SEGMENTED STRINGS AND HOLOGRAPHY

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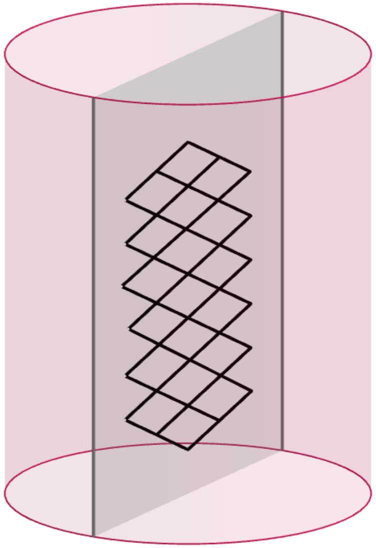

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BME PHYSICS

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2024

Outline

- Introduction
- The AdS_3/CFT_2 correspondence
- The Ryu-Takayanagi formula
- Segmented strings in AdS_3
- Ryu-Takayanagi formula for segmented strings
- Correspondence in even dimensions
- Continuous limit
- Summary and outlook

Introduction

- Electromagnetism
- Weak interaction
- Strong interaction
- Gravity

Quantum gauge theories

(QED, EWT, QCD)

Quantum gravity???

Possible solution – String theory ⇒ **AdS/CFT correspondence**

String theory on curved backround

↕

Field theory in Minkowski space

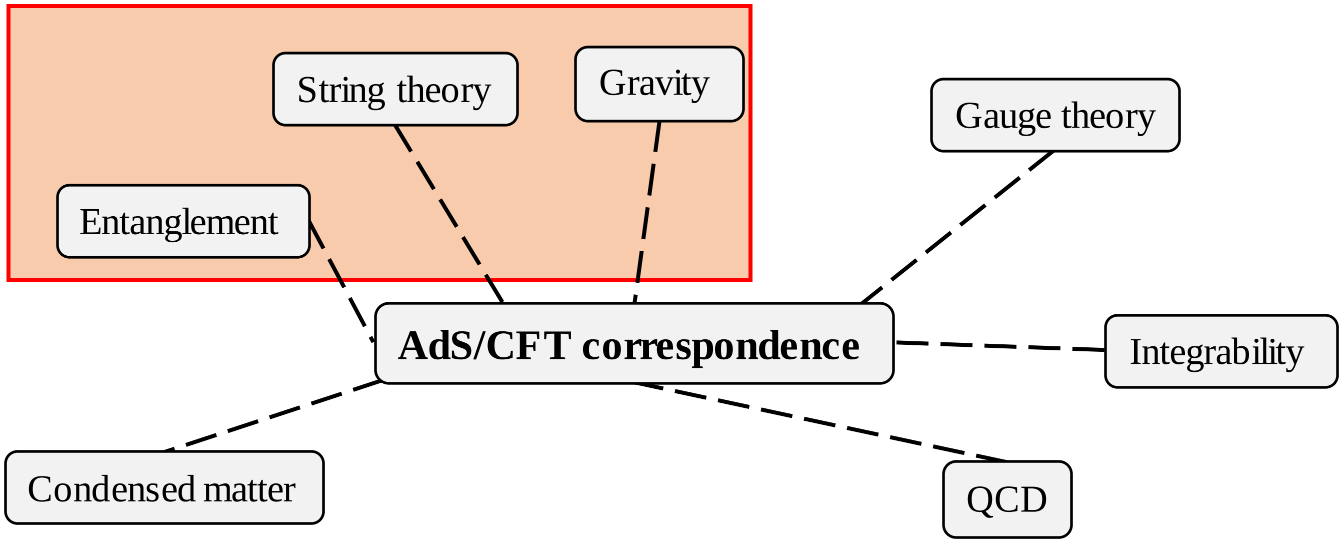
Original form: [Maldacena'97]

- IIB superstring theory on $AdS_5 \times S^5$
- $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

Other aspect: [Ryu,Takayanagi'06] [Raamsdonk'10]

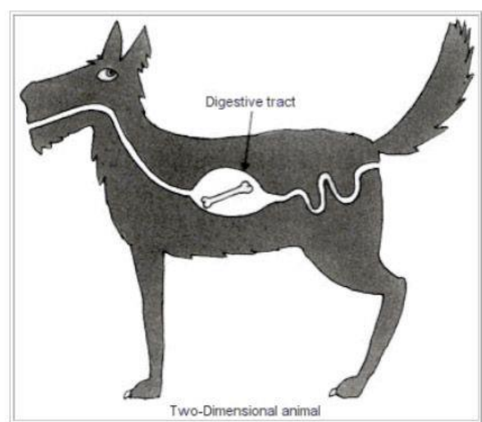
- Spacetime is built up of quantum entanglement

Introduction



In our case:

- Strings in AdS_3
- Entanglement in CFT_2



AdS_3 space

Einstein equations:

- In vacuum: $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$
- Cosmological constant: Λ
  - $\Lambda > 0$: de Sitter space
  - $\Lambda = 0$: Minkowski space
  - $\Lambda < 0$: **Anti-de Sitter (AdS) space**

AdS_3 space:

- Embedding space: $\mathbb{R}^{2,2}$
- 2+1 dimensional subset: AdS_3 space
- Metric:

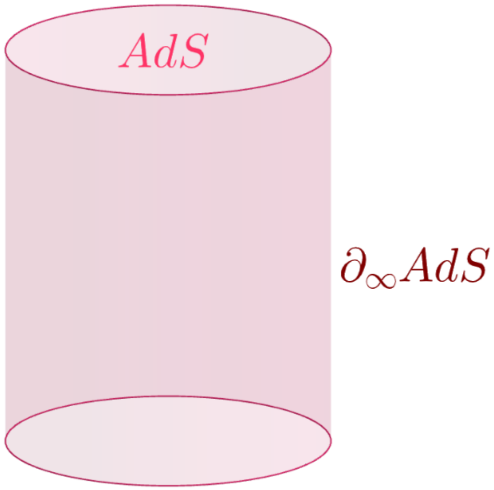
$$ds^2 = -(dX^{-1})^2 - (dX^0)^2 + (dX^1)^2 + (dX^2)^2$$

- Constraint:

$$X \cdot X = -(X^{-1})^2 - (X^0)^2 + (X^1)^2 + (X^2)^2 = -L^2$$

Boundary of AdS_3 *space*:

- $\partial_\infty AdS_3 = \mathbb{P}\{U \in \mathbb{R}^{2,2} | U \cdot U = 0\}$



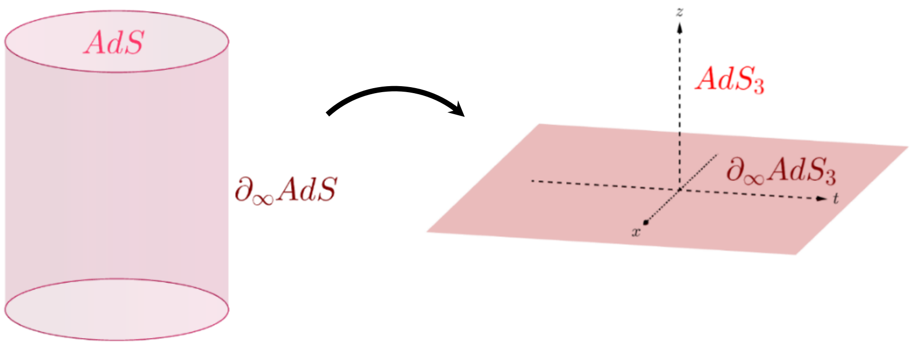
AdS$_3$ space

Poincaré upper half-space model:

- $2+1$ dimensional representation of the AdS_3 space
- Coordinates: (z, x^0, x^1)

$$(X^{-1}, X^\mu, X^d) = \left(\frac{-z^2 - x^2 - L^2}{2z}, \frac{Lx^\mu}{z}, \frac{-z^2 - x^2 + L^2}{2z}\right)$$

- Metric: $ds^2 = L^2 \frac{dz^2 - (dx^0)^2 + (dx^1)^2}{z^2}$



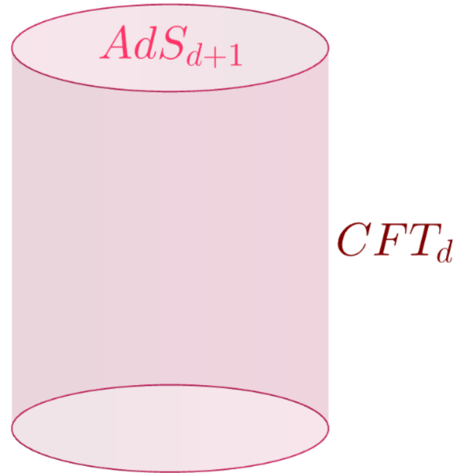
Boundary:

- $z \to 0 \;\Rightarrow$ Coordinates: (x^0, x^1)
- Metric: $ds^2 \propto -(dx^0)^2 + (dx^1)^2$
- Conformally equivalent to the 1+1 dimensional Minkowski space

The AdS/CFT correspondence

Conformal field theory:

- Quantum field theory
- With conformal invariance
- In $d = 2$: infinite-dimensional symmetry algebra
- Exactly solvable!



AdS_3/CFT_2 correspondence:

- $\partial_\infty AdS_3 \sim$ 1+1 dimensional Minkowski space
- Conformal field theory on the boundary

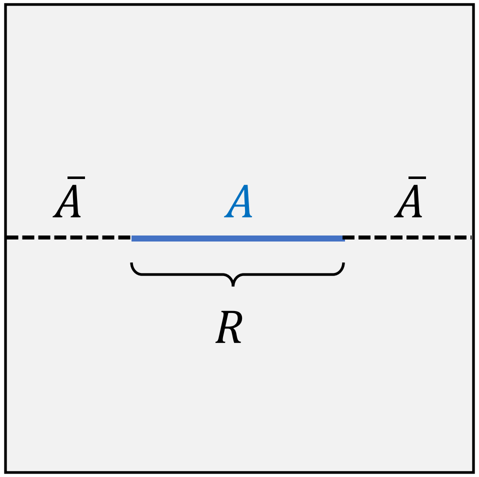
Classical quantities (AdS_3)
↕
Field theoretical quantities (CFT_2)

$$Z_{grav}\left[\Phi\Big|_{\partial AdS} = J\right] = Z_{CFT}[J]$$

Entanglement in CFT_2

Von Neumann entropy:

- Statistical ensemble
- Density matrix ρ
- Von Neumann entropy: $S = -tr\{\rho \log \rho\}$
- Measures indeterminacy of the system



Entanglement in CFT_2:

- 1+1 dimensional CFT in vacuum state
- Observer that only has acces to a region A
- Measures different density matrix: $\rho_A = tr_{\bar{A}}\rho$
- Entanglement entropy: $S(A) = -tr\{\rho_A \log \rho_A\}$
- Measures the entanglement between A and $\bar{A}$

For an interval: [Calabrese,Cardy'18]

- Let A be an interval
- Length R
- *Note:* Cutoff dependent (δ)

$$S(A) = \frac{c}{3}\log\frac{R}{\delta}$$

c: central charge of CFT

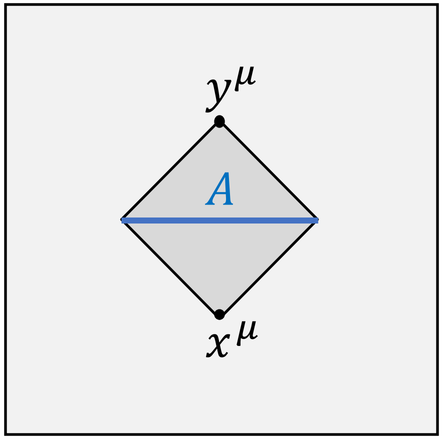
Entanglement in CFT_2

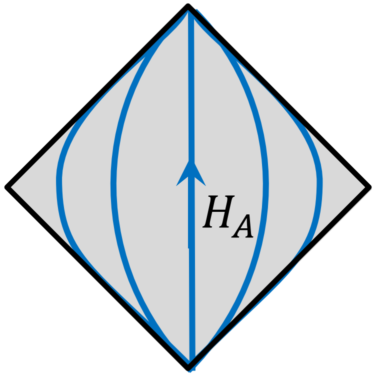
Causal diamonds:

- Causal diamond = causality domain of a subsystem A
- Described by its past and future tips x^μ and y^μ
- Reduced density matrix: $\rho_A = e^{-H_A}$
- Where H_A: **modular Hamiltonian**

Kinematic space: [Boer'16]

- Space of causal diamonds (or subsystems) → Coordinates: (x^μ, y^ν)
- Coset structure: $\frac{SO(2,2)}{SO(1,1)\times SO(1,1)}$ → Invariant metric: $\omega_{\mu\nu}$





Spherical minimal surfaces of AdS_3

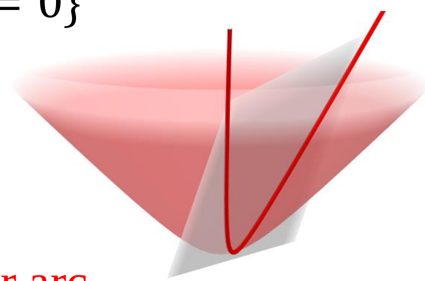
$$x_u^\mu = L\frac{U^\mu}{U^d - U^{-1}}$$

$$x_v^\nu = L\frac{V^\nu}{V^d - V^{-1}}$$

Minimal surfaces of AdS_3:

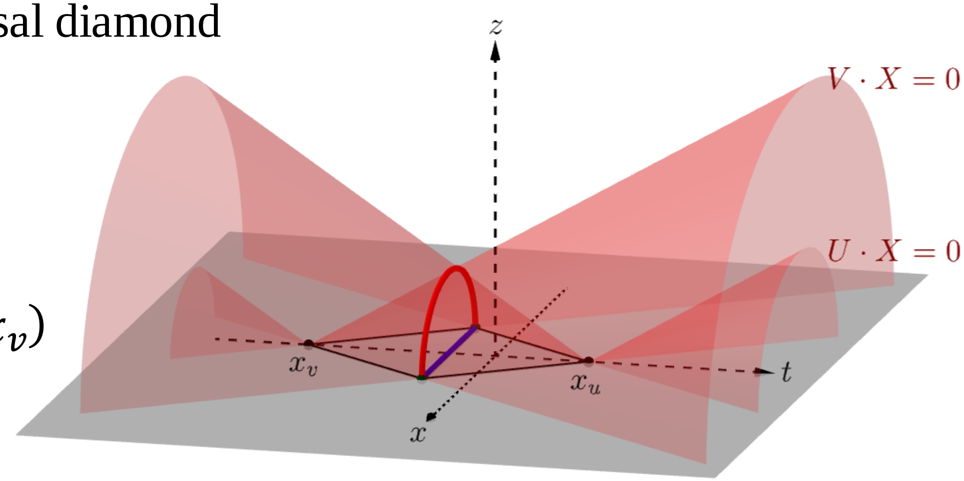
- Two null vectors: $U, V \in \mathbb{R}^{2,2}$: $U \cdot U = V \cdot V = 0$
- Minimal surface= $\{X \mid U \cdot X = 0 \cap V \cdot X = 0\}$

In the Poincaré model:



- $U \cdot X = 0$ and $V \cdot X = 0 \rightarrow$ Two cones
- Tips of cones are on the boundary: x_u, x_v
- Minimal surface: One dimensional circular arc
- Image on the boundary: Interval, causal diamond

Area of minimal surfaces:



- Area: $$A(U,V) = L \log\frac{4R^2}{\delta^2}$$
- Where: $R^2 = -\frac{1}{4}(x_u - x_v)\cdot(x_u - x_v)$
- Cutoff: $z > \delta$

⇓

Proportional to the entanglement entropy of the resulting boundary interval

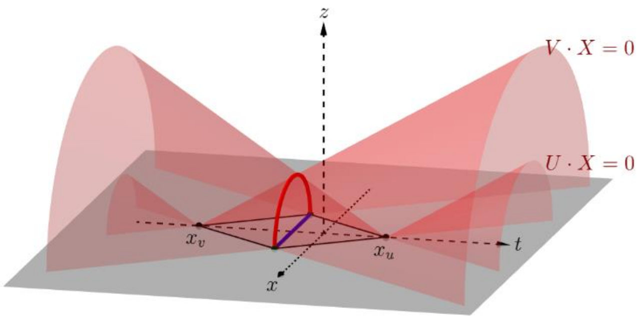
The Ryu-Takayanagi formula

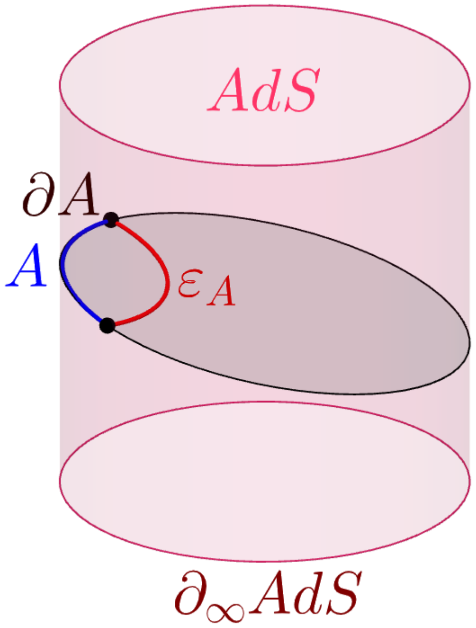
The Ryu-Takayanagi formula: [Ryu,Takayanagi'06]

- AdS_3 space
- Vacuum CFT_2 on the boundary
- Brown-Henneaux formula: $c = \frac{3L}{2G}$ [Brown'86]
- An A spacelike CFT subsystem, border ∂A:
  - Entanglement entropy: $S(A) = -Tr\{\rho_A \log \rho_A\}$
- ε_A AdS minimal surface, border ∂A:
  - Surface area: $A_{min}(A)$

$$S(A) = \frac{A_{min}(A)}{4G}$$

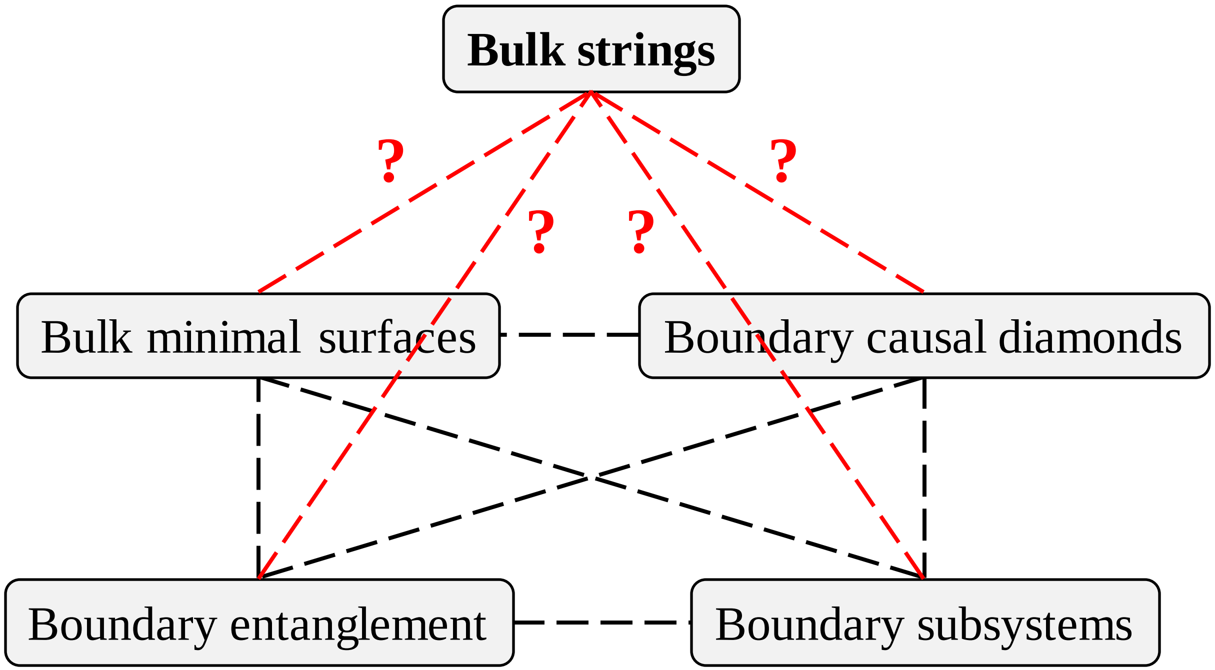
Ryu-Takayanagi formula





In our case:

- Image of minimal surface on the boundary: Interval
- Entanglement entropy: $S(U,V) = \frac{L}{4G} \log \frac{4R^2}{\delta^2}$


Bulk strings
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Bulk minimal surfaces
Boundary causal diamonds
Boundary entanglement
Boundary subsystems

Classical strings in AdS_3

One dimensional classical strings:

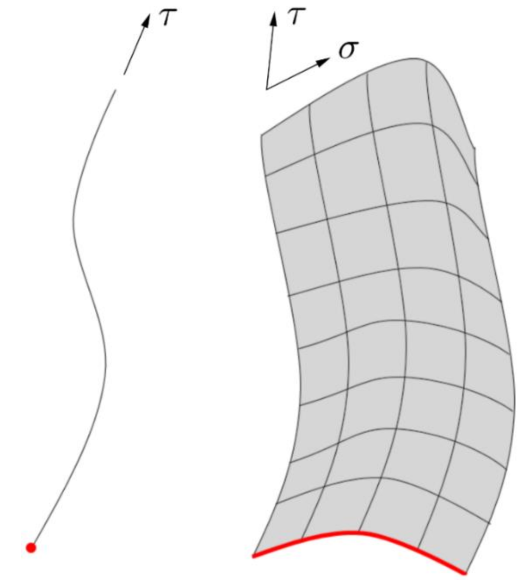
- String: One dimensional object
- Propagation in spacetime: Two dimensional „worldsheet”
- Two parameters: (τ, σ) or (σ^+, σ^-)
- Action: $S = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h} h^{ab} \partial_a X \cdot \partial_b X$ ~ Surface area
- $\delta S = 0$ $\begin{cases} \text{Equation of motion} \\ \text{Virasoro constraints} \end{cases}$

Strings in AdS_3:

- Embedding space: AdS_3
- Equations of motion:

$$\partial_+\partial_- X - (\partial_- X \cdot \partial_+ X) X = 0$$
$$\partial_+ X \cdot \partial_+ X = \partial_- X \cdot \partial_- X = 0$$
$$X \cdot X = -L^2$$

- Normal vector: $N_a = \frac{\epsilon_{abcd} X^b \partial_- X^c \partial_+ X^d}{\partial_- X \cdot \partial_+ X}$



$h^{ab}(\sigma, \tau)$: worldsheet metric

Segmented strings in AdS space

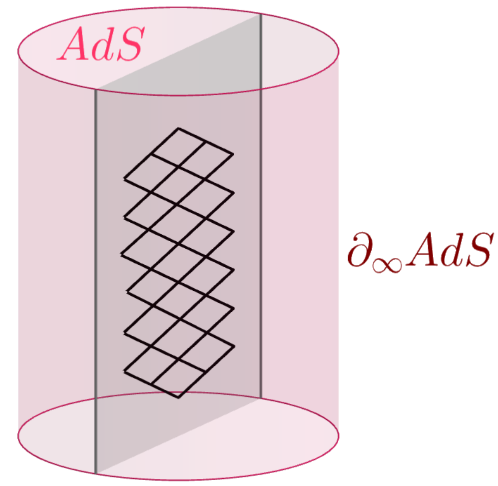
Segmented strings: [Callebaut'15]

- Simplest solution: constant normal vector
- String segment: quadrangle with constant normal vector
- Segmented string: solution built up by segments
- Vertices: $V_i \cdot V_i = -L^2$, $i = 1,2,3,4$
- Edges: $p_i = \pm(V_i - V_{i+1}), i = 1,2,3,4$

$$p_i \cdot p_i = 0$$

⇓

$$X(\sigma^-,\sigma^+) = \frac{L^2 + \sigma^+\sigma^- \frac{1}{2} p_1 \cdot p_4}{L^2 - \sigma^+\sigma^- \frac{1}{2} p_1 \cdot p_4} V_1 + L^2 \frac{\sigma^- p_1 + \sigma^+ p_4}{L^2 - \sigma^+\sigma^- \frac{1}{2} p_1 \cdot p_4}$$

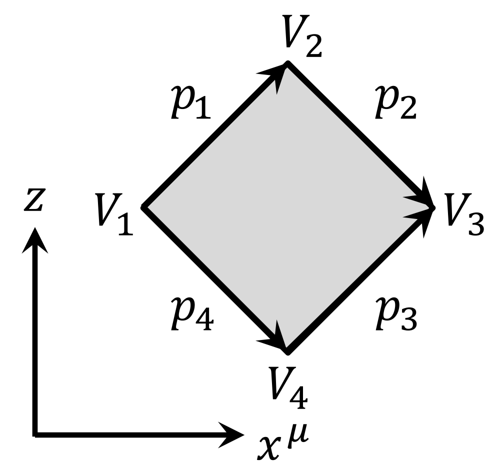


Area of a string segment:

- Evaluating the string action with the segmented solution

$$A_{\boxdot} = L^2 \log \frac{(p_1 \cdot p_4)(p_2 \cdot p_3)}{(p_1 \cdot p_2)(p_3 \cdot p_4)}$$

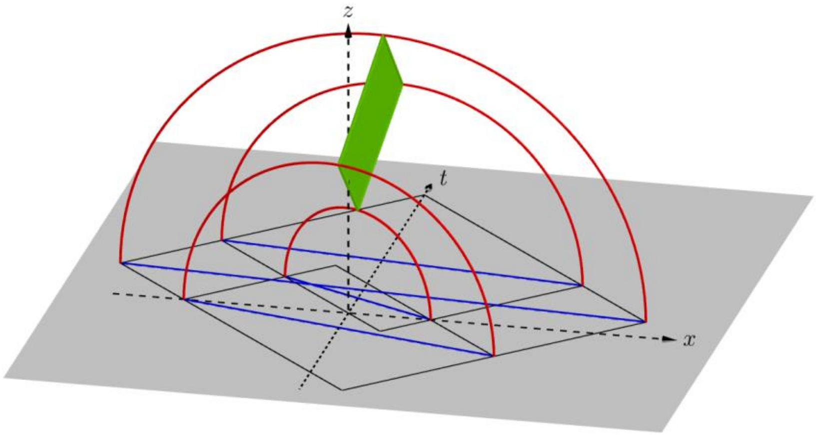
- *Note:* Cutoff independent!



Segmented strings and the boundary

Projection to the boundary:

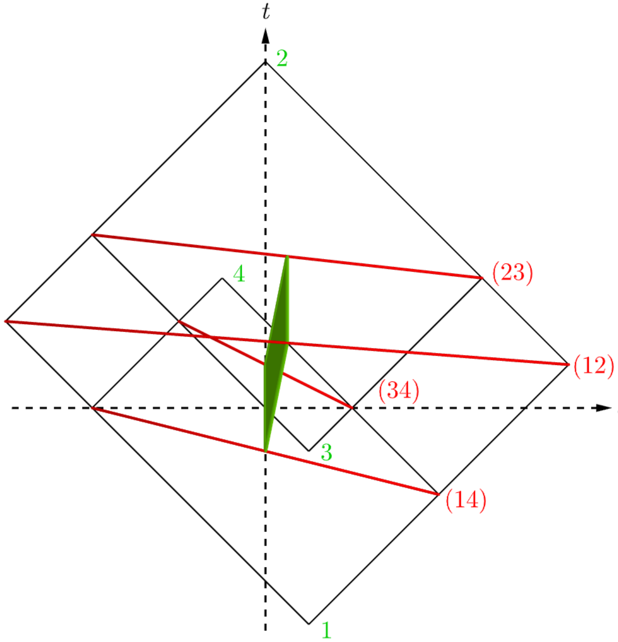
- $p_i \cdot p_i = 0$!
- p_i define four cones in AdS via $p_i \cdot X = 0$
- At each vertex two edges meat
- Therefore each vertex lies on an intersection of a pair of cones
- Let: $x_i^\mu = L \frac{p_i^\mu}{p_i^d - p_i^-}$
- Then $\left(x_i^\mu, x_j^\nu\right)$ defines a causal diamond and a subsystem (ij)



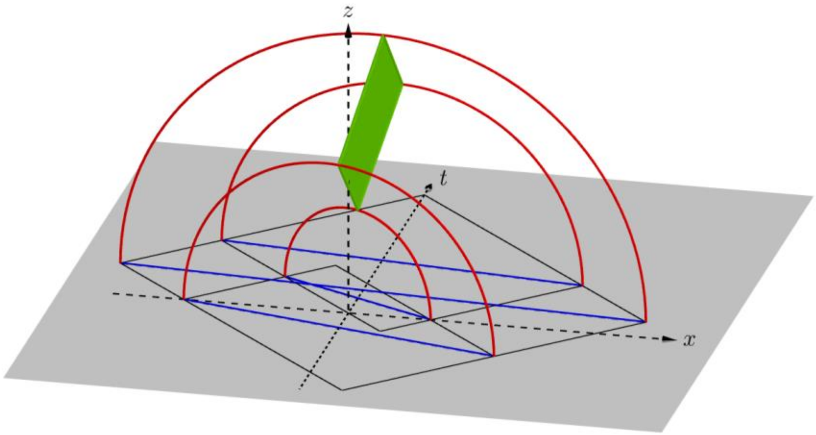
Area of the string segment:

$$A_{\boxdot} = L^2 \log \frac{R_{14}^2 R_{23}^2}{R_{12}^2 R_{34}^2} = A_{14} + A_{23} - A_{34} - A_{12}$$

- Where: $R_{ij}^2 = -\frac{1}{4}\left(x_i - x_j\right)^2$



Ryu-Takayanagi formula for segmented strings

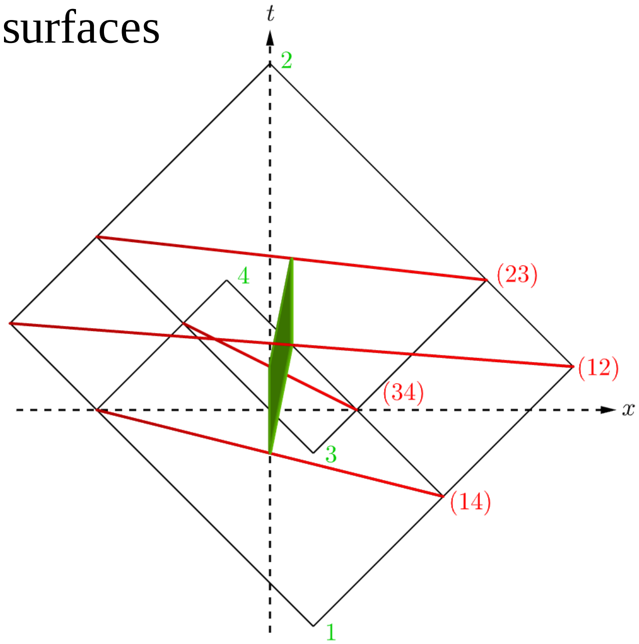


Timelike segmented string:

- Edges: $p_i : p_i \cdot p_i = 0, i = 1,2,3,4$
- For vertices: $p_i \cdot X = 0$
- Area: $A = L^2 \log \frac{R_{14}^2 R_{23}^2}{R_{12}^2 R_{34}^2}$

Spacelike minimal surfaces:

- Cones: $p_i \cdot X = 0$
- Their intersections: $(14), (23), (12), (34)$ minimal surfaces
- Segment vertices lie on them!
- Their areas: $A(ij) = L \log \frac{4R_{ij}^2}{\delta^2}$



CFT vacuum subsystems:

- Images of minimal surfaces: (ij) subsystems
- Their entanglement entropies: $S(ij) = \frac{A(ij)}{4G}$

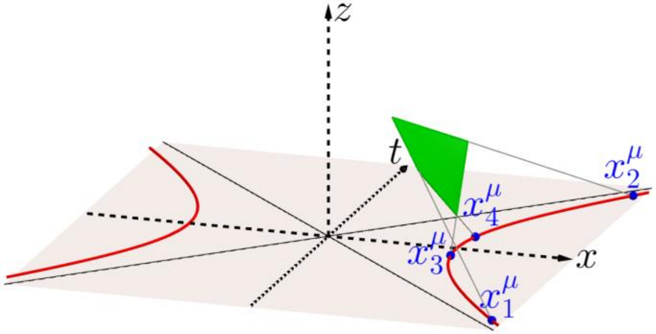
$$A \equiv 4GL(S(14) + S(23) - S(34) - S(12))$$

Interpretation and consequences

Interpretation:

- CFT subsystems
- Flow of boundary causal diamonds

} **Holograms of bulk strings**



Interesting properties: Eg.:

- For a segment: $p_1 + p_2 = p_3 + p_4$
- On the boundary:
  causal diamonds lie on a common hyperbola
- Accelerating frame

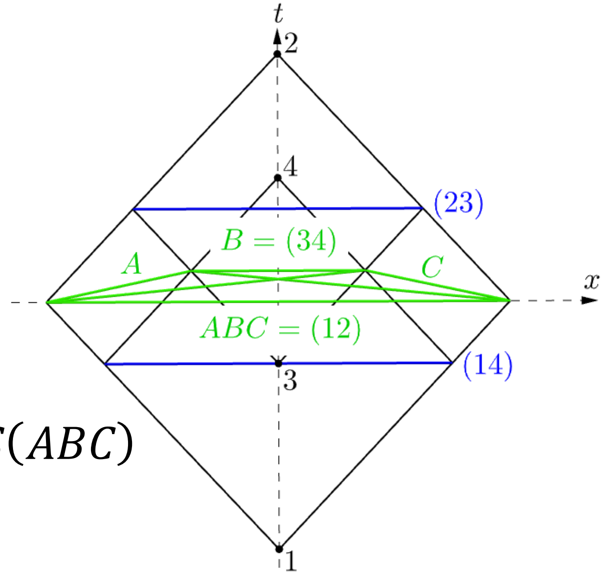
Consequences: Eg.: strong subadditivity:

- Subsystems A, B, C

$$S(AB) + S(BC) - S(B) - S(ABC) \geq 0$$

Strong subadditivity!

- Measuring on a larger subsystem reduces uncertainity
- $\exists A, B, C$ subsystems: $A_{\boxdot} \sim S(AB) + S(BC) - S(B) - S(ABC)$
- Geometrically: Positivity of segment area $A \geq 0$



Minimal surfaces in higher dimensions

Minimal surfaces of the AdS_{d+1}:

- Two null vectors: $U, V \in \mathbb{R}^{2,d}$: $U \cdot U = V \cdot V = 0$
- Minimal surface= $\{X \mid U \cdot X = 0 \cap V \cdot X = 0\}$

In the Poincaré model:

- $U \cdot X = 0$ and $V \cdot X = 0 \rightarrow$ Two cones
- Tips of the cones are on the boundary: x_u, x_v
- Minimal surface: d-1 dimensional sphere
- Image on the boundary: d-2 sphere, causal diamond

Area of minimal surfaces: [Ryu,Takayanagi'06]

- If d =even: $A = \left(Powers\ of\ \frac{R}{\delta}\right) + \boxed{\alpha \log \frac{R^2}{\delta^2}}$

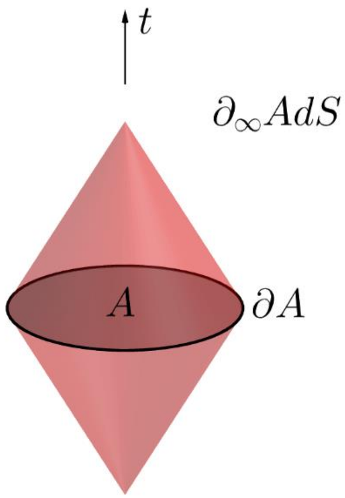
- If d =odd $A = \left(Powers\ of\ \frac{R}{\delta}\right)$

Correspondence in even dimensions

Ryu-Takayanagi proposal in higher dimensions:

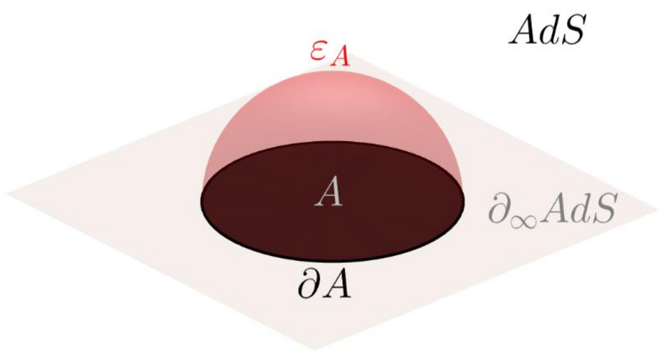
- AdS_{d+1} space
- Boundary CFT_d in vacuum state
- A spacelike CFT subsystem with boundary ∂A
- ε_A AdS minimal surface with boundary ∂A
- Entanglement entropy in general:

$$S(A) = \frac{A_{min}(A)}{4G}$$



Entropy in higher dimensions:

- $S = S^{\partial} + S^{uni} + S^{other}$
- $S \propto \frac{A(\partial A)}{\delta^{d-2}}$
- If $d = even$: $S^{uni} \propto \log\frac{R}{\delta}$
- If $d = odd$: $S^{uni} = const.$



Segmented strings in higher dimensions

Strings in AdS_{d+1}:

- Embedding space: AdS_{d+1}
- Equations of motion:

$$\partial_+\partial_- X - (\partial_- X \cdot \partial_+ X) X = 0$$
$$\partial_+ X \cdot \partial_+ X = \partial_- X \cdot \partial_- X = 0$$
$$X \cdot X = -L^2$$

Segmented strings:

- Vertices: $V_i \cdot V_i = -L^2$, $i = 1,2,3,4$
- Edges: $p_i = \pm(V_i - V_{i+1}), i = 1,2,3,4$
- Same interpolation ansatz

Projection to the boundary:

- Let: $x_i^\mu = L \frac{p_i^\mu}{p_i^d - p_i^-}$

Area of the string segment:

$$A_{\boxdot} = L^2 \log \frac{R_{14}^2 R_{23}^2}{R_{12}^2 R_{34}^2}$$

- Where: $R_{ij}^2 = -\frac{1}{4}\left(x_i - x_j\right)^2$

Correspondence in even dimensions:

- $\boldsymbol{d}$ **= even**
- Edges: $p_i \leftrightarrow$ Cones
- Vertices on minimal surfaces
- Minimal surfaces $\leftrightarrow$ Spherical subsystems
- Area of segment $\leftrightarrow$ Entanglement

$$A \sim \left(S^{uni}(14) + S^{uni}(23) - \\ - S^{uni}(34) - S^{uni}(12)\right)$$

⇩

$\boldsymbol{d}$ **= odd?**

Continuous limit

Continuous limit:

- General AdS_{d+1} space
- Directional derivatives: $\partial_- X$, $\partial_+ X$
  $\longrightarrow$ Virasoro constraints: $\partial_+ X \cdot \partial_+ X = \partial_- X \cdot \partial_- X = 0$
- Poincaré coordinates: $x^\mu = L \frac{\partial_- X^\mu}{\partial_- X^d - \partial_- X^{-1}}, y^\mu = \frac{\partial_+ X^\mu}{\partial_+ X^d - \partial_+ X^{-1}}$
- On the worldsheet: $\partial_- X \cdot X = 0, \partial_+ X \cdot X = 0$
  $\longrightarrow$ Causal diamond with tips x^μ, y^μ
- String action in causal diamond coordinates:

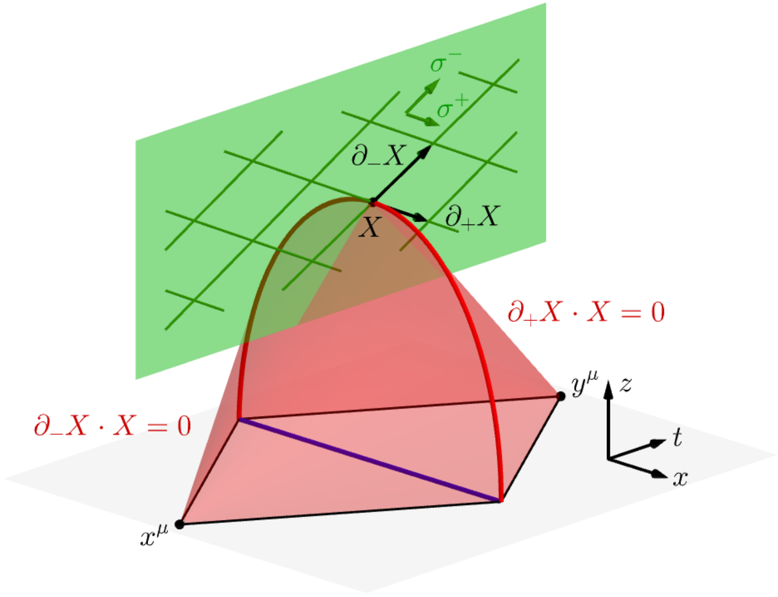
$$S = \int d\sigma^- d\sigma^+ \sqrt{-h} h^{ab} \omega_{\mu\nu} \partial_{(a} x^\mu \partial_{b)} y^\nu$$

$\Downarrow$

$\omega_{\mu\nu} \equiv$ Kinematic space metric

$\Downarrow$

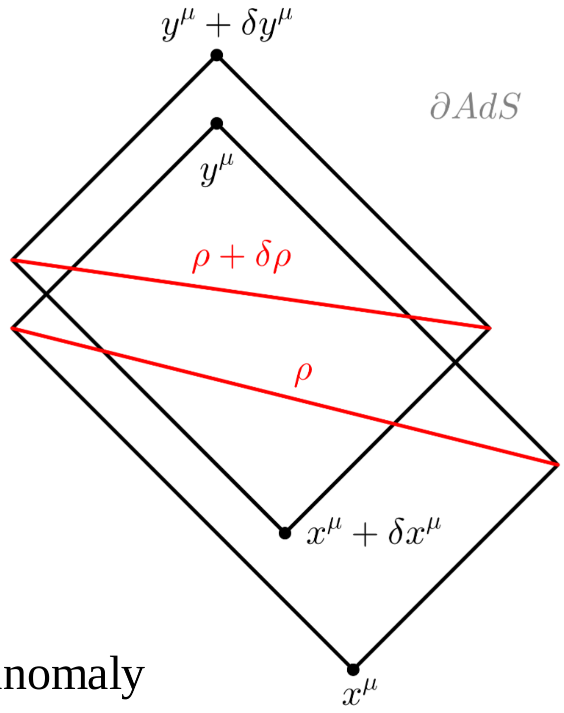
$\frac{SO(2,2)}{SO(1,1)\times SO(1,1)}$ invariant!



Fidelity susceptibility

Infinitesimally close causal diamonds:

- Causal diamonds at (x^μ, y^μ) and $(x^\mu + \delta x^\mu, y^\mu + \delta y^\mu)$
- Density matrices: ρ and $\rho + \delta\rho$
- Parallel purifications: ψ and $\psi + \delta\psi$ [Uhlmann'86]

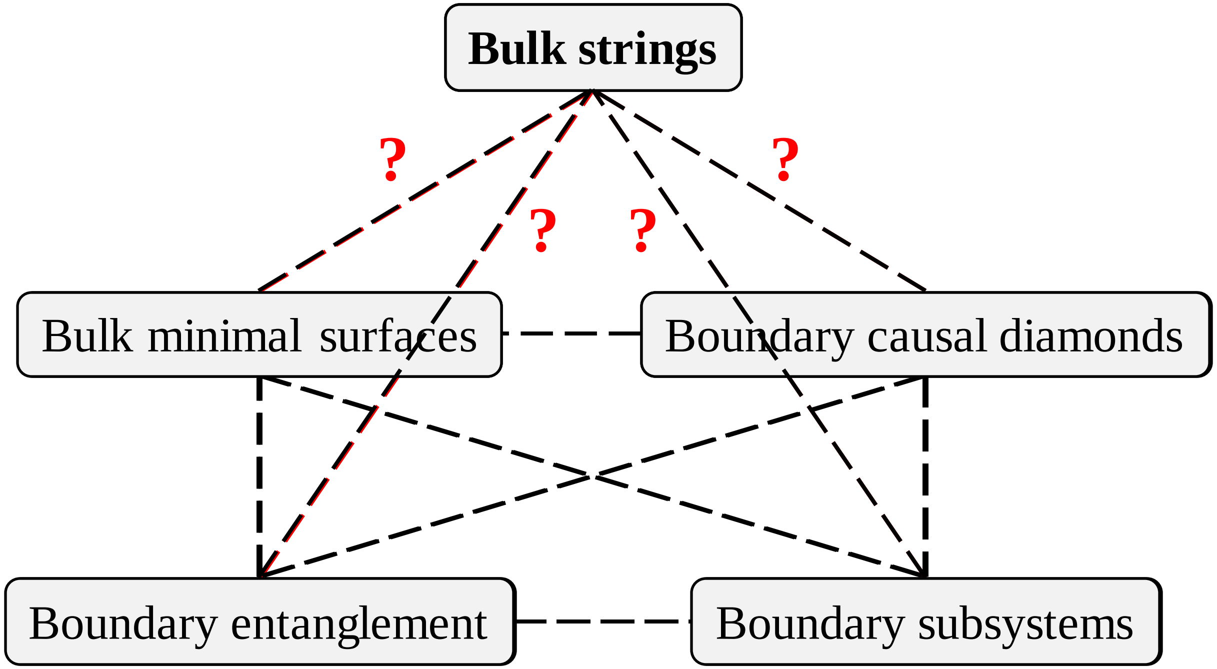


Fidelity susceptibility and complexity:

- Overlap of states: $|\langle\psi|\psi + \delta\psi\rangle| = 1 - \omega^{FS}_{\mu\nu}\delta x^\mu \delta y^\nu + \cdots$
- Where: $\omega^{FS}_{\mu\nu}$: **fidelity susceptibility**
- Kinematic space metric $\omega_{\mu\nu} \propto \omega^{\mathrm{FS}}_{\mu\nu}$
- Where prefactors contain geometric factors and a_d^* trace anomaly

$\Downarrow$

Bulk string geometry ↔ Boundary quantum geometry


Bulk strings
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Bulk minimal surfaces
Boundary causal diamonds
Boundary entanglement
Boundary subsystems

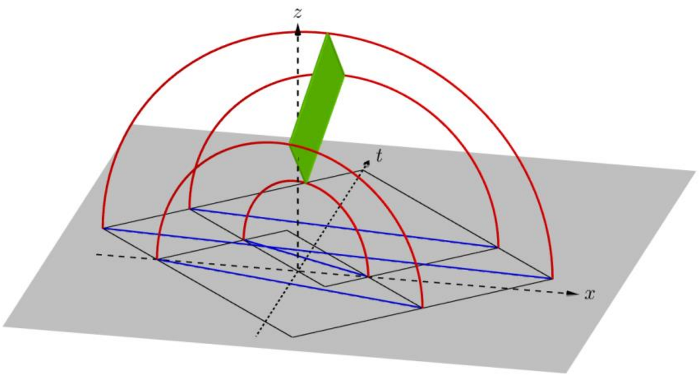
Summary

Summary:

- AdS_3/CFT_2 correspondence
- Ryu-Takayanagi formula → Connection between surfaces of segmented strings, areas of minimal surfaces and entanglement entropies of subsystems in CFT vacuum
- Geometry of classical strings ↔ Field theoretical entanglement
- Dictionary: Eg.:
    - Positivity of area ↔ Strong subadditivity
- Duality for segmented strings in AdS_{d+1}/CFT_d, if d = even
- Continious limit ↔ Quantum geometry

Outlook:

- Entanglement as a glue?
- *CFT* excitations?
- Further entanglement inequalities?



THANK YOU FOR YOUR ATTENTION!

[**B. Boldis**, P. Lévay, Phys. Rev. D **109**, 046002]

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- Ryu-Takayanagi formula → Connection between surfaces of segmented strings, areas of minimal surfaces and entanglement entropies of subsystems in CFT vacuum
- Geometry of classical strings ↔ Field theoretical entanglement
- Dictionary: Eg.:
  - Positivity of area ↔ Strong subadditivity
- Duality for segmented strings in AdS_{d+1}/CFT_d, if $d =$ even
- Continious limit ↔ Quantum geometry

Outlook:

- Entanglement as a glue?
- *CFT* excitations?
- Further entanglement inequalities?