Theory to Enable Practical Quantum Advantage

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&

Quantum Motion

Shallow quantum circuits

- Shor's algorithm, etc. needs $> 10^{12}$ quantum gates

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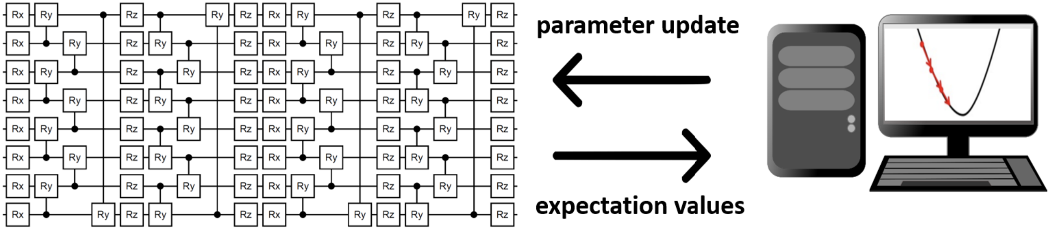
- Shor's algorithm, etc. needs $> 10^{12}$ quantum gates
- extract information $E(\underline{\theta}) = \langle \psi(\underline{\theta}) | \mathcal{H} | \psi(\underline{\theta}) \rangle$ using **error mitigation**
- **training** to find solution parameters $\underline{\theta}^{\star}$



Main challenges

Classical Control

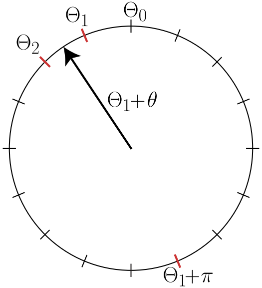
Quantum Error Mitigation

Extracting Classical Information

Training Variational Circuits

Classical control

- most algorithms need $\mathcal{R}(\theta)$ with $0 \leq \theta \leq 2\pi$
- but classical control at cryogenic T
- B bits discretisation: limited precision in θ



→ **B Koczor** J Morton, S C Benjamin *Probabilistic Interpolation of Quantum Rotation Angles* Phys. Rev. Lett. (2024) 132, 130602

→ **B Koczor** *Sparse Probabilistic Synthesis of Quantum Operations* arXiv:2402.15550

$$\mathcal{R}(\Theta_k+\theta) = \gamma_1(\theta)\mathcal{R}_1 + \gamma_2(\theta)\mathcal{R}_2 + \gamma_3(\theta)\mathcal{R}_3. \qquad (1)$$

- analytical solution for $\gamma_l(\theta)$
- $\gamma_3(\theta)$ is negative!

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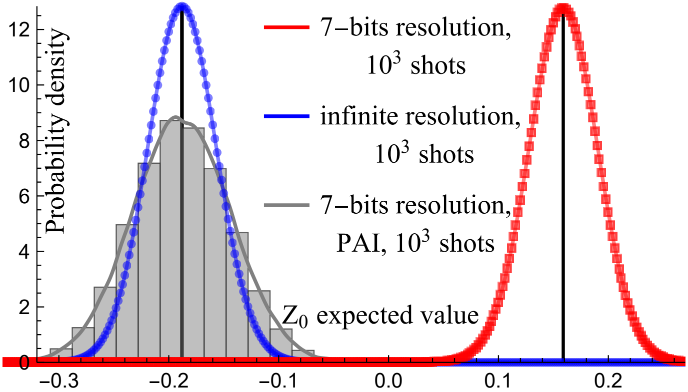
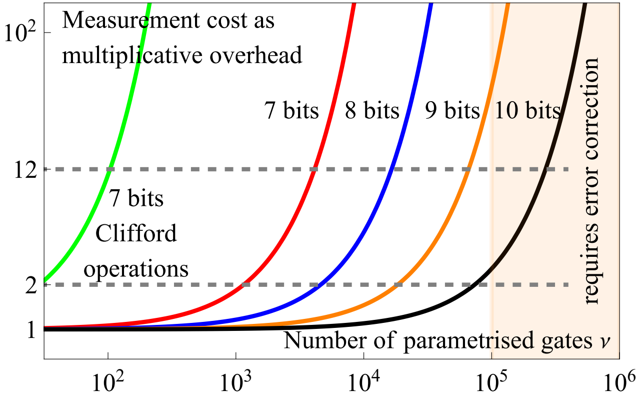


Theorem:

- best possible protocol: Θ_k, Θ_{k+1} and $\Theta_k+\pi+\frac{\Delta}{2}$
- yields minimal overhead $\|\gamma\|_1$ (cost)



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Generalisation

- dictionary $\{\mathcal{U}_l\}$ of N_{dict} different native operations
- overcome hardware limitations to get exact $\mathcal{U}_{desired}$

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$$\min_{\underline{\gamma}} \|\gamma\|_1 \qquad \text{subject to} \qquad \mathcal{U}_{desired} = \sum_{l=1}^{N_{dict}} \gamma_l \mathcal{U}_l. \tag{2}$$

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$\rightarrow$ randomly choosing $\mathcal{U}_l$ yields on average $\mathcal{U}_{desired}$

$\rightarrow$ **B Koczor** *Sparse Probabilistic Synthesis of Quantum Operations* arXiv:2402.15550

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- $\hat{\mathcal{R}}_{\text{desired}}$ is continuous rotation
- $\hat{\mathcal{R}}_l$ are different finite Clifford+T sequences (red and black dots)

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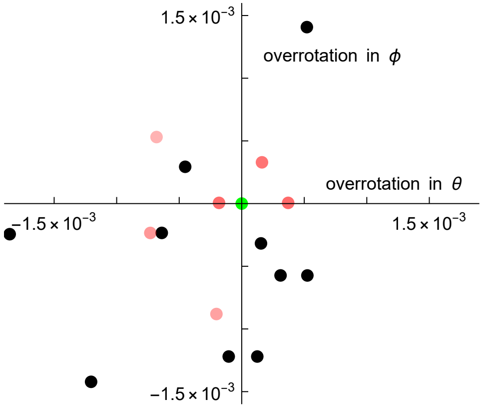
- $\hat{\mathcal{R}}_{\text{desired}}$ is continuous rotation (green dot)
- $\hat{\mathcal{R}}_l$ are different finite Clifford+T sequences (red and black dots)



Example 2: broadband pulses

- $\hat{\mathcal{R}}_{\text{desired}}$ is broadband pulse
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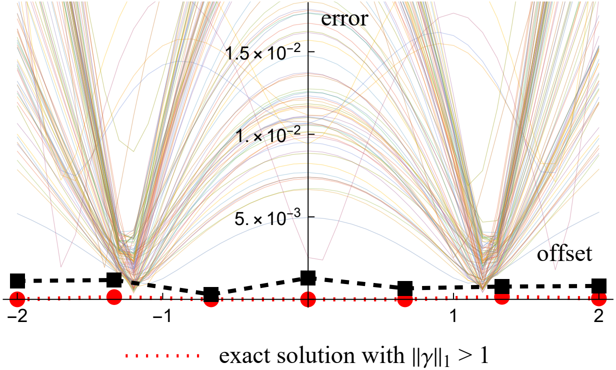
- $\hat{\mathcal{R}}_{\text{desired}}$ is broadband pulse
- $\hat{\mathcal{R}}_l$ are different shaped pulses (solid lines)
- black dots: approximation with no overhead!



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(almost all) algorithms estimate noisy expectation values

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Error mitigation extends the computational reach of a noisy quantum processor

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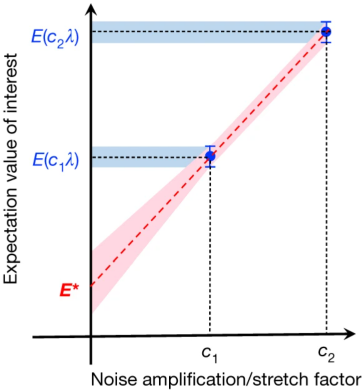

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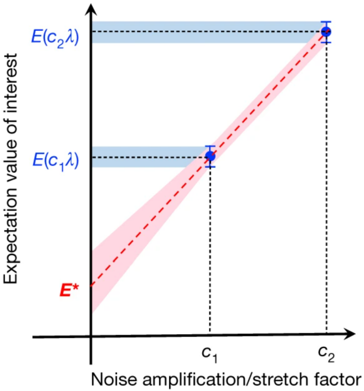
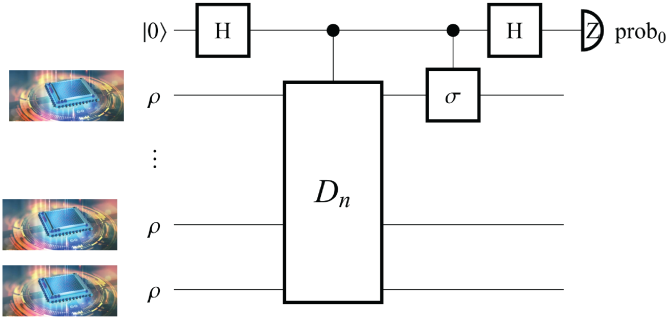


many techniques/tricks – **no theoretical guarantees**

Review paper: Z. Cai et al., *Quantum Error Mitigation* Rev. Mod. Phys. 95, 045005 (2023)

Error Suppression by Derangements (ESD)

- we use n cores to prepare copies of ρ
- want to measure expectation value of σ
- error is suppressed exponentially as Q^n with $Q < 1$

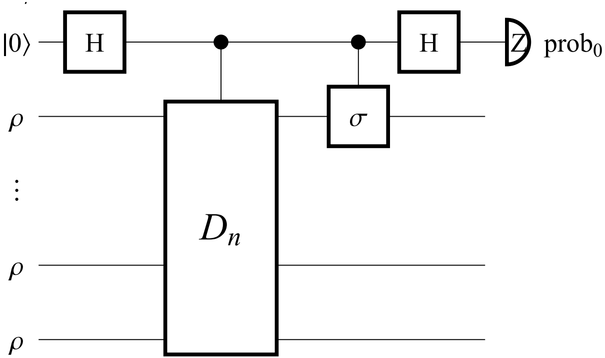


→ **B. Koczor** *Exponential Error Suppression for Near-Term Quantum Devices* (2021 Sept.) Phys Rev X, 11, 031057

→ Huggins et al. *Virtual Distillation for Quantum Error Mitigation* (2021 Nov.) Phys Rev X, 11, 041036

→ **B Koczor** *The Dominant Eigenvector of a Noisy Quantum State* (2021 Dec.) New J Phys, 23, 123047

- derangement is generalisation of SWAP
- excludes states that break permutation symmetry



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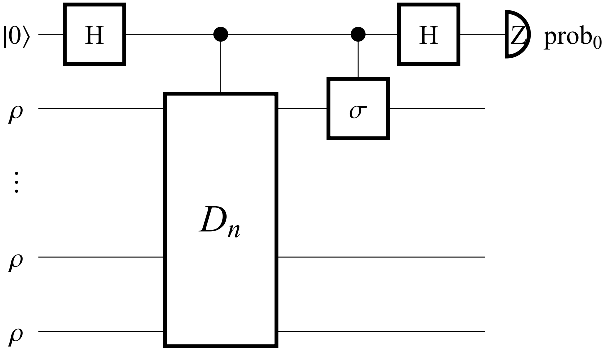
- excludes states that break permutation symmetry
- includes error-free state $|\psi\rangle^{\otimes n}$ with probability λ^n
- includes same errors to all registers $|\psi_k\rangle^{\otimes n}$ with probability $(1-\lambda)^n p_k^n$



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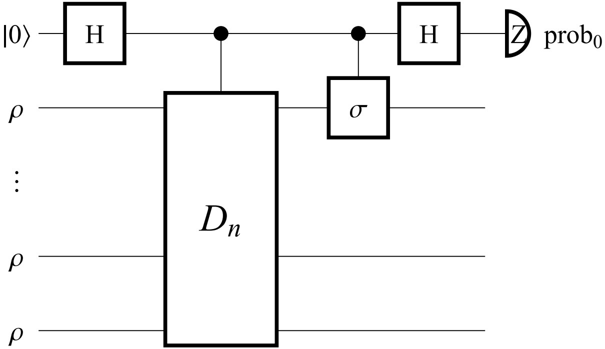
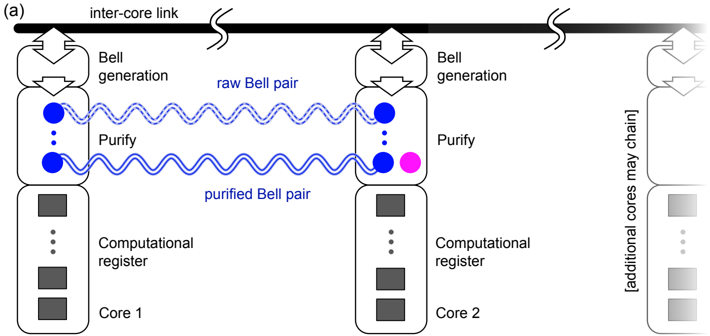


Formally: $\frac{\mathrm{Tr}[\rho^n \sigma]}{\mathrm{Tr}[\rho^n]}$

Multicore architectures

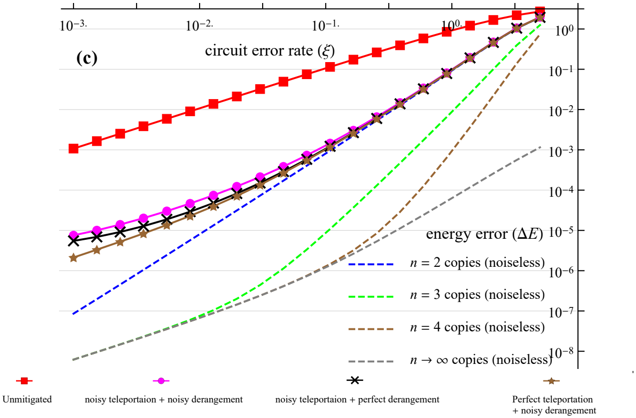
- Bell pairs between quantum cores (silicon, ion traps...)
- quantum teleportation enables error mitigation



→ H Jnane, B Undseth, Z Cai, SC Benjamin, **B Koczor** *Multicore Quantum Computing* (2022) Phys Rev Applied 18, 044064

→ L J Stephenson et al. *High-Rate, High-Fidelity Entanglement of Qubits Across an Elementary Quantum Network* (2020) Phys Rev Lett 124, 110501

- simulation of 6-qubit spin-chain ground state
- near perfect mitigation even with noisy links



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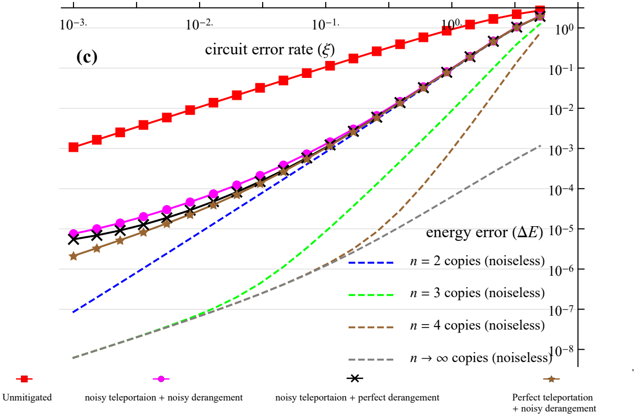
- Google experiment: 140-fold suppression



T E O'Brien et al. *Purification-based quantum error mitigation of pair-correlated electron simulations*
Nature Physics 19, 1787 (2023)

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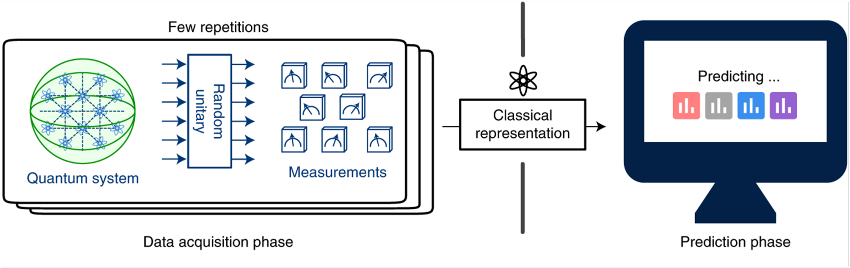
Classical shadows

- randomly rotate every qubit and measure
- efficient representation of the state ρ



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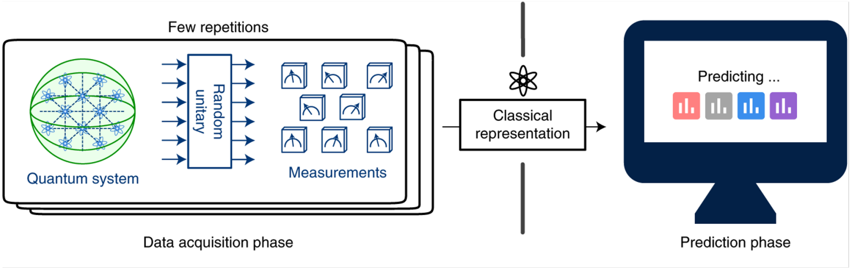
Classical shadows

- randomly rotate every qubit and measure
- efficient representation of the state ρ
- predict many observables O_k classically $\mathrm{tr}[\rho O_k]$
- rigorous performance guarantees



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Error mitigated classical shadows

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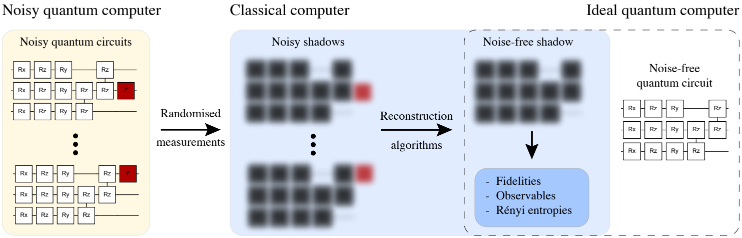
- most natural: Probabilistic Error Cancellation



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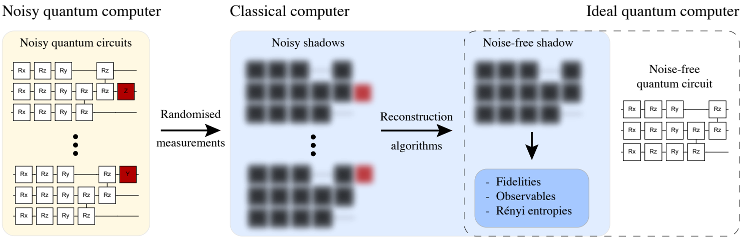
- most natural: Probabilistic Error Cancellation
- estimate M Pauli strings of locality q

$$N_s = 32\epsilon^{-2} \, \log(\frac{M}{\delta}) \, \|g\|_1^2 \, 3^q \tag{3}$$



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Shadow Spectroscopy

- due to phase evolution $|\psi(t)\rangle = \sum_{j=1}^{d} c_j \, e^{-itE_j} |\psi_j\rangle$

HHS Chan, R Meister, ML Goh, **B Koczor** *Algorithmic shadow spectroscopy* arXiv:2212.11036

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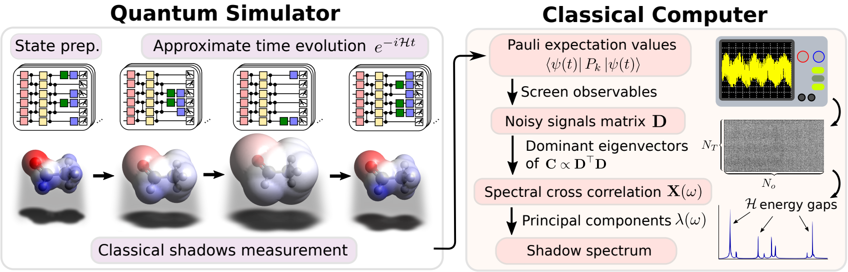

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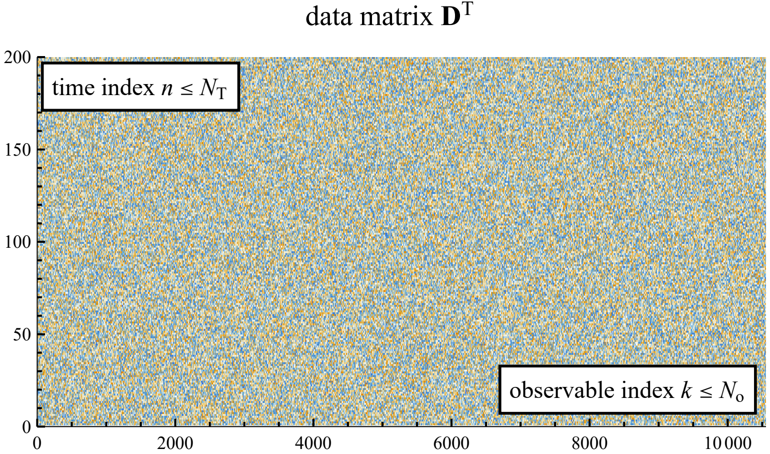

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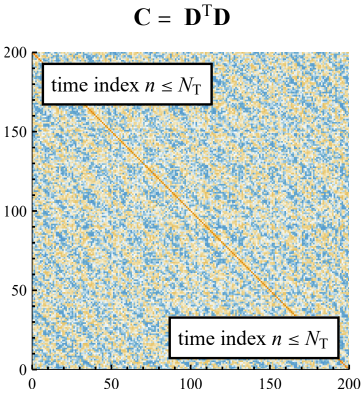
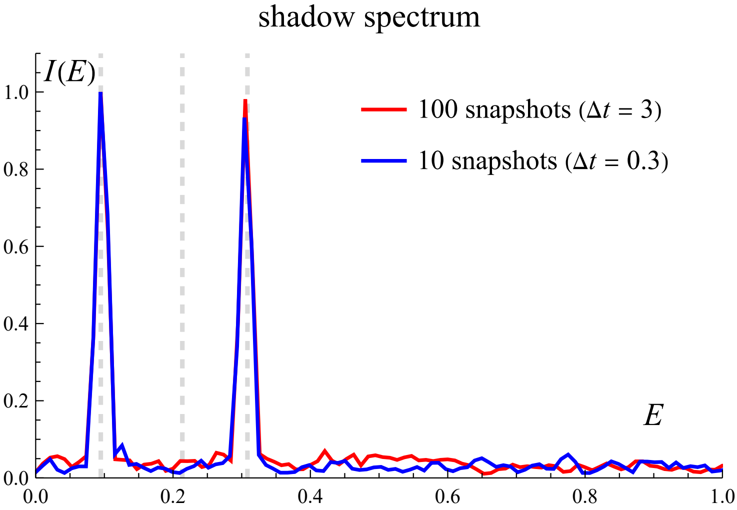


- spectrum via Fourier transform
- peaks are gaps in $\mathcal{H}$ with $|\psi(0)\rangle \propto (1, \frac{1}{10}, \frac{1}{10}, \ldots)$



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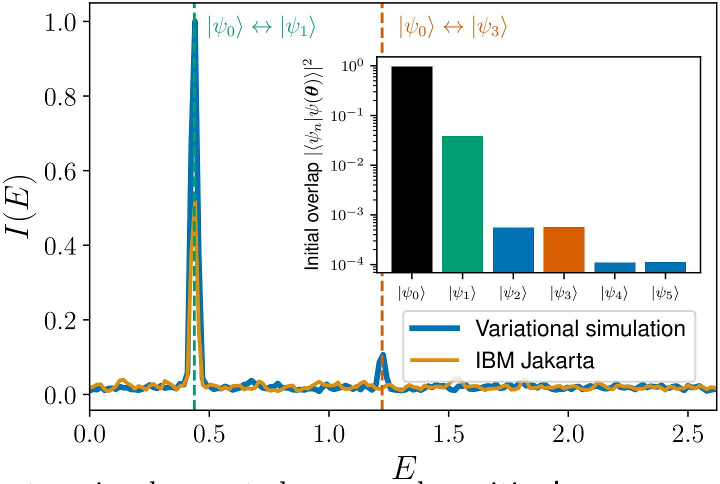
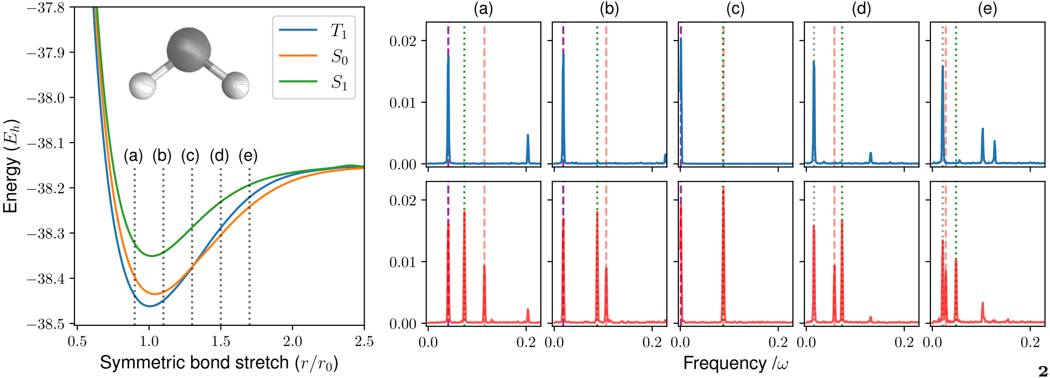


Theorem: gate noise does not change peak position!

Fault tolerant applications

- Fermions (chemistry) is sweet spot
- local observables expected to give intense signals
- rich information: peak to observable assignment



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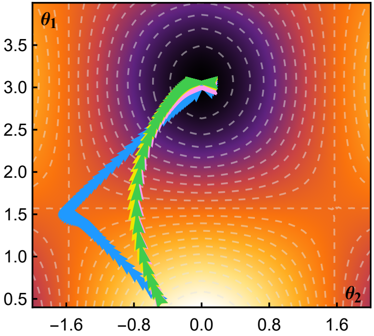

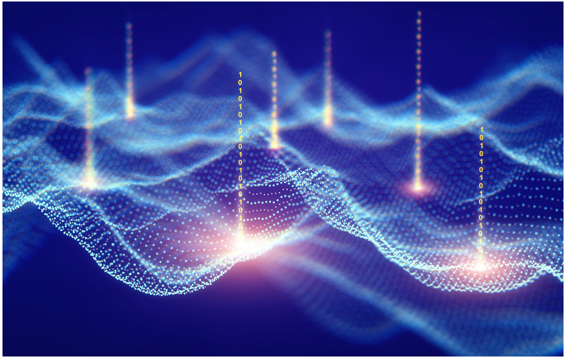

→ B van Straaten and **B Koczor** *Measurement Cost of Metric-Aware Variational Quantum Algorithms* (2021) PRX Quantum 2, 030324
→ **B. Koczor** and S. C. Benjamin, *Quantum Analytic Descent* (2022) Phys Rev Research 4 (2), 023017
→ **B Koczor** and S C Benjamin *Quantum natural gradient generalised to non-unitary circuits* (2022) Phys Rev A

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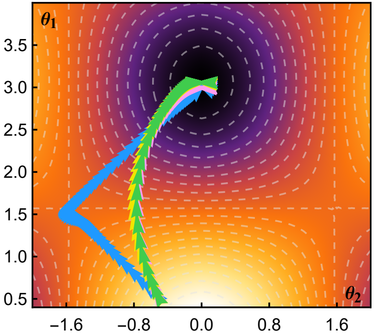
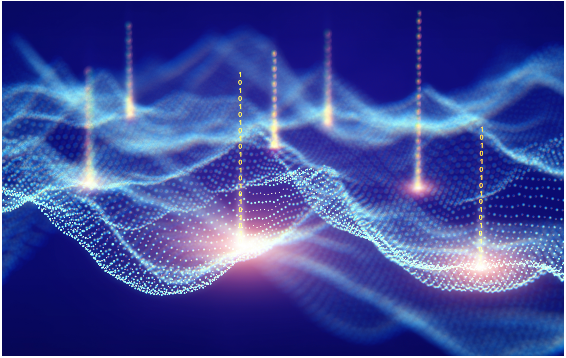
- can have exponentially many traps
- landscape flat almost everywhere: barren plateaus





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CoVAR: Covariances Root Finding

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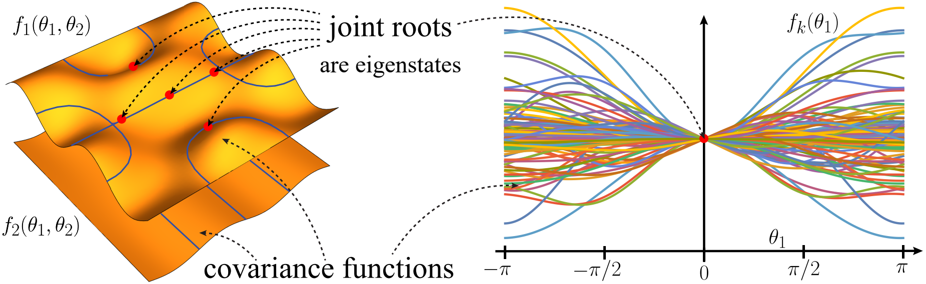
- **here:** simultaneously analyse $10^4 - 10^8$ landscapes



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Phys Rev X, (2022).

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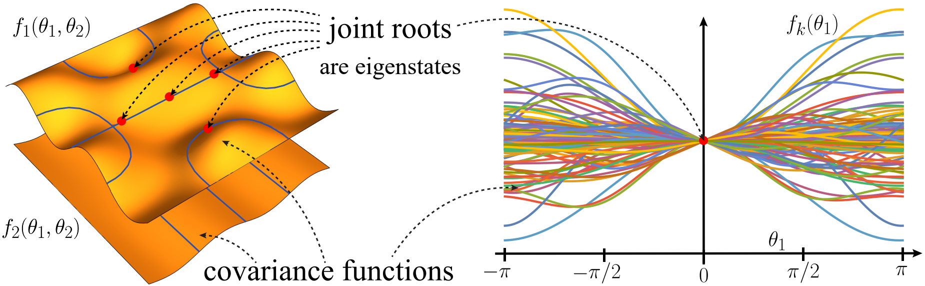
- possible due to powerful classical shadows



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Finding eigenstates by finding roots

covariance between Hamiltonian $\mathcal{H}$ and operator A

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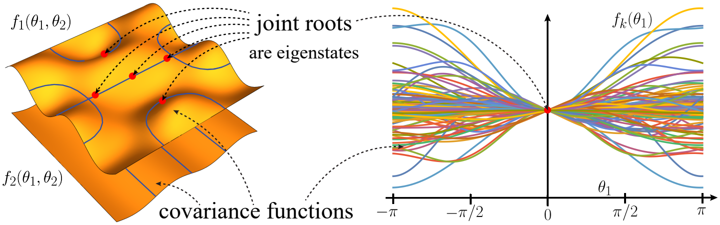
- covariance depends on parameters via state $|\psi(\underline{\theta})\rangle$

task: find parameters θ such that $\mathrm{Cov}_A = 0$ for any A

Newton's root finding

covariances form surfaces $f_k(\underline{\theta})$ whose roots $\underline{\theta}$ are solutions

$$f_1(\underline{\theta}) = 0, \quad f_2(\underline{\theta}) = 0, \quad \dots \quad f_{N_c}(\underline{\theta}) = 0$$



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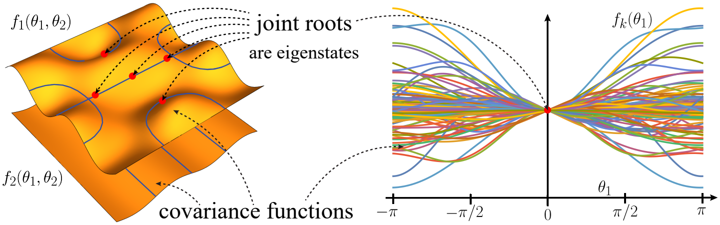
covariances form surfaces $f_k(\underline{\theta})$ whose roots $\underline{\theta}$ are solutions

$$f_1(\underline{\theta}) = 0, \quad f_2(\underline{\theta}) = 0, \quad \dots \quad f_{N_c}(\underline{\theta}) = 0$$

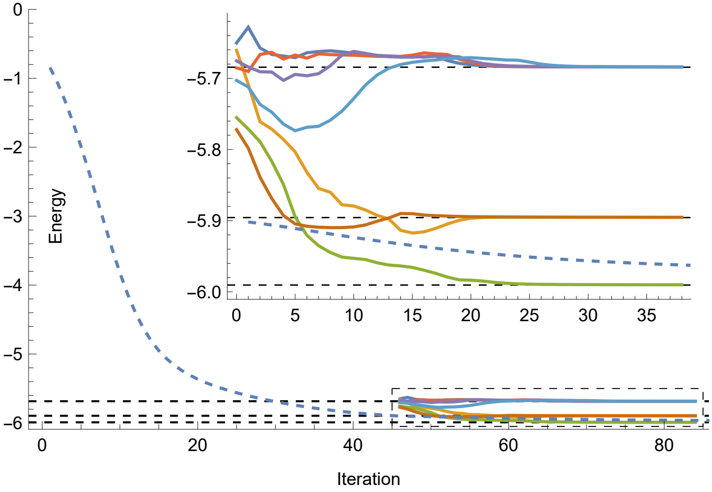
Newton: linearise $\mathbf{f}$ with Jacobian $\mathbf{J}$ and iterate

$$\underline{\theta}_{t+1} = \underline{\theta}_t - \mathbf{J}^{-1}\mathbf{f}$$



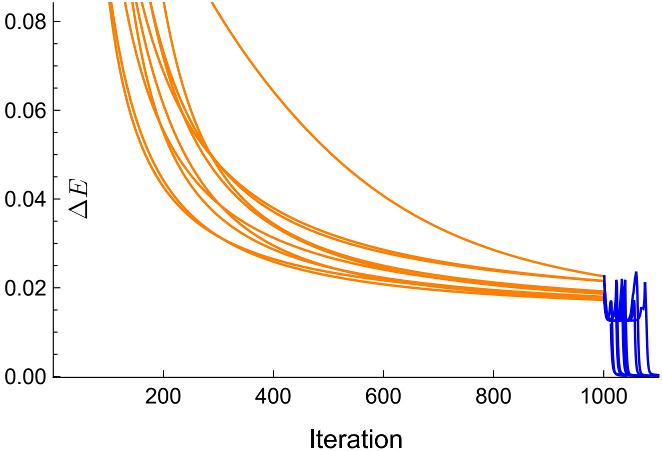
Practical applications

- phase estimation: gets attracted to any eigenstate
- naturally maps out the lowest eigenstates of $\mathcal{H} = J\sum_{i=1}^{N} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + c_i Z_i$



Practical applications

- phase estimation: gets attracted to any eigenstate
- naturally maps out the lowest eigenstates of $\mathcal{H} = J\sum_{i=1}^{N} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + c_i Z_i$
- robust to local traps



Emulation in Mathematica: QuESTLink

Download the QuEST Mathematica package

```
In[1]:= Import["https://quest.qtechtheory.org/QuEST.m"];
```

Connect to the remote Igor server (which must be running)

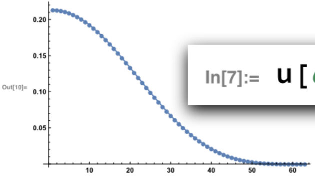
```
In[2]:= env = CreateRemoteQuESTEnv[45];
```

Create two 25-qubit registers, which are stored on Igor

```
In[3]:= ψ = CreateQureg @ 25;
        ϕ = CreateQureg @ 25;

In[5]:= InitPlusState @ ψ;
        InitPlusState @ ϕ;
```

An example computation done lightning fast using Igor's Quadro P6000

```
In[7]:= u[θ_] := H2 T4 H0 Rx1[θ] C3[Ry10[θ]];

In[9]:= fids = Table[
            CalcFidelity[ApplyCircuit[u[θ], ψ, ϕ], ψ],
            {θ, 0, 2π, .1}];

In[10]:= ListLinePlot[fids, PlotMarkers → Automatic]
```




```
In[7]:= u[θ_] := H2 T4 H0 Rx1[θ] C3[Ry10[θ]];
```


Disconnect from Igor

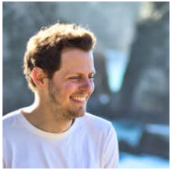
```
In[11]:= DestroyAllQuregs[];

In[12]:= DestroyQuESTEnv @ env;
```

quest.qtechtheory.org

Acknowledgments


Greg Boyd


Hans Chan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