D}^\mu g_{\sigma\nu}D_\rho$$

– in harmonic gauge $\alpha = 1$ and $\beta = 1/2$

Evolution equations

– evolution equations

$$\dot{u}^i(k) = \beta^i(\{u^j\})$$

– Newton's and cosmological couplings

$$\begin{aligned} \dot{g} &= \beta_g(g,\lambda), \\ \dot{\lambda} &= \beta_\lambda(g,\lambda) \end{aligned}$$

– explicit form of the beta functions

$$\begin{aligned} \beta_g(g,\lambda) &= (d-2+\eta)g \\ \beta_\lambda(g,\lambda) &= -(2-\eta)\lambda + \frac{g}{2(4\pi)^{d/2-1}}(2d(d+1)\Phi^1_{d/2}(-2\lambda) \\ &\quad -8d\Phi^1_{d/2}(0) - d(d+1)\eta_N\tilde{\Phi}^1_{d/2}(-2\lambda)) \end{aligned}$$

with the anomalous dimension η

$$\eta(g,\lambda) = \frac{gB_1(\lambda)}{1-gB_2(\lambda)}$$

Evolution equations

– we introduced

$$
\begin{aligned}
B_1(\lambda) &= \frac{1}{3}(4\pi)^{1-d/2}(d(d+1)\Phi^1_{d/2-1}(-2\lambda) - 6d(d-1)\Phi^2_{d/2}(-2\lambda) \\
&\quad -4d\Phi^1_{d/2-1}(0) - 24\Phi^2_{d/2}(0)), \\
B_2(\lambda) &= -\frac{1}{6}(4\pi)^{1-d/2}(d(d+1)\tilde{\Phi}^1_{d/2-1}(-2\lambda) - 6d(d-1)\tilde{\Phi}^2_{d/2}(-2\lambda))
\end{aligned}
$$

– threshold function

$$
\begin{aligned}
\Phi^p_n(w) &\equiv \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}\frac{R^{(0)} - zR^{(0)'}(z)}{(z+R^{(0)}+w)^p}, \\
\tilde{\Phi}^p_n(w) &\equiv \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}\frac{R^{(0)}(z)}{(z+R^{(0)}(z)+w)^p}
\end{aligned}
$$

Evolution equations

for the Litim regulator $R^{(0)}(y)$

$$R^{(0)}(y) = (1-y)\theta(1-y)$$

threshold function

$$\begin{aligned}
\Phi^p_n(w)^{Litim} &= \frac{1}{\Gamma(n+1)}\frac{1}{(1+w)^p} \\
\tilde{\Phi}^p_n(w)^{Litim} &= \frac{1}{\Gamma(n+2)}\frac{1}{(1+w)^p}
\end{aligned}$$

the beta functions become simpler

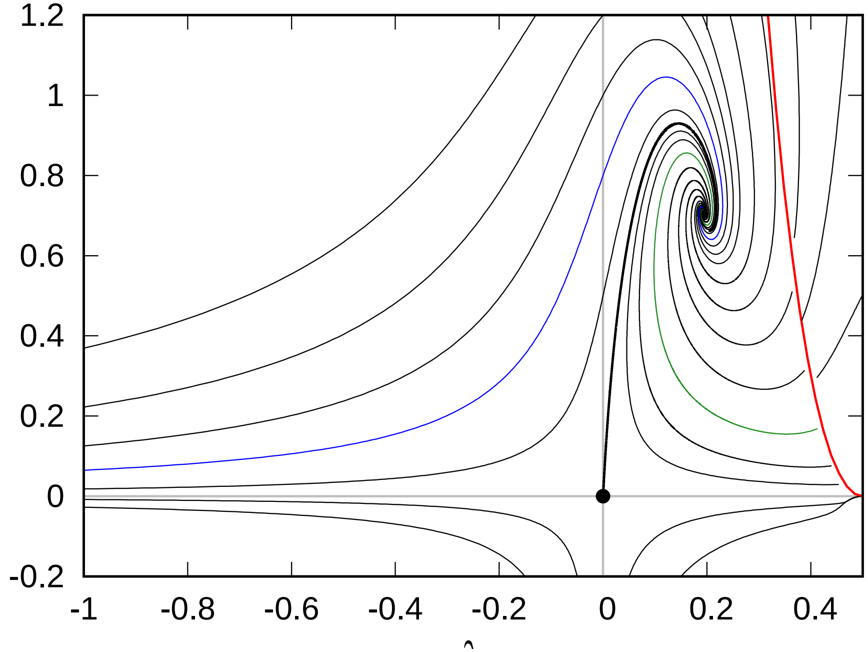
$$\begin{aligned}
\beta_g &= (2+\eta)g, \\
\beta_\lambda &= -(2-\eta)\lambda + \frac{g}{8\pi}\left(\frac{20}{1-2\lambda} - 16 - \frac{10}{3}\eta\frac{1}{1-2\lambda}\right)
\end{aligned}$$

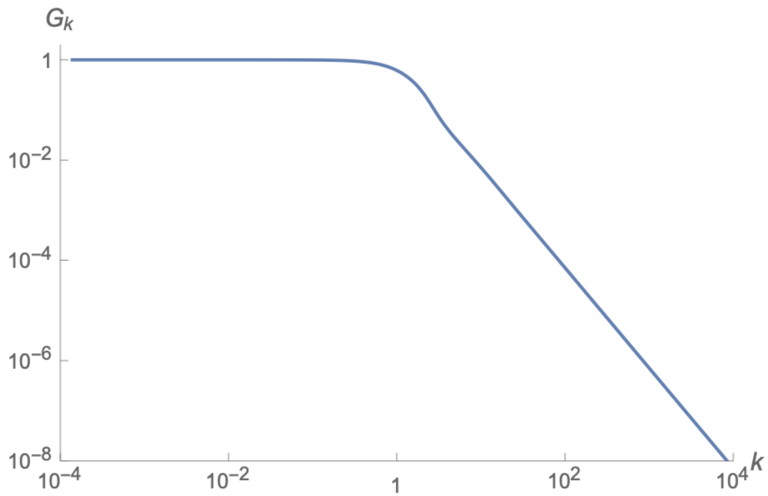
anomalous dimension

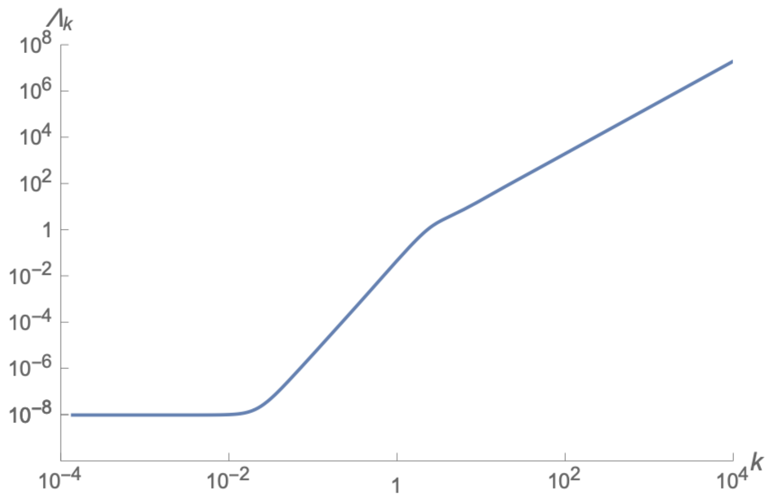
$$\eta = \frac{g(\frac{5}{1-2\lambda} - \frac{9}{(1-2\lambda)^2} - 7)}{3\pi(1 + \frac{g}{12\pi}(\frac{5}{1-2\lambda} - \frac{6}{(1-2\lambda)^2}))}$$

Phase space

- black dots: GFP, Reuter
- GFP, $s_{12} = \pm 2$
- Reuter, $g_* = 0.707, \lambda_* = 0.193,$ $s_{1,2} = -1.48 \pm 3.04i.$
- blue: $\lambda \to -\infty$
- green: $\lambda > 0$
- red: singular η

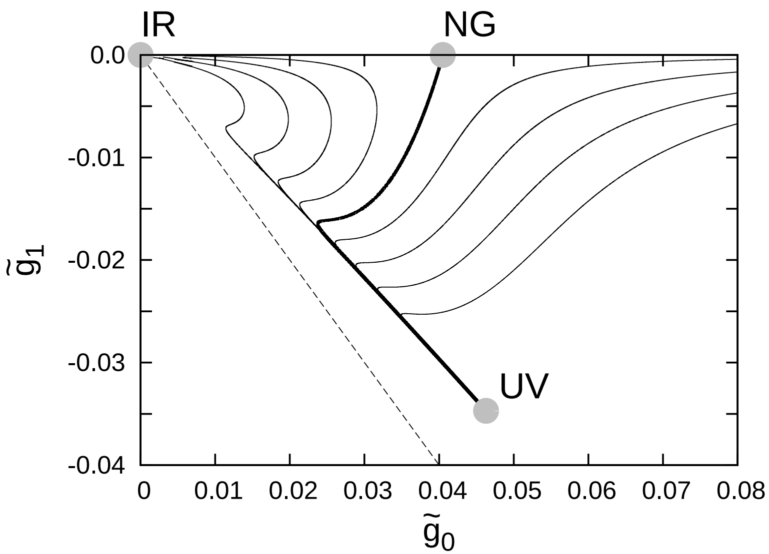




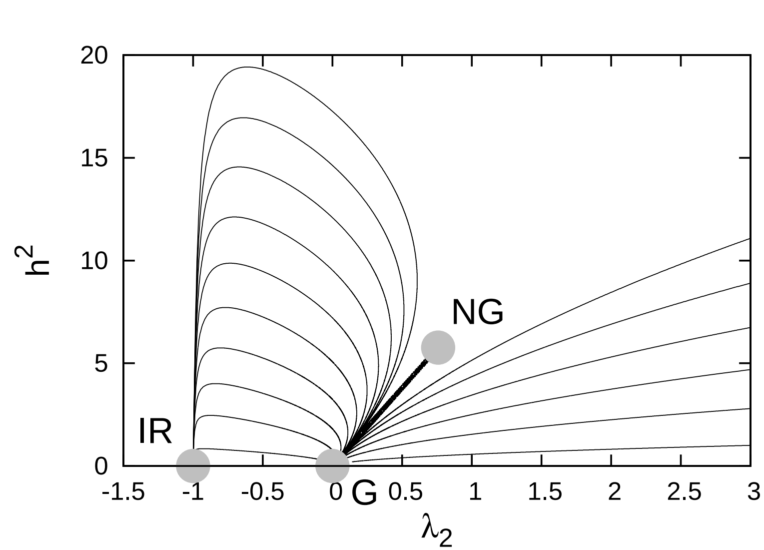


- Reuter: $G_k = g^* k^{-2}$, $\Lambda_k = \lambda^* k^2$, $k > 1$
- near GFP: $G_k = G_0$, $\Lambda_k = \Lambda_0$, $k < 1$

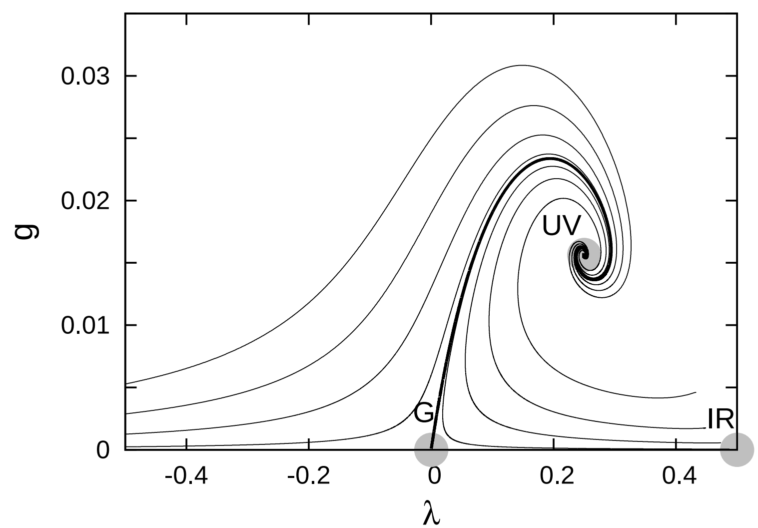
Asymptotically safe models



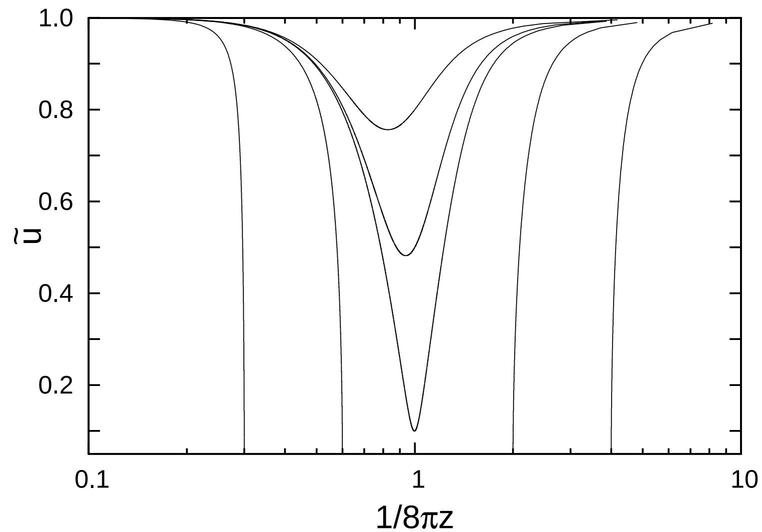
3d nonlinear σ model



3d Gross-Neveu model



4d Asymptotically safe gravity



2d sine–Gordon model

Conformally reduced gravity

– starting with the EH action

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{g}(2\Lambda - R)$$

– we assume that

$$g_{\mu\nu} = \phi^{2\nu(d)}\hat{g}_{\mu\nu}$$

with $\hat{g}_{\mu\nu}$ non-dynamical reference metric, $\phi(x)$ scalar function, $\nu(d) = 2/(d-2)$

– conformally reduced version

$$S_{EH} = \frac{1}{8\pi\xi(d)G}\int d^4x\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\hat{R}\phi^2 - \xi(4d)\Lambda\phi^{2d/(d-2)}\right)$$

with $\hat{R}$ curvature of $\hat{g}_{\mu\nu}$, $\xi(d) = (d-2)/(4(d-1))$

– $d = 4$: $\nu = 1$, $\xi = 1/6$, $g_{\mu\nu} = \phi^2\hat{g}_{\mu\nu}$

$$S_{EH} = -\frac{3}{4\pi G}\int d^4x\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{12}\hat{R}\phi^2 - \frac{1}{6}\Lambda\phi^4\right)$$

Conformally reduced gravity

- conform factor instability: opposite sign kinetic term may cause instabilities if $\phi(x)$ varies fast
- the self-interaction is controlled by Λ

– Wetterich's RG equation in the background-field method for the CREH effective action:

$$\dot{\Gamma}_k[\bar{f};\chi_B] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma^{(2)}[\bar{f};\chi_B] + \mathcal{R}_k[\chi_B]\right)^{-1}\dot{\mathcal{R}}_k[\chi_B]\right]$$

with $\Gamma^{(2)}[\bar{f};\chi_B] = \frac{1}{\sqrt{\hat{g}_x}\sqrt{\hat{g}_y}}\frac{\delta^2\Gamma_k[\bar{f};\chi_B]}{\delta\bar{f}_x\delta\bar{f}_y}$ and $\mathrm{Tr}A = \int_x \sqrt{\hat{g}_x}A_{xx}$.

– the CREH effective action with running couplings

$$\Gamma_k[\bar{f};\chi_B] = -Z_k\int_x \sqrt{\hat{g}_x}\left\{-\frac{1}{2}(\chi_B+\bar{f})\hat{\Box}(\chi_B+\bar{f}) + \frac{1}{12}\hat{R}(\chi_B+\bar{f})^2 - \frac{\Lambda_k}{6}(\chi_B+\bar{f})^4\right\}$$

– background field: χ_B =const, wavefunction renormalization: $Z_k = \frac{3}{4\pi G_k}$

– the potentials are

$$\begin{aligned} V_k(\phi) &= Z_k U_k(\phi),\ \ U_k(\phi) = \frac{\Lambda_k}{6}\phi^4 - \frac{1}{12}\hat{R}\phi^2,\ \ u_k(\phi) = U_k(\phi) + \frac{1}{12}\hat{R}\phi^2 = \frac{\Lambda_k}{6}\phi^4 \\ U_k''(\phi) &= 2\Lambda_k\phi^2 - \frac{1}{6}\hat{R},\ \ U_k'''(\phi) = 4\Lambda_k\phi,\ \ U_k''''(\phi) = 4\Lambda_k \end{aligned}$$

Conformally reduced gravity

– compact form of the CREH effective action

$$\Gamma_k[\bar{f};\chi_B] = \int_x \sqrt{\hat{g}_x}\left\{\frac{1}{2}Z_k\phi\hat{\Box}\phi + V_k(\phi)\right\}$$

where $\phi = \chi_B + \bar{f}$ and

$$V_k(\phi) = V_k(\chi_B) + V_k'(\chi_B)\bar{f} + \frac{1}{2}V_k''(\chi_B)\bar{f}^2 + \frac{1}{6}V_k'''(\chi_B)\bar{f}^3 + \frac{1}{24}V_k''''(\chi_B)\bar{f}^4 + \ldots$$

– flat reference metric $\hat{g}_{\mu\nu} = \delta_{\mu\nu} \to \sqrt{\hat{g}} = 1$ and $\hat{\Box} = \Box$

– the background metric determining the running RG scale k is given as $\bar{g}_{\mu\nu} = \chi_B^2\hat{g}_{\mu\nu}$.

– the regulator is introduced according to the replacement

$$-\bar{\Box} \quad \Longrightarrow \quad -\bar{\Box} + k^2 R^{(0)}\left(\frac{-\bar{\Box}}{k^2}\right)$$

where $\bar{\Box} = \frac{1}{\chi_B^2}\hat{\Box}$ i.e., $\hat{\Box} = \chi_B^2\bar{\Box}$.

Conformally reduced gravity

– the second functional derivative is

$$\left(\Gamma^{(2)}[\bar{f};\chi_B]\right)_{xy} = \sqrt{\hat{g}}\left(Z_k\chi_B^2\bar{\Box}_x + V_k''(\chi_B) + \delta M(\bar{f})\right)\delta_{xy}$$

with

$$\delta M(\bar{f}) = V_k'''(\chi_B)\bar{f} + \frac{1}{2}V_k''''(\chi_B)\bar{f}^2 \equiv Z_k\delta\mu(\langle f|), \;\; \delta\mu(\bar{f}) = U_k'''(\chi_B)\bar{f} + \frac{1}{2}U_k''''(\chi_B)\bar{f}^2$$

– we find after regularization the expression

$$\left(\Gamma^{(2)}[\bar{f};\chi_B] + R_k(\chi_B)\right)_{xy} = \sqrt{\hat{g}}\left[-Z_k\left(-\hat{\Box}_x + k^2\chi_B^2 R^{(0)}\left(\frac{-\hat{\Box}}{k^2\chi_B^2}\right)\right) + V_k''(\chi_B) + \delta M(\bar{f})\right]\delta_{xy}$$

– the regulator form: $R_k(\chi_B) = -Z_k\chi_B^2 k^2 R^{(0)}\left(\frac{-\hat{\Box}}{k^2\chi_B^2}\right)$

– its scale-derivative is given as

$$\dot{R}_k(\chi_B) = -k^2\chi_B^2 2Z_k\left[\left(1-\frac{1}{2}\eta\right)R^{(0)}\left(\frac{-\hat{\Box}}{k^2\chi_B^2}\right) - \frac{-\hat{\Box}}{k^2\chi_B^2}R^{(0)\prime}\left(\frac{-\hat{\Box}}{k^2\chi_B^2}\right)\right]$$

with the anomalous dimension $\eta = -\frac{\dot{Z}_k}{Z_k} = \frac{\dot{G}_k}{G_k}$

Conformally reduced gravity

$$
\begin{aligned}
l.h.s. \quad = \quad & \int_x \Big\{ -\frac{1}{2}\eta \bar{f}\hat{\Box}\bar{f} - \eta\Big(U_k(\chi_B) + U_k'(\chi_B)\bar{f} + \frac{1}{2}U_k''(\chi_B)\bar{f}^2 + \ldots \Big) \\
& + \dot{U}_k(\chi_B) + \dot{U}_k'(\chi_B)\bar{f} + \frac{1}{2}\dot{U}_k''(\chi_B)\bar{f}^2 + \ldots \Big\}
\end{aligned}
$$

and on the right-hand side

$$
r.h.s. = \frac{k^2\chi_B^2}{Z_k}\mathrm{Tr}\{[K - \delta\mu]^{-1}\mathcal{N}(-\hat{\Box})\}
$$

with the notations

$$
\begin{aligned}
K \quad &= \quad \mathcal{A}(-\hat{\Box}) + \frac{1}{6}\hat{R} \\
\mathcal{A}(-\hat{\Box}) \quad &= \quad -\hat{\Box} + k^2\chi_B^2 R^{(0)}\big(\frac{-\hat{\Box}}{k^2\chi_B^2}\big) - u_k''(\chi_B) \\
\mathcal{N}(-\hat{\Box}) \quad &= \quad \Big(1 - \frac{1}{2}\eta_N\Big) R^{(0)}\big(\frac{-\hat{\Box}}{k^2\chi_B^2}\big) - \frac{-\hat{\Box}}{k^2\chi_B^2} R^{(0)\prime}\big(\frac{-\hat{\Box}}{k^2\chi_B^2}\big)
\end{aligned}
$$

and $u_k''(\chi_B) = 2\Lambda_k\chi_B^2$

Conformally reduced gravity

– the Neumann-expansion is

$$[K-\delta\mu]^{-1} = K^{-1} + K^{-1}\delta\mu K^{-1} + K^{-1}\delta\mu K^{-1}\delta\mu K^{-1} + \ldots$$

then

$$r.h.s. \approx \frac{k^2\chi_B^2}{Z_k}(T_0 + T_1 + T_2)$$

with

$$T_0 = \mathrm{Tr}\{K^{-1}\mathrm{N}\},\ \ T_1 = \mathrm{Tr}\{K^{-1}\delta\mu K^{-1}\mathrm{N}\},\ \ T_2 = \mathrm{Tr}\{K^{-1}\delta\mu K^{-1}\delta\mu K^{-1}\mathrm{N}\}$$

up to the order $\bar{f}^2$

– taking the term of the order $\bar{f}^0$ we get

$$\left(-\eta U_k(\chi_B) + \dot{U}_k(\chi_B)\right)\Omega = \frac{k^2\chi_B^2}{Z_k}T_0$$

with $\Omega = \int_x \sqrt{\hat{g}}$

Conformally reduced gravity

– Neumann-expansion of K^{-1}

$$K^{-1} = \left(\text{Ą}(-\hat{\Box}) + \frac{1}{6}\hat{R} \right)^{-1} \approx \text{Ą}^{-1}(-\hat{\Box}) - \frac{1}{6}\text{Ą}^{-1}(-\hat{\Box})\hat{R}\text{Ą}^{-1}(-\hat{\Box}) + \ldots$$

– (assumption: $\hat{R}$ is constant) we find $T_0 = T_{00} - \frac{1}{6}\hat{R}T_{01}$, where

$$\begin{aligned} T_{00} &= \mathrm{Tr} W_{01}(-\hat{\Box}) = (4\pi)^{-2}\left[Q_2[W_{01}] + \frac{1}{6}\hat{R}Q_1[W_{01}] + \ldots \right]\Omega, \\ T_{01} &= \mathrm{Tr} W_{02}(-\hat{\Box}) = (4\pi)^{-2}\left[Q_2[W_{02}] + \ldots \right]\Omega \end{aligned}$$

with $W_{01}(y) = \frac{\text{Ņ}(y)}{\text{Ą}(y)}$, $W_{02}(y) = \frac{\text{Ņ}(y)}{\text{Ą}^2(y)}$, $w = -\frac{u_k''(\chi_B)}{\chi_B^2 k^2}$, the moments are

$$\begin{aligned} Q_n[W] &= = \frac{(\chi_B^2 k^2)^n}{\Gamma(n)} \int_0^\infty dz z^{n-1} W(\chi_B^2 k^2 z) \\ Q_n(W_{0p}) &= \frac{\chi_B^2 k^2)^n}{(n-1)!} \int_0^\infty dz z^{n-1} \frac{(1 - \frac{1}{2}\eta)R^{(0)}(z) - zR^{(0)\prime}(z)}{[z + R^{(0)}(z) + w]^p} \\ &= (\chi_B^2 k^2)^{n-p}[\phi_n^p(w) - \frac{1}{2}\eta\tilde{\phi}_n^p(w)] \end{aligned}$$

Conformally reduced gravity

– threshold functions (same)

$$\phi_n^p(w) = \frac{1}{(n-1)!}\int_0^\infty dz z^{n-1}\frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{[z+R^{(0)}(z)+w]^p},$$

$$\tilde{\phi}_n^p(w) = \frac{1}{(n-1)!}\int_0^\infty dz z^{n-1}\frac{R^{(0)}(z)}{[z+R^{(0)}(z)+w]^p}$$

– the RG equation becomes

$$\frac{1}{6}(\dot{\Lambda}_k - \eta\Lambda_k)\chi_B^4 + \frac{1}{12}\hat{R}\eta\chi_B^2 = \frac{k^2\chi_B^2 G_k}{12\pi}\left[Q_2[W_{01}] + \frac{1}{6}\hat{R}\Big(Q_1[W_{01}] - c_k Q_2[W_{02}]\Big) + \mathcal{O}(\hat{R}^2)\right]$$

– the various powers of $\hat{R}$ are

$$(\dot{\Lambda}_k - \eta\Lambda_k) = \frac{k^2 G_k}{2\pi\chi_B^2}Q_2[W_{01}] = \frac{k^2 G_k}{2\pi\chi_B^2}\chi_B^2 k^2[\phi_2^1(w) - \frac{1}{2}\eta_N\tilde{\phi}_2^1(w)],$$

$$\eta = \frac{k^2 G_k}{6\pi}\Big(Q_1[W_{01}] - Q_2[W_{02}]\Big)$$

$$= \frac{k^2 G_k}{6\pi}\Big(\phi_1^1(w) - \frac{1}{2}\eta_N\tilde{\phi}_1^1(w) - [\phi_2^2(w) - \frac{1}{2}\eta_N\tilde{\phi}_2^2(w)]\Big)$$

Conformally reduced gravity

– the dimensionless couplings are: $\lambda = k^{-2}\Lambda_k, \quad g = k^2 G_k$

– it implies that $\dot{\Lambda}_k = k^2(\dot{\lambda} + 2\lambda), \quad \dot{G}_k = \eta G_k, \quad w = -\frac{u_k''(\chi_B)}{k^2\chi_B^2} = -2\lambda$

– we obtain the flow equations

$$
\begin{aligned}
\dot{g} &= (d - 2 + \eta) g \\
\dot{\lambda} &= -(2-\eta)\lambda + \frac{g}{2\pi}\Big(\phi_2^1(w) - \eta\tilde{\phi}_2^1(w)\Big), \\
\eta &= \frac{g_k}{6\pi}\left[\phi_1^1(w) - \frac{1}{2}\eta\tilde{\phi}_1^1(w) - \Big(\phi_2^2(w) - \frac{1}{2}\eta\tilde{\phi}_2^2(w)\Big)\right] = g_k\Big(B_1(\lambda) + \eta B_2(\lambda)\Big),
\end{aligned}
$$

with

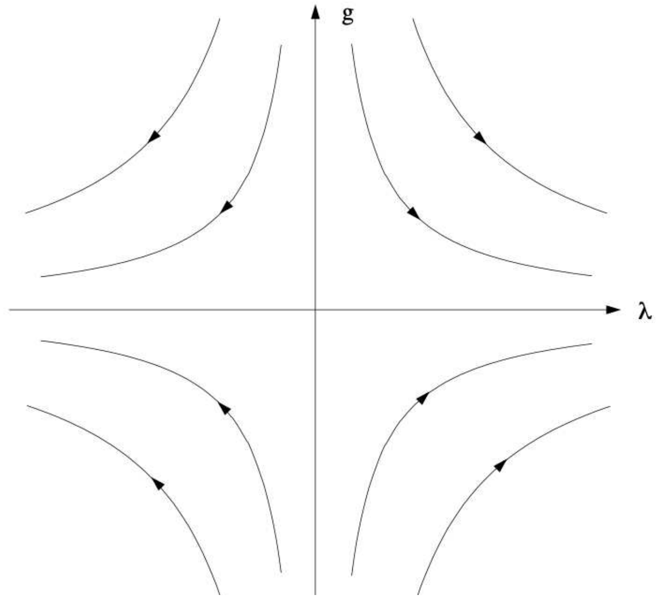
$$
\begin{aligned}
B_1(\lambda) &= \frac{1}{6\pi}\Big(\phi_1^1(w) - \phi_2^2(w)\Big) \\
B_2(\lambda) &= -\frac{1}{12\pi}\Big(\tilde{\phi}_1^1(w) - \tilde{\phi}_2^2(w)\Big)
\end{aligned}
$$

– the anomalous dimension becomes

$$
\eta = \frac{g_k B_1(\lambda)}{1 - g_k B_2(\lambda)}
$$

Region of general relatvity

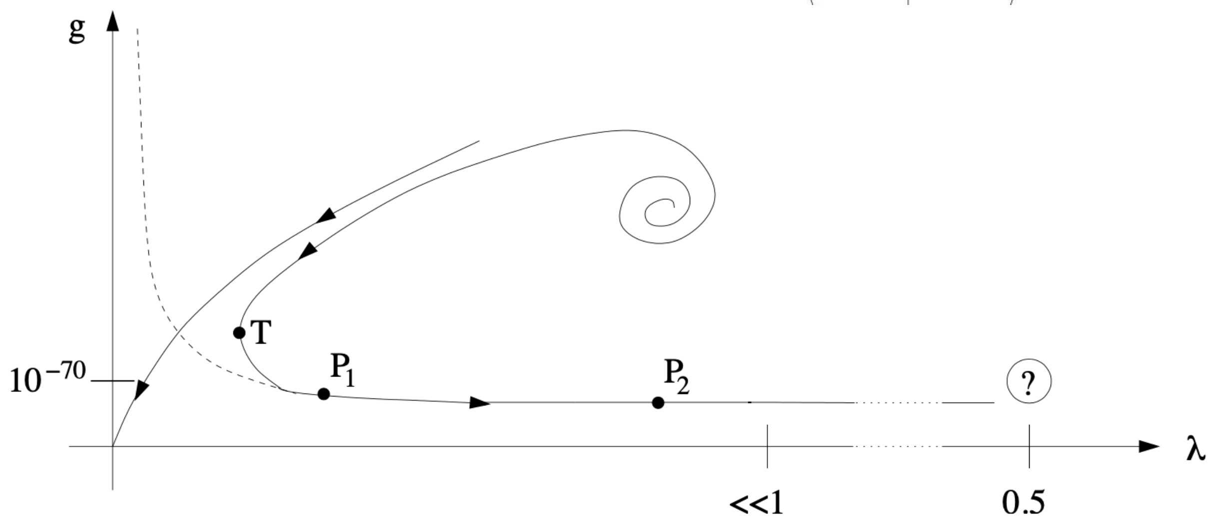
– look for region where G_k and Λ_k are constants



$$G_k \approx G_0, \quad g \sim k^2$$
$$\Lambda_k \approx \Lambda_0, \quad \lambda \sim k^{-2}$$

– $g \sim \frac{1}{\lambda}$ hyperbolas, close to the GFP

– $G_k > 0, \quad \Lambda_k > 0$



- GR regime should be large
- $P_1 < k < P_2$

Which trajectory is chosen by Nature?

estimate of $g(k_{lab})$:

– we measure G_k and Λ_k in these scales: $G(k_{lab})$ and $\Lambda(k_{lab})$
– laboratory scale from meters to AU ($\sim 10^{11}m$): $k_{lab}^{-1} = 1 \ldots 10^{11}$m
– Planck length $\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35}m$
– Planck mass $m_{Pl} = \sqrt{\frac{\hbar c}{G}} = 2.17 \times 10^{-8}kg$
– $\hbar = c = 1$
– $\ell_{Pl} = m_{Pl}^{-1} = \sqrt{G(k_{lab})}$

$$g(k_{lab}) = k_{lab}^2 G(k_{lab}), \quad \lambda(k_{lab}) = \Lambda(k_{lab})/k_{lab}^2$$

– in Planck units

$$g(k_{lab}) = (k_{lab}/m_{Pl})^2 \equiv (l_{Pl}/k_{lab}^{-1})^2$$

– we know that $G(k_{lab}) = G_{lab} \equiv 6,67 \times 10^{-11} m^3 kg^{-1} s^{-2}$
– k_{lab} can be determined

$$g(k_{lab}) \approx 10^{-70} \quad (k_{lab}^{-1} = 1\ m), \ \ g(k_{lab}) \approx 10^{-92} \quad (k_{lab}^{-1} = 1\ AU)$$

Which trajectory is chosen by Nature?

estimate of $\lambda(k_{lab})$:

– $\lambda(k_{lab})$ is measured in cosmological scales
– qualitative relation between the curvature and Λ_k

$$r_c \approx \Lambda(k_{lab})^{-1/2} = \lambda(k_{lab})^{-1/2} k_{lab}^{-1}$$

– almost flat spacetime at $k_{lab}^{-1} = 1\ m \to r_c \gg k_{lab}^{-1}$), so

$$\lambda(k_{lab}) \ll 1$$

– k_{lab} far from the k_{term} of the singularity scale $\lambda \approx 1/2$

GFP scaling

estimate of turning point T:

– $g, \lambda \ll 1 \to \mathcal{O}(g^2), \mathcal{O}(\lambda^2)$ are neglected
– $\dot{G}_k = \eta G_k$ with $\eta = -bg$, $b = (24 - \varphi_2)/3\pi \sim 1$, so

$$-\eta\big|_{GRregime} \approx 10^{-70} \ldots 10^{-92}$$

so $\eta = 0$
– RG equations

$$\begin{aligned} \partial_t \lambda &= -2\lambda + \varphi_2 g/\pi \\ \partial_t g &= 2g \end{aligned}$$

– $G_k = G_0$ constant, Λ_k has a weak evolution
– turning point: λ turn from decreasing to insreasing function, there $\dot{\lambda} = 0 \to$

$$\lambda_T = (\varphi_2/2\pi) g_T, \;\; \lambda_T/g_T = \mathcal{O}(1)$$

– λ is increasing, and at P_1 we have $2\lambda \gg \varphi_2 g/\pi$, so Λ_k becomes constant

GFP scaling

scaling solutions around GFP:

$g(k_T) = g_T$ and $\lambda(k_T) = \lambda_T$

$$\begin{aligned} g(k) &= g_T \left(\frac{k}{k_T}\right)^2 \\ \lambda(k) &= \lambda_T \left(\frac{k}{k_T}\right)^2 \left[1 + \left(\frac{k}{k_T}\right)^4\right] \end{aligned}$$

dimensionful couplings

$$\begin{aligned} G(k) &= \frac{g_T}{k_T^2} = const, \ \ G_{lab} = m_{Pl}^2, \ \ \rightarrow \ \ k_T^2 = g_T m_{Pl}^2 \\ \Lambda(k) &= \frac{1}{2}\lambda_T k_T^2 \left[1 + \left(\frac{k}{k_t}\right)^4\right] \end{aligned}$$

relation of k_T and m_{Pl}:

$$k_T = \sqrt{g_T} m_{Pl} \ \ \rightarrow \ \ m_{Pl} \gg k_T$$

GFP scaling

scaling in the GR regime:

– eliminate g_T

$$g(k) = \left(\frac{k}{m_{Pl}}\right)^2$$

$$\lambda(k) = \frac{1}{2} g_T \lambda_T \left(\frac{m_{Pl}}{k}\right)^2 \left[1 + \frac{k^4}{g_T^2 m_{Pl}^4}\right]$$

– remark:

$$G(k)\Lambda(k) = g(k)\lambda(k) = \frac{1}{2} g_T \lambda_T \left[1 + \left(\frac{k}{k_T}\right)^4\right]$$

– λ runs due to the factor $1 + (k/k_T)^4$, if $(k/k_T)^4 << \ll 1$ then the run is negligible

– definition of k_1 at P_1

$$k_1/k_T = 10^{-\nu}$$

– in GR regime

$$\begin{aligned} g(k) &= (k/m_{Pl})^2 \;\rightarrow\; G(k) = G_{lab} \\ \lambda(k) &= \frac{1}{2} g_T \lambda_T (m_{Pl}/k)^2 \;\rightarrow\; \Lambda(k) = \frac{1}{2}\lambda_T k_T^2 = \frac{1}{2}\Lambda(k_T) \end{aligned}$$

GFP scaling

scaling in the GR regime:

$$(G\Lambda)_{GR} = \frac{1}{2}(G\Lambda)_T$$

$$G(k)\Lambda(k) = g(k)\lambda(k) = \frac{1}{2}g_T\lambda_T$$

– at the beginning of the RG regime

$$g(k_1) = g_T 10^{-2\nu}, \;\; \lambda(k_1) = \lambda_T 10^{2\nu}$$

$$g(k_1) < g_T \quad \text{decreases}$$

$$\lambda(k_1) > \lambda_T \quad \text{increases}$$

– using $g_T \approx \lambda_T$

$$g(k_1) = \lambda(k_1)10^{-4\nu}$$

– so $\lambda_T < \lambda_{k_1} < \lambda_{k_{lab}} \ll 1$

GFP scaling

– it implies that

$$g_T \approx \lambda_T \ll 1$$

– so

$$g(k_1) \ll 1 \quad \text{and} \quad \lambda(k_1) \ll 1$$

– what do we know about the flow that Nature chose?

- at the Reuter fixed point $g^*, \lambda^* \sim \mathcal{O}(0.1)$
- it approaches the GFP (30 orders in k)
- wedge shaped trajectory at the turning point
- it spends long time near the GFP
- the points T and P_1 are located at an extremely short distance to the GFP.
- GR regime is far from UV and IR effects
- GR regime is long

the RG trajectory which Nature has selected is highly ”unnatural”

IR scales

hierarchies:

– we assume that

$$\lambda(k) = \frac{1}{2} g_T \lambda_T (m_{Pl}/k)^2$$

is valid in the IR, and using $\lambda = 1/2$ we get

$$k_{term} = \sqrt{g_T \lambda_T} m_{Pl} = (\varphi_2/2\pi)^{1/2} g_T m_{Pl} \approx g_T m_{Pl}$$

– since $k_T = \sqrt{g_T} m_{Pl}$

$$k_{term} = \sqrt{g_T} k_T$$

– in length scales

$$k_{term}^{-1} = \frac{\ell_{Pl}}{\sqrt{g_T \lambda_T}}$$

– since $g_T \approx \lambda_T \ll 1 \to k_{term}^{-1} \gg \ell_{Pl}$

– double hierarchy

$$\frac{k_{term}}{k_T} = \sqrt{g_T} \ll 1, \quad \frac{k_T}{m_{Pl}} = \sqrt{g_T} \ll 1$$

GFP scaling

– from $k_T = \sqrt{g_T} m_{Pl}$ and $\lambda(k) = \frac{1}{2}\lambda_T k_T^2$

$$\Lambda(k) = \frac{1}{2} g_T \lambda_T m_{Pl}^2 = const$$

– giving

$$\left.\frac{\Lambda}{m_{Pl}^2}\right|_{GR} = g_T^2 \ll 1$$

– **we can rephrase the cosmological constant problem**

- old question: Why is Λ so small?
- new question: Why does gravity behave classically over such a long interval of scales?

– $\lambda(k) = \Lambda_{lab}/k^2$, in the IR $\lambda(k) \approx 1 \rightarrow$

$$k_{term} \approx \sqrt{\Lambda_{lab}}$$

– nature picked trajectory gives small cosmological constant
– matter field cannot modify it
– $\Lambda(H_0)$ might differ from Λ_{lab} although the difference is small (not 120 orders of magnitude)

Hubble scale

estimate of g_T:

– in general

$$\Lambda = 3\Omega_\Lambda H_0^2 \approx H_0^2$$

– this gives a possible definition of the GR regime limit

$$k_{term} \approx H_0$$

– in the GR regime G and Λ do not vary too much

$$G(H_0) \sim G_{lab} \quad \Lambda(H_0) \sim \Lambda_{lab}$$

– then, using $\Lambda(k) = H_0^2$ and $G(k) = m_{Pl}^2$

$$\begin{aligned} \frac{1}{2} g_T \lambda_T &= g(k)\lambda(k) = (H_0/m_{Pl})^2 \\ g_T &\approx \lambda_T \approx H_0/m_{Pl} \approx 10^{-60} \end{aligned}$$

– consistent with $g_T > g_{lab} \approx 10^{-70}$
– 30 order of magnitude between (k_{term}, k_T) and (k_T, m_{Pl})

Hubble scale

– since $k_{term} = \sqrt{g_T \lambda_T} m_{Pl} \approx g_T m_{Pl} = H_0$ and $k_T = \sqrt{g_T} m_{Pl}$ then

$$k_T = \sqrt{H_0/m_{Pl}} \approx 10^{-30} m_{Pl}$$

– associated length scale

$$k_T^{-1} \approx 10^{30} \ell_{Pl} \approx 10^{-3} cm$$

is macroscopic

– if $\nu = 1$ then

$$k_1^{-1} \approx 10^{-2} cm$$

– if $\ell > k_1^{-1}$ then Λ is constant, otherwise

$$\Lambda(k) = H_0^2 \left[1 + (k/k_T)^4\right]$$

– we cannot detect the change of Λ in milli/ or micrometer scales since H_0^2 is extremely small

Hubble scale

estimate of g and λ in the IR:

$g(k) = (k/m_{Pl})^2 \rightarrow$

$$g(H_0) = (H_0/m_{Pl})^2$$

$\lambda_k = \frac{1}{2} g_T \lambda_T (m_{Pl}/H_0)^2 \rightarrow$

$$\lambda(H_0) = \frac{1}{2} g_T \lambda_T = \frac{1}{2}(H_0/m_{Pl})^2 (m_{Pl}/H_0)^2 = \frac{1}{2}$$

at the Hubble scale we expect

$$g(H_0) \approx 10^{-120}$$

$$\lambda(H_0) \approx \frac{1}{2}$$

IR fixed point

it is expected that new interactions become relevant in the IR, it can create a new IR attractive fixed point at

$$g^* \approx 10^{-120}, \quad \lambda^* \approx \mathcal{O}(0.1)$$

the fixed point scaling

$$G(k) = g_*^{IR}/k^2, \quad \Lambda_*^{IR} k^2$$

the cosmological time is

$$k = \hat{\xi}/t$$

then $G = G(t)$ and $\Lambda = \Lambda(t)$
– RG improved field equations can be derived
– RG trajectory $\to$ time evolution of Universe
– the IR fixed point is beyond the GR regime, and the Einstein-Hilbert truncation cannot be applied
– in the IR we expect deviation from classical cosmology at large distance scales
– discrepancy between the observable mass and the one got from observed motion for galactic systems.
– explanation can be the dark matter, however the IR fixed point due to the new physics there can account for the discrepancy

Summary of the Nature picked trajectory

- at $k = \infty$ it is infinitesimally close to the Reuter fixed point, $g \sim \mathcal{O}(1)$, $\lambda \sim \mathcal{O}(0.1)$
- runs along the separatrix till k_T, g, λ decreases
- at the turning point T $g_t \approx \lambda_T \approx 10^{-60}$
- turns at T and then it hits the GR regime at k_1
- between k_1 and k_2 G and Λ are constants
- when k approaches the deep IR regime, then $\lambda \to 1/2$ and the Einstein-Hilbert truncation is not reliable anymore
- it is expected that new interactions become relevant in the IR, creating a new IR attractive fixed point at $g^* \approx 10^{-120}$ and $\lambda^* \approx \mathcal{O}(0.1)$

Questions

- UV Reuter fixed point, however the blocking is meaningful into IR
- only relevant couplings
- UV Landau pole, QED + AB gravity may give a UV NGFP, but not for all initial values
- Euclidean vs Lorentz signature
  are there Lorentz covariant form of RG method?
  Lorentz symmetry violation at UV?
- new physics may arise beyond the Planck scale, however the Reuter fixed point disables it
- quantum – classical transition, AS gravity, QED
- open dynamics in RG?